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Extremal Fuzzy Space

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Abstract

The extremal topology on an arbitrary set X was defined as a maximal non-discrete topology [1] and [2]. In this paper we introduce an extremal topology on a fuzzy set X, which is also a maximal non-discrete topology, and it has to be in a specific form. This form depends on some ultrafilters $\Im$. We consider some properties for this kind of topologies when $\Im$ is free. The subspaces and the base of this topology is also considered.

Keywords

An extremal topology; Fuzzy space; ultrafilter; door space.

1. Introduction

If X is a set, in 1991 , the extremal topology was defined by Papazyan, see [1], which is a maximal non-discrete topology τ on X, so every topology which is strictly finer than τ is discrete.

In 1965 Zadeh [6], introduced the fundamental concept of fuzzy sets, which formed the backbone of fuzzy mathematics. After that some studies presented by Chang and Wong investigated the basic concepts and general properties of fuzzy topologies.

On the other side, some studies are presented by, for instance, Sola and me about the concept of the extremal topologies.

In this paper we will consider an extremal topology on a fuzzy set X and prove that this topology has to be in a specific form τ_{x_0}, for each $x_0 \in X$. After that we will consider some topological properties on it.

2 Preliminaries

The material of this section mostly comes from [6], [7], [10], [8] and [11].

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Definition 2.1. *An extremal topology on a nonempty set X is a maximal non-discrete topology τ.*

Let X be non-empty set and $I = [0, 1]$ unit interval, we denoted all functions from X to I by I^X.

Definition 2.2. *A Fuzzy set A in X is every element of I^X, so a fuzzy set A in X is characterized by a membership function $\mu_A : X \to I$ which associates with each point $x \in X$ and it is denoted by $A = \{(x, \mu_A(x)) : x \in X\} \subseteq X \times I$.*

For fuzzy sets A and B in X and $x \in X$,we have

- $A = B \iff \mu_A(x) = \mu_B(x)$.
- $C = A \cup B \iff \mu_C(x) = max\{\mu_A(x), \mu_B(x)\}$.
- $C = A \wedge B \iff \mu_C(x) = min\{\mu_A(x), \mu_B(x)\}$.
- $\mu_{A^c}(x) = 1 - \mu_A(x)$.

More generally, for a family of fuzzy sets, $\wp = \{A_i : i \in I\}$, the union $C = \cup_i A_i$, and the intersection, $D = \cap_i A_i$, are defined by

$$\mu_C(x) = sup\{\mu_{A_i}(x) : i \in I, x \in X\}, and \mu_D(x) = inf\{\mu_{A_i}(x) : i \in I, x \in X\}$$

The symbol $\widetilde{0}$ will be used to denote the empty fuzzy set, so $\mu_{\widetilde{0}}(x) = 0$ for all $x \in X$, whereas $\widetilde{1}$ will be denoted for the set X where $\mu_{\widetilde{1}}(x) = 1$ for all $x \in X$.

For $\lambda \in (0, 1]$, a fuzzy point $P^{\lambda}_{x_0}$ in X is a fuzzy set on X with support x_0 and value λ, i.e:

$$P^{\lambda}_{x_0} = \begin{cases} \lambda, & \text{if } x = x_0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

so it can be written as $P^{\lambda}_{x_0} = \{(x_0, \lambda)\}$.

Definition 2.3. . *A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies following conditions:*

- $\widetilde{1}, \widetilde{0} \in \tau$.
- *If $A, B \in \tau$ then $A \cap B \in \tau$.*
- *If $A_i \in \tau$ then $\cup_i A_i \in \tau$*

the pair (X, τ) is called a fuzzy topological space where its members are called open fuzzy sets. A fuzzy set is closed iff its complement A^c is open. As in general topology, the indiscrete fuzzy topology contains only $\widetilde{0}$, and $\widetilde{1}$, while the discrete fuzzy topology contains all fuzzy sets.

A fuzzy Filter on a fuzzy set X is a nonempty collection $\Im \subset I^X$ of nonempty fuzzy sets on X which satisfies that $F_1 \cap F_2 \in \Im$ for all $F_1, F_2 \in \Im$ and if $F_1 \subseteq F_2$ and $F_1 \in \Im$ then $F_2 \in \Im$. The maximal, with respect to set inclusion, fuzzy filter on X is called a fuzzy ultrafilter on X. A collection $\mathfrak{B}$ of subsets of $\Im$ is a base for $\Im$ iff for each $F \in \Im$ there is some $B \in \mathfrak{B}$ such that $B \leq F$.

