Scitech Research Organisation

Journal of Progressive Research in Mathematics
E-ISSN: 2395-0218

On the vanishing of the cohomology of stable C^*-algebras

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Received: April 25, 2021; Accepted: May 20, 2021; Published: June 3, 2021

Cite this article: Alrashidi, Y. (2021). On the vanishing of the cohomology of stable C*- algebras. *Journal of Progressive Research in Mathematics*, *18*(2), 39-44.
Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2048.

Abstract.

We concerned with the vanishing of the dihedral and reflexive cohomology groups of stable C^*-algebra. Wodzicki has proved that, the cyclic cohomology of stable C^*-algebra is vanished. We extend this fact to prove that the reflexive and dihedral cohomology of this class are also vanish.

Key words: Dihedral homology – Stable algebras - C^*-algebra -cohomology.

Mathematics Subject Classification: 55Q05, 57Q10

1- Introduction.

There are many studies interested in vanishing the cohomology group of operator algebras. For example, thethird cohomology group $H^3(l^1(Z_+), l^1(Z_+)`) = 0$ where $l^1(Z_+)$ is a unital semi-group algebra of $N$ [15], also the third cyclic cohomology group $HC^3(I, I`) = 0$ where I is a nonunital Banach algebra $l^1(Z)$ [15]. The dihedral cohomology $^{\varepsilon}HD^n(A) = 0$, $n \in N$, n is odd, $\varepsilon = \pm 1$, where A is biflat algebra [4]. The class of algebra called Amenable algebras, that, is all continuous derivation from an algebra A into A-bimodule M are inner, is a good result of the vanishing of the I-St.dimensional cohomology of a Banach algebra A, with coefficient in A-bimodule M [8]. If A is a C^*-algebra without bounded traces or a nuclear C^*-algebra, the Hochschild and dihedral cohomology groups vanish ([12],[13]).

In the paper we study the vanishing cohomology groups (Reflexive and Dihedral) of some classes of C^*-algebra and give examples of nontrivial dihedral cohomology groups of a commutative Banach algebra under special condition.

2- Dihedral (Co)homology of operator algebra.

We recall the definition properties of Banach algebra and its homology from[1],[3] and [11]. Let A be a unital Banach algebra over a commutative ring $k(k = \mathbb{C})$. A complex $C(A) = (C^*(A), b_*)$, where $C_n(A) = A \otimes \ldots \otimes A$ is the tensor product of algebra $(n + 1$ times)and,$b_*: C_n(A) \to C_{n-1}(A)$ is the boundary operator

$$b_n(a_* \otimes ... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i \; a_0 \otimes .. \otimes a_i \, a_{i-1} \otimes .. \otimes a_{n-1}.$$

It is well known that $b_{n-1}b_n = 0$, and hence $kerb_n \supset Imb_{n+1}$.

$H_n(A) = H\big(C(A)\big) = \frac{ker\, b_n}{Im\, b_{n-1}}$ (1)

Is called the Hochschild homology of unital Banach algebras A with involutive and denote by $(HH_*(A))$.

If A is an unital Banach algebras, one acts on the complex $C(A)$, by the cyclic group of order $(n+1)$ by means of the operator $t_n: C_n(A) \to C_n(A)$

$$t_n(a_0 \otimes ... \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes ... \otimes a_{n-1}.$$

The quotient complex $CC_n(A) = \frac{C_n(A)}{Im(1-t_n)}$ is a subcomplex of a complex $CC_*(A)$.

Consider the chain complex $CC_*(A) = (C\mathcal{H}_\bullet(A), b_\bullet^{`})$ and the Connes-Tsygan bicomplex $CC_*(A)$ (see [5]). Then the subcomplex $(\ker(1-t_\bullet), b_\bullet^{`}) \subset (C\mathcal{H}_\bullet(A), b_\bullet^{`})$ has the same homology as the complex $(CC_\bullet(A), b_\bullet)$, that is

$$\mathcal{H}_\bullet(CC_\bullet(A) = \mathcal{H}_\bullet(CH_\bullet(A), b_\bullet)/\operatorname{Im}(1-t_\bullet)) = \mathcal{H}_\bullet(CH_\bullet(A), b_\bullet)/Ker\ N)$$
$$= \mathcal{H}_\bullet(\operatorname{Im} N, b_\bullet^{`}) = \mathcal{H}_\bullet(Ker(1-t_\bullet), b_\bullet^{`}), \qquad (2)$$

where

$$CH_n(A) = A^{\otimes n+1} = A \otimes ... \otimes A \;\; (n+1\; times),$$
$$b_n, b_n^{`}: CH_n(A) \to CH_{n-1}(A),$$

Such that:

$$b_n^{`}(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n),$$
$$b_n(a_0 \otimes \ldots \otimes a_n) = b_n^{`} + (-1)^n (a_n a \otimes \ldots \otimes a_{n-1}),$$
$$t_n: CH_n(A) \to CH_n(A),$$

Such that $t_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \ldots \otimes a_{n-1})$ and $N_n = 1 = t_n^1 + \ldots + t_n^n$.

