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Study of the homology theory of Hecke Algebras

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Abstract.

In the ready product, we study the simplicial and cyclic homology of a unital Z-graded Hecke algebras$H$ over $k = \mathbb{C}$ and consider a couple of properties of it. Along these lines, we given a relation between simplicial and cyclic homology of graded Hecke algebras.

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1. Introduction

Hecke algebras is that algebra defined on the Hecke operator. In 1937, Hecke operator has introduced by E. Hecke. After that, L. J. Mordell studied the Hecke operator. In the sixties, Shimura showed some notations in abstract Hecke algebras. In [7], Nistor introduced the crossed product of a ring $\mathcal{O}(X)$ and the smooth Γ, and he computed the Hochschild homology for it. Also, he introduced some definitions about Cyclic homology.

In [8], Solleveld studied the graded Hecke algebras, since he introduced the description for the spectrum of graded Hecke algebras.

Here, we will study the unital Z-graded Hecke algebras H and the Simplitial and Cyclic homology of H.

In section 2: we study the crossed product and graded Hecke algebras with some important definitions and related properties.

In section 3: we study and introduce the Cyclic homology of graded Hecke algebras. Finally, we give and prove the relation between simplicial and cyclic homology of graded Hecke algebras and also, we prove the *Mayer-vietories sequence* of graded Hecke algebras.

In this segment, we demonstrate some essential considerations and convictions concerning graded Hecke algebras.

Definition (1.1): [2]

Let G be a group and A be an F-algebra ($F$is a field). Let $\beta: G \to Aut(A)$ be the nimbleness of Gon A by algebra auto-morphisms. Build vector space$A \underset{F}{\otimes} FG$ with the multiplication;

$$(a \otimes g).(a^{/} \otimes g^{/}) = a\beta_g(a^{/}) \otimes gg^{/}, \text{ for all } a, a^{/} \in A, g, g^{/} \in G.$$

This characterizes an associative F-algebra, denoted$(A \rtimes G)$or$(G \ltimes A)$, called crossed product of $A$and G.

We can use Xas topological space and Asub-algebra of $C(X;\mathbb{C})$whose maximal ideal spectrum is definitely X .Let G be a limited group and acts on X by homeomorphisms, such an extent that the actuated activity on $C(X;\mathbb{C})$ preserves A. Let $\mathbb{C}_x$be the one-dimensional A-module with character $x \in X$. We compose

$G_x := \{g \in G: g(x) = x\}, I_x := Ind_A^{A\rtimes G}\mathbb{C}_x.$

Theorem (1.2): [9]

(a) $I_x \cong I_{x^{/}}$If$Gx = Gx^{/}$.

(b) $I_x \cong Ind_{A\rtimes G_x}^{A\rtimes G}(\mathbb{C}[G_x])$, where A follows up on $\mathbb{C}[G_x]$ through the assessment at$x$. (c) I_x is completely reducible.

We can depict crossed product of its(co)homology by the extended quotients.Let

$$\tilde{X} = \{(g, x) \in G \times X: g(x) = x\}. \quad (1.1)$$

and characterize G is acts on $\tilde{X}$ by $g(g^{/}, x) = (gg^{/}g^{-1}, g(x))$. The broadened quotients of X by G is characterized as $\tilde{X}/G$. We compose

$$X^g = \{x \in X: g(x) = x\}. \quad (1.2)$$

A form $Z_G(g)$ be the center of g in G, let $G/\sim$ is accumulation of accountancy classes in G. The expanded quotient can be additionallydeveloped as a disjoint union:

$$\tilde{X}/G = \left(\bigcup_{g\in G}(g, X^g)\right)/G = \bigcup_{c\in G/\sim}\left(\bigcup_{g\in c}(g, X^g)/G\right)$$

$$\cong \bigcup_{g/\sim \in G/\sim} (g, \frac{X^g}{Z_G(g)}) \cong \coprod_{g/\sim \in G/\sim} X^g/Z_G(g)$$

Let $A = O(X)$, the algebra of regular functions on X. Both X^g and $\tilde{X}$are nonsingular affine varieties. An algebra$A = O(X)$is smooth as it has no singularity at 0, hence a crossed product $O(X) \rtimes G$is smooth. Let $\Omega^n(X)$is the space of all algebraic n-forms on $X$and $H^n_{DR}(X)$be the De-Rhamcohomology of X.

For a unital F-algebra$A$and A-bimodule Mthere is a differential complex$(C_\bullet(A, M), b)$, where $C_n(A, M) = M \otimes_F A^{\otimes n}$ and $b: C_n(A, M) \to C_{n-1}(A, M)$ is characterized on basic tensors as:

$$b\left(m \otimes a_1 \bigotimes \dots \otimes a_n\right) = ma_1 \bigotimes \dots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \bigotimes \dots \otimes a_i a_{i+1} \bigotimes \dots \otimes a_n$$

$$+(-1)^n a_n m \otimes a_1 \bigotimes \dots \otimes a_{n-1} \tag{1.3}$$

The complex $(C_\bullet(A, M), b)$ is known as the simplicial complex whose homology is known as the simplicial homology of A with coefficients in M. Indicated by $H_\bullet(A, M) = H_\bullet(C_\bullet(A, M), b)$. To characterize cyclic homology, we put $M = A$overlook it from the documentation and get $HH_n(A) = H_n(C_\bullet(A), b)$. By stretching out the complex $C_\bullet(A)$ to the mixed complex $B_\bullet(A)$:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ C_2(A) & \overset{B}{\leftarrow} & C_1(A) & \overset{B}{\leftarrow} & C_0(A) \\ \downarrow b & & \downarrow b & & \\ C_1(A) & \overset{B}{\leftarrow} & C_0(A) & & \\ \downarrow b & & & & \\ C_0(A) & & & & \end{array} \tag{1.4}$$