A fuzzy topological space X is said to be a fuzzy door space if every fuzzy subset of X is either fuzzy open or fuzzy closed. Whereas, a topological space X is called a $T_{1/2}-$space if every one-point set is either open or closed. Clearly a door space is a $T_{1/2}-$space.

A point P^r_y is said to be an accumulation point of A iff for every open set U contains P^r_y satisfies $U \cap A_p \neq \emptyset$, where $\mu_{A_p}(y) = 0$ and $\mu_{A_p}(x) = \mu_A(x)$ otherwise.

3 Extremal Fuzzy Topology

Definition 3.1. *Let $r \in I$, we define*

- $P(X\mid_{x_0}^{r})$ *be the collection of all fuzzy subsets $A \subseteq X$ where $\mu_A(x_0) = r$, so $\mu_{X\mid_{x_0}^{r}}(x_0) = r$ and otherwise $\mu_{X\mid_{x_0}^{r}}(x) = 1$.*
- $\Im \subseteq P(X\mid_{x_0}^{r})$ *be a filter on X. So for any $F \subseteq \Im$, $\mu_F(x_0) = r$.*

Theorem 3.2. *Let X be a fuzzy space, $P_{x_0}^{r}$ be a fuzzy point in X and $\Im$ be a filter as above then:*

$$\tau_{x_0} = P(X\mid_{x_0}^{0}) \cup \{P_{x_0}^{r} \cup F : F \in \Im\}$$

is a topology on X .

Proof. It is clear to prove τ_{x_0} is a topology, where $\widetilde{0} \in P(X\mid_{x_0}^{0})$ and $\widetilde{1} = X\mid_{x_0}^{0} \cup (P_{x_0}^{1} \cup F)$ for some $F \in \Im$. □

Corollary 3.3. *The topology $\tau_{x_0} = P(X\mid_{x_0}^{0}) \cup \{P_{x_0}^{r} \cup F : F \in \Im\}$ is a door space and hence a $T_{1/2}$ space.*

Proof. If $A \subseteq X$ such that $\mu_A(x_0) = 0$, then $A \in P(X\mid_{x_0}^{0})$ is open. If $\mu_A(x_0)$ and $\mu_A(y)$ are not vanish for some $y \neq x_0$, then A has the form $P_{x_0}^{r} \cup F$ for some $F \in \Im$ and $\mu_F(y) \neq 0$, so A is also open. But in case $\mu_A(x_0) \neq 0$ and $\mu_A(y) = 0$ for all $y \neq x_0$, in this case A is closed. therefore τ_{x_0} is a door space and hence a $T_{1/2}$ space. □

Lemma 3.4. *Let $\mathfrak{B}$ be a filter base for a filter $\Im$ and let $x_0 \in X$. Then $\mathfrak{B}_{x_0} = \{P_x^i\} \cup \{P_{x_0}^{r} \cup B : B \in \mathfrak{B}\}$ be a base for the previous topology τ_{x_0} for each $x \in X$ and $i, r \in (0, 1]$.*

Proof. If u be an open set in τ_{x_0} which contains P_x^r. In case $u \in P(X\mid_{x_0}^{0})$, the one point set P_x^r is contained in u. But if u has the form $P_{x_0}^{r} \cup F$ for some $F \in \Im$, so there is $B \in \mathfrak{B}$ such that $B \subseteq F$, and this yields to $P_x^r \in P_{x_0}^{r} \cup B \subseteq P_{x_0^r} \cup F$. □

Corollary 3.5. *Let X be a non-empty fuzzy set and $\tau_{x_0} = P(X\mid_{x_0}^{0}) \cup \{P_{x_0}^{r} \cup F : F \in \Im\}$ for some $x_0 \in X$. If $\Im$ is a free filter on $X\mid_{x_0}^{0}$, then τ_{x_0} is a Hausdorff space, furthermore, it has at most one accumulation point.*

Proof. Let $P_x^r \neq P_y^i$ for each $x \neq y$ in X and $r, i \in (0, 1]$. If $x \neq x_0$ and $y \neq x_0$, then $P_x^r, P_y^i \in P(X\mid_{x_0}^{0})$ are disjoint open set. But if $x = x_0$ and $y \neq x_0$. Since $\Im$ is a free filter, so $\cap_{F \in \Im} F = \emptyset$, then there is $F \in \Im$ such that $\mu_F(y) = 0$. Let $u = P_{x_0}^{r} \cup F$. Therefore u and P_y^i are disjoint open sets contains, respectively, P_x^r and P_y^i. Thus τ is a Hausdorff space.