Therefore, the complex $(ker(1-t_\bullet), b_\bullet^{`})$ is isomorphic to complex $(CC_\bullet(A), b_\bullet)$. The isomorphism between them is given by the operator $N_\bullet : CC_\bullet(A) \to (\ker(1-t_\bullet), b_\bullet^{`})$. Consequently, the action of the group $\mathbb{Z}/2$ on the complex $CC_\bullet(A)$, by means of the operator ${}^{\varepsilon}h$ is equal to action of $\mathbb{Z}/2$ on the complex $(\ker(1-t_\bullet), b_\bullet^{`})$ by means of the operator

$${}^{\varepsilon}r: a_0 \otimes a_1 \otimes \ldots \otimes a_n \to (-1)^{\frac{n(n+1)}{2}} \varepsilon a_n^* \otimes a^*{}_{n-1} \otimes \ldots \otimes a_0^*,$$

where a^* is the image of element in $a \in A$ under involution $*: A \to A$, $\varepsilon = \pm 1$. Since ${}^{\varepsilon}h_\bullet t_\bullet = t_\bullet^{-1}\; {}^{\varepsilon}h_\bullet$ then we have $N_\bullet({}^{\varepsilon}h) = ({}^{\varepsilon}h_\bullet)N_\bullet$.

On the other hand, since ${}^{\varepsilon}r_\bullet = t_\bullet\, {}^{\varepsilon}h_\bullet$ then

$${}^{\varepsilon}h_\bullet N_\bullet = N_\bullet \; {}^{\varepsilon}h_\bullet = (N_\bullet t_\bullet) \; {}^{\varepsilon}h_\bullet = N_\bullet(t_\bullet {}^{\varepsilon}h_\bullet) = N_\bullet \; {}^{\varepsilon}r_\bullet.$$

So, the dihedral homology of A is given by formula:

$$\varepsilon HD_\bullet(A) = H_\bullet(ker(1-t_\bullet)/(Im(1-\ ^{\varepsilon}\hbar_\bullet) \cap ker(1-t_\bullet))). \quad (3)$$

For a commutative unital Banach algebra *A*. We denote by$C^n(A)(n = 0,1,\dots)$the Banach space of continuous $(n+1)$-linear functionals on A*;* these functionals we shall later call *n*-dimensional cochains.

We let $t_n: C^n(A) \to C^n(A), (n = 1,2,\dots$denote the operator given by

$$t_n f(a_0, a_1, \dots, a_n) = (-1)^n = (-1)^n f(a_1, \dots, a_n, a_0)$$

and we set $t_0 = I$. We shall write t instead of t_n if it is clear which *n* we mean.

A cochain f satisfying $tf = f$ is called cyclic. We let $CC^n(A)$ denote the closed subspace of $C^n(A)$ formed by the cyclic cochains. (In particular, $CC^0(A) = C^0(A) = A^*$where A^* is the dual Banach space of$A$).

by proposition (4) in [4], $Im(1-t_n)$ is closed in $C^n(A)$ and $CC^n(A) = C^n(A)/Im(1-t_n)$. The induce operator$dc_n: CC^{n+1}(A) \to CC^n(A)$ in the respective quotient spaces. Thus,we obtain a quotient complex $CC^*(A)$ of complex $CC(A)$. The cohomology of $CC^*(A)$, denoted by $HC^n(A)$ is called the *n*-dimensional Banach cyclic cohomology group of A. We let $r_n: C_n(A) \to C_n(A), n = 0,1,\dots$ denote the operator given by the formula:

$$r_n(a_0 \otimes \dots \otimes a_n) = (-1)^{\frac{n(n+1)}{2}} \in a_0^* \otimes a_n^* \otimes \dots \otimes a_1^*, \in = \pm 1,$$

where * is an involution on *A*.

Note that: $Im(id_{t_n(A)} = 1 - t_n)$ is closed in $C^n(A)$.

The quotient complex,

$$CD^n(A) = \frac{C^n(A)}{Im(1-t_n) + Im(1-r_n)}$$

of a complex $C^n(A)$. The *n*-dimensional cohomology of $CD^n(A)$ denoted by $HD^n(A)$ is called *n*-dimensional dihedral cohomology group of a unital Banach algebra *A*.