Where $C_n(A) = A^{\otimes(n+1)}$, $n = 0,1,2, \dots$. The operator $B: C_n(A) \to C_{n+1}(A)$ is Cone's operator and fulfills: $Bb + bB = 0$, $B^2 = 0$ is given as the form,

$$B\left(a_1 \bigotimes \dots \otimes a_n\right) = \sum_{i=0}^{n} (-1)^{ni} \left(1 \otimes a_i - a_i \bigotimes 1\right) \otimes a_{i+1} \bigotimes \dots \otimes a_n \otimes a_1 \bigotimes \dots \otimes a_{i-1}$$

Then, the form of the cyclic homology of A is$HC_n(A) = H_n(B_\bullet(A), b, B)$.

Presently, we describe the simplicial homology of crossed products. Let A and G, the complex $C_n(A \rtimes G)$ has a subspace

$$C_n(A)_g \coloneqq span\left\{ga_0 \otimes a_1 \bigotimes \dots \otimes a_n : a_i \in A\right\}$$

Notice:

1- The complex $(C_\bullet(A)_g, b)$is a subcomplex of $(C_\bullet(A \rtimes G), b)$, whose homology is $H_n(C_\bullet(A)_g, b) = H_n(A, A_g)$, where the A-bimodule structure on $M = A_g$is given by

$$a \cdot m \cdot a' = g^{-1}(a)ma', \quad \text{for all} \quad a, a' \in A, m \in A_g$$

2- $(C_\bullet(A \rtimes G), b) = (\oplus_{g \in G} C_\bullet(A)_g\,, b)$, hence

$$HH_n(A \rtimes G) = H_n(C_\bullet(A \rtimes G), b) \cong H_n\left(\bigoplus_{g \in G} C_\bullet(A)_g\,, b\right)_G$$

$$= \bigoplus_{g \in G} H_n(C_\bullet(A)_g, b) \cong \left(\bigoplus_{g \in G} H_n(A, A_g)\right)^G \tag{1.5}$$

where the subscripts and superscripts G mean covariant and invariants, individually. The G-action is characterized by methods of incorporation$C_\bullet(A)_g \to C_\bullet(A \rtimes G)$:

$$h \cdot \left(ga_0 \otimes a_1 \bigotimes \ldots \otimes a_n\right) = (hgh^{-1})h(a_0) \otimes h(a_1) \bigotimes \ldots \otimes h(a_n) \tag{1.6}$$

Lemma (1.4):

Let A algebra over field F and $A_g$an$A$-bimodule. Then,$H_0(A, A_g) = A_g/[A, A_g]$where $[A, A_g]$is the subspace spanned by all commutators $[ga_0, a_1] = ga_0a_1 - a_1ga_0$ in A then we get:$a_1 \in A, ga_0 \in A_g$.

Example (1.5):

We demonstrate that $H_1(A, A_g) = \frac{A_g \otimes A}{im(b_2)}$. Simplicial complex ends up being:

$$\ldots A_g \otimes A \otimes A \xrightarrow{b_2} A_g \otimes A \xrightarrow{b_1} A_g \xrightarrow{b_0} 0.$$

$H_1(A, A_g) = coker(b_2) = \frac{A_g \otimes A}{im(b_2)}$. We have $\ker(b_1) = A_g \otimes A$,since$b_1(ga_0 \otimes a_1) = ga_0a_1 - a_1ga_0 = 0$, then $ga_0a_1 = a_1ga_0$ for all$a \in A, ga_0 \in A_g$, as it were, A must be commutative.We have thusly,

$$HH_1(A \rtimes G) \cong \left(\bigoplus_{g \in G} H_1(A, A_g)\right)^G = \bigoplus_{g \in G} \left(\frac{A_g \otimes A}{im(b_2)}\right)$$

To describe, consider the Simplicial and cyclic homology for algebras of regular functions. The inclusion $X^g \to X$ induces an isomorphism

$$H_n(O(X), O(X)) \cong HH_n(O(X^g)) \cong \Omega^n(X^g), \tag{1.7}$$

Where, $X^h = \{x \in X, g(x) = x\}$, following [10].

Notice that, a part of $HH_n(O(x)^g) \to HH_n(O(X)^g)$ is surjective. The complex$(C_\bullet(O(X)^g)$is embedded in $C_\bullet(O(X))_g$, by writing a$g$on the left. The inclusion map $\oplus_{g\in G}\, gC_\bullet(O(X)^g \to C_\bullet(O(X) \rtimes G)$ induces a surjection on the homology theory, i.e., a splitting of

$$H_n\left(\bigoplus_{g\in G} gC_\bullet(O(X)^g\right) \to H_n(C_\bullet(O(X) \rtimes G), b) \to HH_n(O(X) \rtimes G)(1.8)$$

is surjective [9].

For $A = O(X)$, and $\langle g\rangle \subset G$ be the cyclic group produced by $g \in G$ so that$g$and $O(X)^g$ lie in the commutative algebra $O^g \coloneqq C[\langle g\rangle] \otimes O(X)^g$ We have the characteristic surjection

$$\pi_n: C_n(O^g) \to \Omega^{\mathrm{n}}(O^g);$$

$$\pi_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 da_1 \ldots da_n,$$

Where, $d: \Omega^n(O^g) \to \Omega^{n+1}(O^g)$ is the de-Rham differential given by;

$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n, \text{ such that } d^2 = 0.$$

For instance, $d(a_0 da_1) = da_0 da_1$for all $a_0, a_1 \in O^g$. From [5], we have $\pi_n b = 0$ and $\pi_{n+1} B = (n+1) d\pi_n$.

