Journal of Progressive Research in Mathematics
Vol 18,1 - 2021 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

Solvability of Forward-Backward Stochastic Partial Differential Equations with Non-Lipschitz Coefficients

Hong Yin

Department of Mathematics, State University of New York, Brockport, NY 14420..
Email: hyin@brockport.edu

Received: April 2, 2021; Accepted: April 20, 2021; Published: April 29, 2021

Abstract

In this paper we study the solvability of a class of fully-coupled forward-backward stochastic partial differential equations (FBSPDEs). Lipschitz conditions are usually required for the well-posedness of such FBSPDEs. We showed that the Lipschitz conditions can actually be removed by the Yosida Approximation Scheme.

2010 AMS Mathematics Subject Classification.

Primary: 60H15; Secondary: 35R60.

Keywords

Forward-backward stochastic partial differential equations, Yosida Approximation.

1. Introduction

In this paper we consider the following general quasilinear *forward-backward stochastic partial differential equations* (FBSPDE for short) of parabolic type:

$$\begin{cases} \mathbf{u}(t,x) = \mathbf{u}_0(x) + \displaystyle\int_0^t \Big\{ L_F(s,x,X) + f(s,x,Y) \Big\} ds + \int_0^t \langle\, \sigma(s,x,Y), dW(s)\,\rangle \\ \mathbf{v}(t,x) = g(x,\mathbf{u}(T,x)) + \displaystyle\int_t^T \Big\{ L_B(s,x,X) - h(s,x,Y) \Big\} ds - \int_t^T \langle Z(s), dW(s)\rangle, \\ \qquad\qquad t \in [0,T],\ x \in G. \end{cases} \tag{1}$$

In the above, $T > 0$ is a finite time duration; $G \subseteq \mathbb{R}^d$ is a bounded domain; $X \triangleq (\mathbf{u}, \mathbf{v}, Z)$; $Y \triangleq$

Cite this paper: Yin, H. (2021). Solvability of Forward-Backward Stochastic Partial Differential Equations with Non-Lipschitz Coefficients. Journal of Progressive Research in Mathematics, 18(1), 87-98. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2040

$(\mathbf{u}, \mathbf{v}, \nabla\mathbf{u}, \nabla\mathbf{v}, Z)$; denoting $\mathfrak{R} \triangleq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$,

$$\begin{cases} f, h : [0,T] \times G \times \mathfrak{R} \times \Omega \to \mathbb{R}, \\ \sigma : [0,T] \times G \times \mathfrak{R} \times \Omega \to \mathbb{R}^d, \\ g : G \times \mathbb{R} \times \Omega \to \mathbb{R}, \end{cases}$$

are random fields satisfying appropriate measurability and regularity conditions; $\mathbf{u}_0 \in L_0^2(G)$; and

$$\begin{cases} L_F(t,x,X) \triangleq \nabla \cdot (A\nabla\mathbf{u} + B\nabla\mathbf{v} + CZ); \\ L_B(t,x,X) \triangleq \nabla \cdot (-\Phi\nabla\mathbf{u} + \Psi\nabla\mathbf{v} + \Upsilon Z), \end{cases} \tag{2}$$

where

$$\begin{cases} \nabla\varphi = (\partial_{x_1}\varphi, \partial_{x_2}\varphi, \cdots, \partial_{x_n}\varphi)^T, \ \forall \varphi \in C^1(\mathbb{R}^n; \mathbb{R}), \\ \nabla\psi = (\nabla\psi_1, \nabla\psi_2, \cdots, \nabla\psi_k), \ \forall \psi \in C^1(\mathbb{R}^n; \mathbb{R}^k), \\ \nabla \cdot \varphi = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial x_i}, \ \forall \varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n), \\ \nabla \cdot \psi = (\nabla \cdot \psi_1, \nabla \cdot \psi_2 \cdots, \nabla \cdot \psi_k)^T, \ \forall \psi \in C^1(\mathbb{R}^n; \mathbb{R}^{n\times k}), \\ D^2 \text{ denotes the Hessian operator}, \end{cases}$$

and

$$\begin{cases} A, \Psi : [0,T] \times G \times \Omega \to S^n, \\ B, \Phi : [0,T] \times G \times \Omega \to \mathbb{R}^{n\times n}, \\ C, \Upsilon : [0,T] \times \Omega \to \mathbb{R}^{n\times d}, \end{cases}$$