We can similarly get the reflexive cohomology $HR^n(A)$.

3- Maine result.

In this part we prove the main theorem of our study. We prove the vanishing state of C^*-algebra.

Definition 3.1:

AC^*-algebra A is called stable if it isomorphic to the tensor product algebra $(K \otimes A)$, where K is the algebra of compact operators on a separable infinite-dimensional Hilbert space.

In ([2], [6]) we find the definitions of the simplicial, cyclic, reflexive and dihedral cohomology of operator algebra. Following [10] the relation between Hochschild, cyclic, reflexive and dihedral cohomology is given by the following commutate diagram $\mathfrak{C}(A)$:

$$\begin{array}{ccccccccc}
\dots \to & {}^{-\alpha}HR^{n+1}(A) & \to & {}^{-\alpha}HD^{n+1}(A) & \to & {}^{\alpha}HD^{n+3}(A) & \to & {}^{-\alpha}HR^{n+2}(A) & \to \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots \to & {}^{\alpha}HR^{n-1}(A) & \to & {}^{\alpha}HD^{n-1}(A) & \to & {}^{-\alpha}HD^{n+1}(A) & \to & {}^{\alpha}HR^{n}(A) & \to \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots \to & \mathrm{H}^{n-1}(\mathrm{A}) & \to & \mathrm{HC}^{n-1}(\mathrm{A}) & \to & \mathrm{HC}^{n+1}(\mathrm{A}) & \to & \mathrm{H}^{n}(\mathrm{A}) & \to \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

$$\begin{array}{ccccccccc}
\ldots \to & {}^{-\alpha}HR^{n-1}(A) & \to & {}^{-\alpha}HD^{n-1}(A) & \to & {}^{\alpha}HD^{n+1}(A) & \to & {}^{-\alpha}HR^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{\alpha}HR^{n-3}(A) & \to & {}^{\alpha}HD^{n-+3}(A) & \to & {}^{-\alpha}HD^{n-1}(A) & \to & {}^{\alpha}HR^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

Suppose that M_m is the algebra of matrices of ordered m with m coefficients in algebraA over ring k with identity. Then the natural isomorphism $HH^*(M_m(A)) \approx HH^*(A)$ holds [7]. It is called a Morita equivalence. Following [44] the cyclic cohomology is Morita equivalence. If A be involutive algebra with identity, the following assertion holds [see [9]].

Proposition3.2:

There exists an isomorphism;

$$Tr_*: \ {}^{\alpha}HD^*(M_m(A)) \to \ {}^{\alpha}HD^*(A)$$

for all and $m > 1$ and $n > 0$.

We shall denote by the $B^*(A)$ the reflexive or dihedral cohomology $(\ {}^{\alpha}HR^*(A)$ or $\ {}^{\alpha}HD^*(A))$ of algebra A.

Our aim now is to prove the following assertion [14].

Theorem3.3:

Let A be a stable C^*-algebra, then the reflexive and dihedral cohomology of A vanishes, i.e

$${}^{\alpha}HR^*(A) = 0, \qquad {}^{\alpha}HD^*(A) = 0, \qquad \alpha = \pm 1.$$

Firstly, we need the following facts:

Lemma 3.4: [4]

Let Abe a C^*-algebra without unit, and for $k > 0$, let$M_k(A)$ is the C^*-algebra of matrices over A, and $i: A \to M_k(A)$ is an inclusion maping such that,

$$a \to \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

is a quasi-isomorphism.

Proof:

Let A be a C^*-algebra without unit. If we adjoint A with an identity element we get $\overline{A} = A \oplus \mathbb{C}$. Consider the following short exact sequence

$$0 \to A \to \overline{A} \to \mathbb{C} \to 0 \qquad (1)$$

where $\overline{A}$ is algebra A with unit. We have the corresponding inclusion of algebra extensions

$$\begin{array}{ccccc}
A & \to & \overline{A} & \to & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
M_k(A) & \to & M_k(\overline{A}) & \to & M_k(\mathbb{C})
\end{array} \qquad (2)$$

Following [13] and [14], since $M_k(A)$ *is* C^*-algebra, it is excision in Hochschild and cyclic homology, this fact is extended to reflexive and dihedral cohomology,

$$\begin{array}{ccccc}
0 \to B_*(A) & \to & B_*(\overline{A}) & \to & B_*(\mathbb{C}) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to B_*(M_k(A)) & \to & B_*\left(M_k(\overline{A})\right) & \to & B_*(M_k(\mathbb{C})) \to 0
\end{array} \qquad (3)$$

Where $B_*(\overline{A}) \to B_*(M_k(A))$ and $B_*(M_k(\mathbb{C})) \to B_*(\mathbb{C})$ are isomorphisms in view of the Morita invariance in reflexive and dihedral cohomology, then $B^*(A) \xrightarrow{\sim} B^* M_k(A)$.