Notes:

(1) $C_{n+1}(O^g) \xrightarrow{b} C_n(O^g) \xrightarrow{\pi_n} \Omega^n(O^g)$ Suggests that,$\pi_n b = 0: C_{n+1}(O^g) \to \Omega^{\mathrm{n}}(O^g)$, where$C_n(O^g) = (O^g)^{\otimes(n+1)},\ n \geq 0$.

(2) $\pi_{n+1} B = d\pi_n$, Suggests that, the accompanying diagram commutes:

$$\begin{array}{ccc} C_n(O^g) & \xrightarrow{\pi_n} & \Omega^n(O^g) \\ \downarrow B & & \downarrow d \\ C_{n+1}(O^g) & \xrightarrow{\pi_{n+1}} & \Omega^{n+1}(O^g) \end{array}.$$

In this manner $(\pi_n/n!)$ instigates a map from the mixed complex $(B_\bullet(O^g), b, B)$ to the mixed complex,$B_\bullet(O^g), 0, d)$:

$$\theta: (B_\bullet(O^g), b, B) \to \left(\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2(O^g) & \overset{d}{\leftarrow} & \Omega^1(O^g) & \overset{d}{\leftarrow} & \Omega^0(O^g) \\ \downarrow b & & \downarrow b & & \\ \Omega^1(O^g) & \overset{d}{\leftarrow} & \Omega^0(O^g) & & \\ \downarrow b & & & & \\ \Omega^0(O^g) & & & & \end{array}\right) \qquad (1.9)$$

Presently,$(\Omega^\bullet(O^g), d)$is a cochain complex whose cohomology $H^n(\Omega^\bullet(O^g), d)$ $H^n(\Omega^*(O^g), d)$ is known as the de-R hamcohomology of an algebra O^g:

$$H_{DR}(O^g) = H^n((\Omega^\bullet(O^g), d).$$

The surjective map: $\Omega^n(O(X)) \to \Omega^n(O(X^g)) = \Omega^n(X^g)$and homology of mixed complex $H_n(M_\bullet(O^g), 0, d) = HC_n(O^g)$ we have surjection

$$HC_n(O^g) \to HC_n\big(O(X^g)\big). \tag{1.10}$$

Lemma (1.6): [9]

All segments in $HH_n(O(X) \rtimes G)$ and in $HC_n(O(X) \rtimes G)$ as in Theorem (1.3) are cycles in $\oplus_{g\in G}\, gC_\bullet(O(X)^g)$.

2 - Crossed product and Graded Hecke algebras

Let $t^\bullet$ characterize a complex of vector space which containing root system R with Weyl group, W. Then, Wacts on the symmetric algebra $S(t^\bullet)$ of $t^\bullet$, we can build the crossed product algebra $W \ltimes S(t^\bullet)$. Graded Hecke algebras is deformations of $W \ltimes S(t^\bullet)$,relying upon a few parameters$k_\alpha \hat{I}C$. Lusztig demonstrated that graded Hecke algebras assume an essential part in the portrayal hypothesis of affine Hecke algebras and of basic p-adic groups. Each graded Hecke algebra H is invested with a characteristic filtration, whose associated graded algebra is $W \ltimes S(t^\bullet)$. This offers ascend to spectral sequences focalizing to various homologies of H.The fundamental references are [6], [8] and [9].

Definition (2.1): [6]

The root system $R = (X, Y, R, R^\vee, \pi)$ consists of X, Y two free abelian groups limited rank with a given perfect pairing,$\langle\ ,\ \rangle: X \times Y \to Z$,$R \subset X, R^\vee \subset Y$ two finite subsets with given bijection $R \leftrightarrow R^\vee$, denoted $\alpha \leftrightarrow \alpha^\vee$and $\pi \subset R$ a subset. These data are subject to requirements**:**

(1) $\langle \alpha, \alpha^\vee\rangle = 2$ for all $\alpha \in R$.

(2) For any $\alpha \in R$, the reflection $s_\alpha: X \to X, x \to x - \langle x, \alpha^\vee\rangle\alpha$ and $s_\alpha: Y \to Y, y \to y - \langle \alpha, y\rangle\alpha^\vee$ leaves stable $R$and $R^\vee$ respectively.

(3) Any $\alpha \in R$ can be composed extraordinarily as $\alpha = \sum_{\beta\in\pi} n_{\alpha,\beta} \cdot \beta$ where $n_{\alpha,\beta} \in Z$, $n_{\alpha,\beta} \geq 0 \;\; or \;\; \leq 0$, we have accordingly $\alpha \in R^+ \;\; or \;\; \alpha \in R^-$.

Note:

A degenerate root datum $\tilde{R} = (a^\bullet, R, a, R^\vee, \pi)$ consists of**:**

a**:** a limited dimensional real inner product space, $a^\bullet$**:** linear dual of a,R**:** reduced root system in $a^\bullet$, $R^\vee$**:** dual root system in a, and $\boldsymbol{\pi}$**:** basis of R. In reality R is evenconsented to be empty.

The degenerate root datum $\tilde{R} = (a^\bullet, R, a, R^\vee, \pi)$ offers ascend to**:**

$t, t^\bullet$: complexifications of a and $a^\bullet$,$S(t^\bullet)$**:** symmetric algebra of $t^\bullet$, W**:** Weyl group of R, $S = \{s_\alpha : \alpha \in \pi\}$**:** class of simple reflections in W, $C[W]$**:** complex group algebra.