By 3.3, τ is a door space, and by [4], A Hausdorff door space has at most one accumulation point. □

Corollary 3.6. *If $\Im \subseteq P(X\mid_{x_0}^{0})$ be an ultrafilter on a fuzzy set X, then the collection of the nonempty sets $\Im_A = \{A \cap F : F \in \Im\} \subseteq P(A\mid_{x_0}^{0})$ be an ultrafilter on $A \subseteq X$.*

Proof. It is easy to prove that. □

In case $\Im$ is an ultrafilter, we can deduce the following result which is the main aim of this paper:

Theorem 3.7. *Let X be a fuzzy space and $\Im$ be an ultrafilter, then any extremal topology on X has the following form:*

$$\tau_{x_0} = P(X\mid_{x_0}^{0}) \cup \{P_{x_0}^{r} \cup F : F \in \Im\}$$

Proof. suppose that τ_{x_0} be an extremal topology on a fuzzy set X, so there exists a fuzzy point, lets call it $P^r_{x_0}$, which is not in τ_{x_0}, for $r \in (0,1]$. Let $\mathfrak{B} = \{B \in P(X\ |^0_{x_0}) : P^r_{x_0} \cup B \in \tau_{x_0}\}$ be a filter base for some a filter $\wp$ on $X\ |^0_{x_0}$. Let $\Im$ be an ultrafilter contains $\wp$.

Assume that η be a topology on X which is generated by $\tau_{x_0} \cup \{P^r_{x_0} \cup F : F \in \Im\}$. Since $P^r_{x_0}$ is not belong to τ_{x_0} and $F \neq \emptyset$ for all $F \in \Im$, so $P^r_{x_0}$ also is not belong to η. This shows that η is non-discrete, and since τ_{x_0} is extremal, this leads to $\{P^r_{x_0} \cup F : F \in \Im\} \subset \tau_{x_0}$. Also we have $P^r_y \in \tau_{x_0}$ for all $y \neq x_0$ because τ_{x_0} is extremal, therefore $P(X\ |^0_{x_0}) \cup \{P^r_{x_0} \cup F : F \in \Im\} \subseteq \tau_{x_0}$.

On the other hand, let $u \in \tau_{x_0}$ such that u is not in $P(X\ |^0_{x_0}) \cup \{P^r_{x_0} \cup F : F \in \Im\}$, then $u = P^r_{x_0} \cup A$ where $r \in (0,1]$ such that $A \cap F = \emptyset$ for all $F \in \Im$.

Since $P^r_{x_0} \cup F \in \tau_{x_0}$, then $P^r_{x_0} = (P^r_{x_0} \cup A) \cap (P^r_{x_0} \cup F) \in \tau_{x_0}$, this contradicts the extremality of τ_{x_0}, therefore $u \in P(X\ |^0_{x_0}) \cup \{P^r_{x_0} \cup F : F \in \Im\}$. Thus $\tau_{x_0} = P(X\ |^0_{x_0}) \cup \{P^r_{x_0} \cup F : F \in \Im\}$.

Conversely, assume that $\tau_{x_0} = P(X\ |^0_{x_0}) \cup \{P^r_{x_0} \cup F : F \in \Im\}$ and assume that $\tau_{x_0} \subsetneq \tau$ for some a non-discrete topology τ on X. So there exists $u \in \tau$ which is not in τ_{x_0}, then u has the form $P^r_{x_0} \cup A$ where $r \in (0,1]$ and $A \cap F = \emptyset$ for all $F \in \Im$. Similar to the previous discussing we get a contradiction. therefore $\tau = \tau_{x_0}$, and this shows that τ_{x_0} is an extremal topology.

□

Corollary 3.8. *Let $\emptyset \neq A \subseteq X$. The subspace τ_A of an extremal topology τ_{x_0} on X is an extremal topology on A.*

Proof. For a nonempty fuzzy subset A of X, we have $\tau_A = \{A \cap u : u \in \tau_{x_0}\} = P(A\ |^0_{x_0}) \cup \{A \cap (P^r_{x_0} \cup F) : F \in \Im\}$ Since $A \cap (P^r_{x_0} \cup F) = P^i_{x_0} \cup (A \cap F)$, where $i = min\{r, \mu_A(x_0)\} \in (0,1]$. By 3.6, $A \cap F = F_A \in \Im_A$. Then τ_A has the form $P(A\ |^0_{x_0}) \cup \{P^i_{x_0} \cup F_A : F_A \in \Im_A\}$ which is an extremal topology on A. □

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