Proposition 3.5:

Suppose that Abe a C^*-algebra, then the following isomorphism is exists $B^{n+1}(K\otimes A) \approx B^n(K\otimes q_1\otimes A)$.

Where $q_n,\ n = 0,1, \ldots$ is the algebra of continuous functions on the n-sphere which vanish at the Northern pole.

Proof:

Consider the following exact sequence

$$0 \to q_1 \to J \xrightarrow{p} \mathbb{C} \to 0 \qquad (4)$$

where J is the algebra of continuous functions on the unit interval [0,1], that vanish at the left end, $ker\, p = q_1$.

Tensoring the sequence (4) by $(K\otimes A)$ we get the following exact and split sequence

$$0 \to (K\otimes q_1\otimes A) \to (K\otimes J\otimes A) \to (K\otimes A) \to 0 \qquad (5)$$

the sequence (5) induces the long exact sequence in dihedral and reflexive cohomology (see [9]).

$$\ldots \to B^{n+1}(K\otimes J\otimes A) \to B^{n+1}(K\otimes A) \xrightarrow{\partial} B^n(K\otimes q_1\otimes A) \to B^n(K\otimes J\otimes A) \to \cdots \qquad (6)$$

where the connecting homomorphism ∂ is commute with the canonical maps: $HR^n \xrightarrow{I} HD^n,\ HR^n \to HR^n$, and $HD^n \to HD^n$. To show that $B^*(K\otimes J\otimes A) = 0$, consider for a C^*-*algebra* A a functor $F(A) = F(K\otimes A)$ from a category of C^*-algebra to a category of graded complex vector spaces, clearly F is stable and split-exact on the collection of the split C^*- extensions (see [8]). It is known that any functor with these two properties (stable and split-exact) is homotopy invariant. Since the identity and zero endomorphisms of $(J\otimes A)$ are homotopic, then $F(J\otimes A) = B^*(K\otimes J\otimes A) = 0$. using this result and sequence (6) we can easily deduce $B^{n+1}(K\otimes A) \approx B^n(K\otimes q_1\otimes A)$.

Proof theorem 3.3:

From the above proposition we obtain the following commutative diagram,

$$\begin{array}{ccc} {}^{\alpha}HR^n(K\otimes A) & \xrightarrow{I} & {}^{\alpha}HD^n(K\otimes A) \\ \downarrow & & \downarrow \\ {}^{\alpha}HR^0(K\otimes q_n\otimes A) & = & {}^{\alpha}HD^0(K\otimes q_n\otimes A) \end{array} \qquad (7)$$

From the above diagram we obtain thus the isomorphism:

$$I:\ {}^{\alpha}HR^*(K\otimes A) \xrightarrow{I} {}^{\alpha}HD^*(K\otimes A).$$

The Connes long exact sequence related the reflexive and dihedral cohomology is given by,

$$\begin{aligned} \ldots \to\ & {}^{\alpha}HR^1(K\otimes A) \to {}^{\alpha}HD^0(K\otimes A) \to {}^{-\alpha}HD^2(K\otimes A) \to {}^{\alpha}HR^2(K\otimes A) \\ & \to {}^{\alpha}HD^1(K\otimes A) \xrightarrow{s} {}^{-\alpha}HD^3(K\otimes A) \to \cdots \to {}^{\alpha}HR^n(K\otimes A) \\ & \to {}^{\alpha}HD^{n-1}(K\otimes A) \xrightarrow{s} {}^{-\alpha}HD^{n+1}(K\otimes A) \to \cdots \end{aligned} \qquad (8)$$

where s is a periodic operator. From the diagram (7) and the sequence (8) we have;

$${}^{\alpha}HD^*(K\otimes A) = {}^{\alpha}HR^*(K\otimes A) = 0, \qquad \alpha = \pm 1$$

Example 3.6:

Let $u = \mathcal{F}(H)/k$ be the Calkin algebra then,

$${}^{\alpha}HR^*(u) = {}^{\alpha}HD^*(u) = 0.$$

Example 3.7:

Let $\mathcal{F}$(H)denote the algebra of bounded operators on an infinite dimensional Hilbert space H. Then

$${}^{\alpha}HR^*(\mathcal{F}(H)) = 0 \text{ and } {}^{\alpha}HD^*(\mathcal{F}(H)) = 0.$$

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