The formal parameters considered are K_α for $\alpha \in \pi$, with the end goal that $K_\alpha = K_\beta$ if α and β conjugate under W. We describe the graded Hecke algebra $\tilde{H}(\tilde{R})$ comparing to $\tilde{R}$.

Definition (2.2): [9]

Let F be field $(F = C)$ and $\tilde{R} = (a^\bullet, R, a, R^\vee, \pi)$ be a degenerate root datum. A graded Hecke C-algebra $\tilde{H}(\tilde{R})$ relating to $\tilde{R}$ is a graded C-vector space $\tilde{H}(\tilde{R}) = \oplus_{i\in C}\left(\tilde{H}(\tilde{R})\right)_i = C[W]\otimes S(t^\bullet)\otimes C[\{K_\alpha : \alpha \in \pi\}]$, outfitted with an associative graded multiplication

$$\pi : \left(\tilde{H}(\tilde{R})\right)_i \otimes \left(\tilde{H}(\tilde{R})\right)_j \to (\tilde{H}(\tilde{R})_{i+j};\ \pi(h_1 \otimes h_2) = h_1 h_2 \in \left(\tilde{H}(\tilde{R})\right)_{i+j},$$

for all,

$$h_1 \in \left(\tilde{H}(\tilde{R})\right)_i, h_2 \in \left(\tilde{H}(\tilde{R})\right)_j \text{ and } \deg(h_1 h_2) = |h_1 h_2| = |h_1| + |h_2| = i + j,$$

Such that, $\pi(1\otimes\pi) = \pi(\pi \otimes 1)$, where $1 = id_{\tilde{H}(\tilde{R})} : \tilde{H}(\tilde{R}) \to \tilde{H}(\tilde{R})$.

The multiplication in $\tilde{H}(\tilde{R})$ is given by following rules:

(a) $[W], S(t^\bullet)$ and $C[\{K_\alpha, \alpha \in \pi\}]$ are sub-algebras in $\tilde{H}(\tilde{R})$,
(b) The K_α are central in $\tilde{H}(\tilde{R})$,
(c) For $x \in t^\bullet$ and $s_\alpha \in S$ we get cross relation $x s_\alpha - s_\alpha s_\alpha(x) = K_\alpha \langle x, a^\vee\rangle$.

The Z-grading on $\tilde{H}(\tilde{R})$ is characterized by $|t^\bullet| = |K_\alpha| = 1$ while $|W| = 1$.

Indeed, we will just examination specializations of the algebra $\tilde{H}(\tilde{R})$. We characterized the graded Hecke algebra $H(\tilde{R}, k)$ corresponding to $\tilde{R}$ with parameter k. Pick complex numbers $k_\alpha \in C$ for $\alpha \in \pi$, such that $k_\alpha = k_\beta$ if α and β are conjugate under W. Let C_k be a $C[\{K_\alpha : \alpha \in \pi\}]$-moduleof 1-dimension such that K_α acts as multiplication by k_α.

Definition (2.3): [9]

We characterize $H = H(\tilde{R}, k) = \tilde{H}(\tilde{R}) \otimes_{C[\{K_\alpha : \alpha\in\pi\}]} C_k$ as a graded Hecke algebra. $H = H(\tilde{R}, k) = C[W]\otimes S(t^\bullet)$ as a vector space. For $x \in t^\bullet$ and $s_\alpha \in S$ we get crossconnection:

$$x s_\alpha - s_\alpha s_\alpha(x) = k_\alpha \langle x, a^\vee\rangle \tag{2.1}$$

Since $S(t^\bullet)$ is Noetherian and W is finite, $H = H(\tilde{R}, k)$ is Noetherian as well. We characterize a grading on H by $|x| = 1\ \ \forall x \in t^\bullet$ and $|W| = 0\ \ \forall x \in W$. Give an opportunity to determine some outstanding cases in which is $H = H(\tilde{R}, k)$ is graded:

(1) If $R = \phi$ then, $H = H(\tilde{R}) = S(t^\bullet)$,

(2) If $k_\alpha = 0 \forall\alpha \in \pi$, then $H = H(\tilde{R}, k) = W \ltimes S(t^\bullet)$, is crossed product with cross connection $w \cdot x = w(x) \cdot w$ for $x \in t^\bullet, w \in W$. By expanding the maps, we get:

$$t^\bullet \xrightarrow{bijection} t^\bullet \Rightarrow S(t^\bullet) \xrightarrow{algebra\ \ automorp\ hism} S(t^\bullet) \Rightarrow H(\tilde{R}, zk) \xrightarrow{algebra\ \ isomorp\ hism} H(\tilde{R}, k)$$

for,$z \in C^\times$. The algebra isomorphism $\omega_z: H(\tilde{R}, zk) \to H(\tilde{R}, zk)$ is the identity on$C[W]$. Extraordinarily, if all$\alpha \in R$are conjugate under W, then there are basically just two graded Hecke algebras attached to $\tilde{R}$:

(a) $H = H(\tilde{R}, 0) = W \ltimes S(t^\bullet)$ with $k = 0$,
(b) $H = H(\tilde{R}, k)$ with $k \neq 0$.

Definition (2.4): [8]

Let(V, π) be an H-module and pick $\lambda \in t$. The λ-weight space of V is$V_\lambda = \{\lambda \in V: \pi(x)v = \langle x, \lambda\rangle v \ \ \forall x \in t^\bullet\}$,and the generalized λ-weight space is $V_\lambda^{generalized} = \{\lambda \in V: \exists n \in N: (\pi(x) - \langle x, \lambda\rangle)^n v = 0 \ \ \forall x \in t^\bullet$. λ is call λ a $S(t^\bullet)$-weight of V if $V_\lambda^{generalized} \neq 0$ or $V_\lambda \neq 0$.

Definition (2.5):

If $\mathrm{V} = \bigoplus_{\lambda \in \mathrm{t}} \mathrm{V}_\lambda^{\mathrm{generalized}}$ the direct sum of finite dimension space V.The center of$H$is$Z(H) = S(t^\bullet)^W$. Extraordinarily,H is finite rank as a$Z(H)$-module, so all its irreducible modules have finite dimension. Moreover, the central character of an irreducible H-module can be versed as component of t/W. We characterize the extended graded Hecke algebra. Let(R, π) be a based root system whose Dynkinscheme automorphism,$\gamma: \pi \to \pi$ is bijection such an extent that;

$$\langle \gamma(\alpha), \gamma(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \qquad \forall \alpha, \beta \in \pi \subset R.$$

LetΓ be finite group of the automorphisms$\gamma: \pi \to \pi$.Groups appear$W^/ := \Gamma \ltimes W$ ordinarily emerge from larger Weyl groups as the isotropy groups of points in some torus, or as normalizers of some parabolic subgroup [8].Assume that $k_{\gamma(\alpha)} = k_\alpha \quad \forall \alpha \in \pi, \gamma \in \Gamma$. Then$\Gamma$ acts on H by the algebra homomorphisms

$$\psi_\gamma H \to H,$$

$$\psi_\gamma(x s_\alpha) = \gamma(x) s_{\gamma(\alpha)} \qquad \forall \alpha \in \pi, x \in t^\bullet \tag{2.2}$$

Definition (2.6): [9]

A crossed product

$$H^/ := \Gamma \ltimes H = \Gamma \ltimes H(\tilde{R}, k) \tag{2.3}$$

expanded graded Hecke algebra. The Z-grading on $H^/$ is characterized by$|x| = 1 \ \forall x \in t^\bullet$ and $|W| = 0 \ \ \forall w \in W^/$be that as it may, the algebra $H^/$is by and large not evaluated, just sifted.

Remark:

The product $h_1 h_2 = \pi(h_1 \otimes h_2) \in H^/$ of two homogeneous components$h_1, h_2 \in H^/$ require not be homogeneous, but rather all its homogeneous segments have degree$\le |h_1| + |h_2|$. All the more accurately, the part of$h_1 h_2 \in H^/$depending on the parameters k_α has degree strictly$< |h_1| + |h_2|$. $H^/$ is a graded if $\deg(h_1 h_2) = |h_1 h_2| = |h_1| + |h_2|$.

Some unprecedented cases, in which$H^/ = \Gamma \ltimes H(\tilde{R}, k)$is graded:

(1) If $R = \varphi$ then, $H^/ = \Gamma \ltimes H(\tilde{R}, k) = \Gamma \ltimes H(\tilde{R}) = \Gamma \ltimes S(t^\bullet)$.
(2) If $k_\alpha = 0 \ \forall \alpha \in \pi$, then $H^/ = \Gamma \ltimes H(\tilde{R}, k) = W^/ \ltimes S(t^\bullet)$, is crossed product with cross relation$w = w(x) \cdot w$ for $x \in t^\bullet, w \in W^/$.

By expanding the maps, we get:

$$t^\bullet \xrightarrow{bijection} t^\bullet \Rightarrow S(t^\bullet) \xrightarrow{algebra\ \ automorp\ hism} S(t^\bullet) \Rightarrow \Gamma \ltimes H(\tilde{R}, zk) \xrightarrow{algebra\ \ isomorp\ hism} \Gamma \ltimes H(\tilde{R}, k)$$

For $z \in C^\times$. The algebra isomorphism$\omega_z : \Gamma \ltimes H(\tilde{R}, zk) \to \Gamma \ltimes H(\tilde{R}, zk)$ is identity on$C[W^/]$, it stays all around characterized for$z = 0$, just on the off chance that it is bijective. Extraordinarily, if all$\alpha \in R$are conjugate under $W^/$, then there are essentially only two graded Hecke algebras attached to $(\tilde{R}, \Gamma)$:

(a) $H^/ = \Gamma \ltimes H(\tilde{R}, 0) = W^/ \ltimes S(t^\bullet)$ With parameter $k = 0$,
(b) $H^/ = \Gamma \ltimes H(\tilde{R}, 0)$ with parameter $k \neq 0$.

Remarks:

The extended graded Hecke algebra is,

$$H^/ = \Gamma \ltimes H(\tilde{R}, k) = \Gamma \ltimes \left(\tilde{H}(\tilde{R}) \bigotimes_{C[\{K_\alpha : \alpha \in \pi\}]} C_K \right)$$

$$= \Gamma \ltimes \left((C[W] \otimes S(t^\bullet) \otimes C[\{K_\alpha : \alpha \in \pi\}]) \bigotimes_{C[\{K_\alpha : \alpha \in \pi\}]} C_K \right)$$

1- Since$\alpha \in \pi \subset R = \varphi$, we have no parameters $k_\alpha$and K_α, thus
$C[\{K_\alpha : \alpha \in \pi\}] = C[\{ \quad \}] = C[f] = f = C[W]$. Hence, $H^/ = \Gamma \ltimes H(\tilde{R}) = \Gamma \ltimes S(t^\bullet)$.

2- Since$K_\alpha = 0 \ \ \forall \alpha \in \pi$we get parameters $k_\alpha = K_\alpha = 0$, thus

$$C_K = C[\{0 : \alpha \in \pi\}] = C[\{0\}] = \varphi, \qquad C[W] = W.$$

Hence $H^/ = \Gamma \ltimes (W \otimes S(t^\bullet)) = (\Gamma \ltimes W) \ltimes S(t^\bullet) = W^/ \ltimes S(t^\bullet)$.

3- The cyclic homology theory of graded Hecke algebras

The periodic cyclic homology of finite Weyl group andgraded Hecke algebra is the same [9], that is particularly summed up to the expanded graded Hecke algebras from definition (2.5).We utilize the documentations from Section 2.

Lemma (3.1):

Consider the extended graded $H^{/} = \Gamma \ltimes H(\tilde{R},0) = W^{/} \ltimes S(t^{\bullet})$ with parameter:

(a) $HH_n(W^{/} \ltimes S(t^{\bullet})) = 0$, for all$n > \dim_C(t^{\bullet})$.
(b) $HC_n\left(W^{/} \ltimes S(t^{\bullet})\right) = HP_n\left(W^{/} \ltimes S(t^{\bullet})\right) = 0$, for all odd$n > \dim_C(t^{\bullet})$.
(c) The consideration $C[W^{/}] \to W^{/} \ltimes S(t^{\bullet})$ actuates isomorphism on periodic cyclic homology.

Lemma (3.3):

$$HH_n\left(\Gamma \ltimes H(\tilde{R},k)\right) = 0, \qquad for\ n > \dim_C(t^{\bullet}).$$

Proof.

The simplicial complex$(C_{\bullet}(H^{/}), b)$ from (1.3) is sifted by (3.1),were,

$C_n(H^{/}) = (H^{/})^{\otimes(n+1)}, n \geq 0$ and $b: C_n(H^{/}) \to C_{n-1}(H^{/})$ characterized by;

$$b(h_0 \otimes \ldots \otimes h_n) = \sum_{i=0}^{n-1}(-1)^i h_0 \otimes \ldots \otimes h_i h_{i+1} \otimes \ldots \otimes h_n + (-1)^n h_n h_0 \otimes h_1 \otimes \ldots \otimes h_{n-1},$$

for all$h_0, h_1, \ldots, h_n \in H^{/}$. By putting

$$F_p C_n(H^{/}) := (H^{/})^{\otimes(n+1)}_{\leq p} \tag{3.1}$$

This offers ascend to a spectral sequence$E^r_{\bullet,\bullet}$merging to$HH_{\bullet}(H^{/})$whose initially term is

$$E^1_{p,q} = H_{p+q}\left(\frac{F_p C_{\bullet}(H^{/})}{F_{p-1} C_{\bullet}(H^{/})}\right) \tag{3.3}$$

Let$C_{\bullet}(H^{/})$ denote the complex $\left(\frac{F_p C_{\bullet}(H^{/})}{F_{p-1} C_{\bullet}(H^{/})}\right)$ the boundary map on $C_{\bullet}(H^{/})$given by(1.3), but with ignoring all terms which are not in top degree $(\partial: C_{\bullet}(H^{/}) \to C_{\bullet-1}(H^{/})$. From equations (2.1) and (2.2) the resulting map is independent of k. So$E^1_{p,q}$ is given for all parameters from condition $k = 0$. But for$k = 0$, the algebra $H^{/} = \Gamma \ltimes H(\tilde{R},k) = W^{/} \ltimes S(t^{\bullet})$ is graded, so the filtration is trivial, and the spectral sequence $E^r_{\bullet,\bullet}$ stabilizes at $E^1_{\bullet,\bullet}$ .From Lemma (3.1) we reason that $E^1_{\bullet,\bullet} = 0$ if$\underbrace{p+q}_{n} > \dim_C(t^{\bullet})$.For general parameters k_{α}we can't state promptly whether $E^r_{\bullet,\bullet}$ stabilizes at $E^1_{\bullet,\bullet}$, yet regardless $E^r_{\bullet,\bullet}$ is a sub-quotient of $E^1_{\bullet,\bullet}$. Thus $E^{\infty}_{p,q} = 0$ if $\underbrace{p+q}_{n} > \dim_C(t^{\bullet})$.

********Remark:**

From Conne`s periodicity exact sequence identifying with simplicial and cyclic homology [3], and [4], we have

$$HC_n\left(\Gamma \ltimes H(\tilde{R},k)\right) \cong HP_n\left(\Gamma \ltimes H(\tilde{R},k)\right) \;\; for \;\; n > \dim_C(t^\bullet) \tag{3.4}$$

Presently, we process cyclic homology with a spectral sequence.

Theorem (3.4):

For$n > \dim_C(t^\bullet)$ we have

$$HC_n\left(\Gamma \ltimes H(\tilde{R},k)\right) \cong HC_n(C[\Gamma \ltimes W]).$$

The inclusion$C[\Gamma \ltimes W] \to \Gamma \ltimes H(\tilde{R},k)$ induces isomorphism of the periodic cyclic homology.

Proof.

From (1.4), the cyclic homology of$H^/$is$HC_n(H^/) = H_n(B_\bullet(H^/), b, B)$were,$C_n(H^/) = H^{/\otimes(n+1)}, n = 0,1,2, \ldots$. The operator$B: C_n(H^/) \to C - (n+1)(H^/)$ is called Conne's operator and satisfies:$Bb + bB = 0, B^2 = 0$and

$$B_\bullet(H^/) := (H^/)^{\otimes(n+1)} \oplus (H^/)^{\otimes n} \oplus \ldots \oplus H^/ \tag{3.5}$$

Notice that,$B_\bullet(H^/) := \oplus_{i=1}^{n}\left(C_i(H^/)\right) = C_n(H^/) \oplus C_{n-1}(H^/) \oplus \ldots \oplus C_0(H^/)$.

The differential complex$\left(B_\bullet(H^/), b\right)$ is filtrated by

$$F_p B_n(H^/) := (H^/)^{\otimes(n+1)}_{\leq p} \oplus (H^/)^{\otimes n}_{\leq p} \oplus \ldots \oplus H^/_{\leq p} \tag{3.6}$$

There exists spectral sequence$E^r_{\bullet,\bullet}$ converges to$HC_{p+q}(H^/)$, whose initially term is

$$E^1_{p,q} = H_{p+q}\left(\frac{F_p B_\bullet(H^/)}{F_{p-1} B_\bullet(H^/)}\right) \tag{3.7}$$

By (2.1) and (2.2) the boundary maps in$C_{\bullet\bullet}(H^/) = \left(\frac{F_p B_\bullet(H^/)}{F_{p-1} B_\bullet(H^/)}\right)$ are independent of parametersk_α. So, vector spaces$E^1_{p,q}$ don`t depend on$k$ and given from the case $k = 0$. But for $k = 0$, the algebra$H^/ = \Gamma \ltimes H(\tilde{R},0) = W^/ \ltimes S(t^\bullet)$ is graded, so spectral sequence$E^r_{\bullet,\bullet}$stabilizes at$E^1_{\bullet,\bullet}$ and $E^1_{p,q}$ is a part of $HC_{p+q}\left(W^/ \ltimes S(t^\bullet)\right)$such that$\left|E^1_{p,q}\right| = p$.

Unlike Hochschild homology,$HC_n\left(W^/ \ltimes S(t^\bullet)\right) \neq 0$ for large$n$, from Lemma (3.1) we get$HC_n\left(W^/ \ltimes S(t^\bullet)\right) = 0$, for all odd$n > \dim_C(t^\bullet)$.For even$n > \dim_C(t^\bullet)$, Theorem (1.3) says that

$$HC_n\left(W^/ \ltimes S(t^\bullet)\right) = HP_n\left(W^/ \ltimes S(t^\bullet)\right).$$

From Lemma (3.1) we have that, for all $n > \dim_C(t^\bullet)$, $HC_n\left(W^/ \ltimes S(t^\bullet)\right)$ has no parts in degrees$p > 0$, and $HH_0(C[W^/])$ is its part of degree$p = 0$.

By returning to general k and considering various p and q. By definition

$$E^1_{p,q} = 0, \qquad if \begin{cases} p < 0 \\ or, \quad p + q < 0 \end{cases}.$$

We skip the event $0 \leq p + q \leq dim_C(t^{\bullet})$. In fine, we pick $p, q \in Z$ such that $p + q > dim_C(t^{\bullet})$. Hence $E^1_{p,q} = 0$ unless $p = 0$ and q is even, in which case $E^1_{p,q} = HH_0(C[W^{/}])$. For every $r \in Z$ there exists boundary map $\partial^r_{p,q}: E^r_{p,q} \to E^r_{p-1,q+r-1}$ and $E^{r+1}_{\bullet,\bullet} = Homology\ of\ (E^r_{\bullet,\bullet}, \partial^r_{p,q})$.

By claiming that $\partial^r_{p,q} = 0$ when $r \geq 1$ and $p + q > dim_C(t^{\bullet})$. Surely, for $p > 0$, $Domain = E^r_{p>0,q} = 0$, while for $p = 0$, $Range = E^r_{-r,q+r-1} = 0$, as $-r < 0$.

Hence, in the range $p + q > dim_C(t^{\bullet})$, a spectral sequence $E^r_{p,q}$ stabilizes at $r = 1$, for all k. Since vector spaces $E^1_{p,q}$ don`t depend on $k$ and given from condition $k = 0$. Let limit $r \to \infty$ and using the convergence, we find $HC_{p,q}(H^{/})$ doesn`t depend on k. In view (3.4), we have $HP_{p+q}(H^{/}) \cong HC_{p+q}(H^{/})$ does not too. Since the function $HP_{\bullet}$ is 2-periodic, so $HP_n(H^{/})$ is independent of k for all $n \in Z$, we have thus

$$HP_n\left(\Gamma \ltimes H(\tilde{R}, k)\right) \cong HP_n(C[\Gamma \ltimes W]), \qquad for\ all\ n \in Z.$$

Theorem (3.5):

For all $n \in Z_{\geq 0}$ there are isomorphisms:

$$HH_n\left(\Gamma \ltimes H(\tilde{R}, k)\right) \cong HH_n\left(W^{/} \ltimes S(t^{\bullet})\right),$$

$$HC_n\left(\Gamma \ltimes H(\tilde{R}, k)\right) \cong HC_n\left(W^{/} \ltimes S(t^{\bullet})\right),$$

$$HP_n\left(\Gamma \ltimes H(\tilde{R}, k)\right) \cong HP_n\left(W^{/} \ltimes S(t^{\bullet})\right).$$

Definition (3.6):

Consider the graded Hecke algebra $H = H(\tilde{R}, k) = \widetilde{H}(\tilde{R}) \otimes_{C[\{K_\alpha : \alpha \in \pi\}]} C_k$ (with involution) with coefficients in H-bimodule and M is the bimodule. Let The matrices $\mathcal{M}_r(M)$ with degree $r \times r$. Then the inclusion is defined as

$$inc: \mathcal{M}_r(M) \to \mathcal{M}_{r+1}(M)$$

as

$$\alpha \mapsto \begin{bmatrix} & & & 0 \\ & \alpha & & \bullet \\ 0 & \bullet & 0 & 0 \end{bmatrix}$$

Since the trace map$tr: \mathcal{M}_r(M) \to M$ is given by $tr(\alpha) = \sum_{i=1}^{r} \alpha_{ii}$. And the generalized trace map $tr: \mathcal{M}_r(M) \otimes \mathcal{M}_r(S)^{\otimes n} \to M \otimes S^{\otimes n}$ as $tr(\alpha \otimes \beta \otimes \cdots \otimes \eta) = \sum(\alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes \ldots \otimes \eta_{i_n i_0})$.

Theorem (3.7):

Let$H = H(\tilde{R}, k) = \widetilde{H}(\tilde{R}) \otimes_{C[\{K_\alpha : \alpha \in \pi\}]} C_k$ graded Hecke algebra with unity, then the relation between the cyclic and simplicial cohomology group is the sequence:

$$\cdots \to HH^n(\mathrm{H}) \overset{B}{\to} HC^{n-1}(\mathrm{H}) \overset{S}{\to} HC^{n+1}(\mathrm{H}) \overset{I}{\to} HH^{n+1}(\mathrm{H}) \overset{B}{\to} \cdots \quad (3.9)$$

Proof:

Consider the bi-complex$CC(H)^{\{2\}}$ which contains the first and second columns of $CC(H)$, where $C[2,0]_{pq} = C_{p-2,q}$. Then from the exact sequence we get;

$$0 \to CC(H)[2,0] \to CC(H) \to CC(H)^{\{2\}} \to 0$$

which related between the Hochschild and Cyclic cohomology of graded Hecke algebra, then we get the required**.**

Theorem (3.8):

Let$H$be a graded Hecke algebra with unity $r \geq 1$ the maps

$$tr^*: HC^*(\mathcal{M}_r(H)) \to HC^*(H)$$

and

$$inc^*: HC^*(H) \to HC^*(H)$$

are isomorphisms and both is inverse to each other.

Proof:

Consider the pre-simplicial homotopy,$h = \sum(-1)^i h_i$since;

$h_i: \mathcal{M}_r(M) \otimes \mathcal{M}_r(H)^{\otimes n} \to \mathcal{M}_r(M) \otimes \mathcal{M}_r(H)^{\otimes n+1}$which defined as**;**

$$h_i(a^0, \ldots, a^n) = \sum E_{j1}(a_{jk}^0) \otimes E_{11}(a_{km}^1) \otimes \ldots \otimes E_{11}(a_{pq}^i) \otimes E_{1q}(1) \otimes a^{i+1} \otimes \ldots \otimes a^n$$

Where$a^0 \in \mathcal{M}_r(M)$, $a^s \in \mathcal{M}_r(H)$,and $h = \sum_{i=0}^{n}(-1)^i h_i$. for$n = 0$, $h(a) = E_{j1}(a_{jk}) \otimes E_{1k}(1)$. If $n = 1$, $h(a,b) = E_{j1}(a_{jk}) \otimes E_{1k}(1) \otimes b - E_{j1}(a_{jk}) \otimes E_{11}(b_{ki}) E_{1l}(1)$. Then we get;$hd + dh = d_0 h_0 - d_{n+1} h_n$. Such that $id = d_0 h_0, d_{n+1} h_n = inc \circ tr$, then $id \ \ and \ \ inc \circ tr$ are homotopic with each other.

Theorem (3.9):

For the graded Hecke algebra; we have

$$\cdots \to C_{n+1}(H) \xrightarrow{d_n} C_n(H) \xrightarrow{d_{n-1}} C_{n-1}(H) \to \cdots \tag{3.10}$$

Consider we have the subsequence X, Y of C as:

$$X: \cdots \xrightarrow{d_{n+1}} X_{n+1}(H) \xrightarrow{d_n} X_n(H) \xrightarrow{d_{n-1}} X_{n-1}(H) \to \cdots \tag{3.11}$$

$$Y: \cdots \xrightarrow{d_{n+1}} Y_{n+1}(H) \xrightarrow{d_n} Y_n(H) \xrightarrow{d_{n-1}} Y_{n-1}(H) \to \cdots \tag{3.12}$$

Then we have the *Mayer-vietories sequence* as the sequence;

$$\cdots \xrightarrow{f_{n+1}^*} HC_{n+1}(X \oplus Y) \xrightarrow{g_{n+1}^*} HC_{n+1}(X+Y) \xrightarrow{h_{n+1}^*} HC_n(X \cap Y) \xrightarrow{f_n^*} HC_n(X \oplus Y)$$

$$\xrightarrow{g_n^*} HC_n(X+Y) \xrightarrow{h_n^*} HC_{n-1}(X \cap Y) \xrightarrow{f_{n-1}^*} HC_{n-1}(X \oplus Y) \xrightarrow{g_{n-1}^*} \cdots \tag{3.13}$$

Proof:

To prove (3.13); we study the long exact sequence which is relating among (3.11), (3.12) and (3.10). Then we get the sequence (3.13), since $f_n: C_n(X \cap Y) \to C_n(X \oplus Y)$, $f_n(x) = (x, -x),\ g_n: C_n(X \oplus Y) \to C_n(X+Y)\ since\ g_n(x, y) = x + y$ and $x \in X_n$ s.h. $z - x \in Y_n$.

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