are random fields (S^n is the set of all $n \times n$-symmetric matrices) satisfying appropriate measurability and regularity conditions. Let $W(t)$ be a standard d-dimensional Brownian motion. We note that if A, B, Φ, and Ψ are differentiable in x, then the divergence form of the operators L_F and L_B would be equivalent to the standard form:

$$\begin{cases} \tilde{L}_F(t,x,X) \triangleq \mathrm{tr}\,\{AD^2\mathbf{u} + BD^2\mathbf{v} + C^T\nabla Z\}; \\ \tilde{L}_B(t,x,X) \triangleq \mathrm{tr}\,\{-\Phi D^2\mathbf{u} + \Psi D^2\mathbf{v} + \Upsilon^T\nabla Z\}, \end{cases} \tag{3}$$

with some corresponding changes in the functions f and h. Our goal is to find a triplet of progressively measurable random fields $(\mathbf{u}, \mathbf{v}, Z) : [0,T] \times G \times \Omega \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ that satisfies (2) in a certain sense.

Linear backward stochastic differential equations (BSDEs) were introduced by Bismut [1, 2] in 1973. A systematic study of BSDEs was initiated by Pardoux-Peng [22] and several other authors. Ma-Yong [20] have studied linear degenerate BSDEs motivated by stochastic control theory. Later, Hu-Ma-Yong [11] considered the semi-linear equations as well. BSPDEs were shown to arise naturally in stochastic versions of the Black-Scholes formula by Ma-Yong [19]. There are also tremendous interest and developments of BSDEs in infinite dimensional settings. The readers may refer to Confortola [5] and Hu-Peng [12] for details, and Chow [4], Da Prato-Zabczyk [8, 9], Kallianpur-Xiong [13] for systematic review of infinite dimensional SDEs.

The FBSPDEs (1) are clearly a natural extension of the (strongly coupled) *forward-backward stochastic differential equations* (FBSDEs), which has been studied extensively in the past two decades, due to its frequent occurrence in many applied fields such as stochastic control theory and mathematical finance. Many seemingly irrelevant problems eventually come down to the solvability of a certain type of strongly coupled FBSDEs. Some typical examples include, but are not limited to: stochastic Maximum Principle in stochastic control theory (which covers all utility optimization problems in finance/insurance. See, e.g., [14]); Principle-agent problem (or contract theory in general. See, e.g., [7]); or more directly, many hedging problems involving large investors ([6], [3]), to mention just a few. But on the other hand, in light of the nonlinear Feynman-Kac formula, or the Four Step Scheme (cf. [15]), it is not hard to imagine that the solution of a backward SPDE (or a decoupled form of (1)) could be a crucial device for solving an FBSDE with random coefficients (cf. e.g., [19, 20]). In fact, in [18], it has been shown that the solvability of a large class of non-Markovian FBSDEs is almost equivalent to the solvability of the corresponding backward stochastic partial differential equations (BSPDEs). Furthermore, in [24], under Lipschitz conditions, the solvability of general strongly coupled FBSDEs with random coefficients has been studied using the method of contraction mapping and the method of continuation.

To further relax the Lipschitz conditions and in light of Hu [10], we then turn to the Yosida Approximation Scheme. By considering a slightly different set of non-degeneracy conditions and assuming monotonicity conditions, we successfully removed the Lipschitz conditions. The rest of the paper is organized as follows. We unify some of the notations and list some results from Yosida approximations in section 2. In section 3, we give a priori estimates and show the well-posedness of the FBSPDEs via the Yosida Approximation Scheme.

2 Formulation of the Problem

Let G be an open bounded domain in $\mathbb{R}^n$ with smooth boundary conditions. For any $k \in \mathbb{N}$, let $\|\cdot\|_{L^2}$ be the norm of $L^2(G;\mathbb{R}^k)$ and $\|\cdot\|$, $\|\cdot\|_{H^1}$ be norms of $H_0^1(G;\mathbb{R}^k)$. They are given as follows:

$$\|\mathbf{u}\|_{L^2} \triangleq \Big(\int_G |\mathbf{u}|^2 dx\Big)^{\frac{1}{2}}, \quad \|\mathbf{u}\| \triangleq \Big(\int_G |\nabla \mathbf{u}|^2 dx\Big)^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{u}\|_{H^1} \triangleq \sqrt{\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|^2}.$$

Note that the norm $|\cdot|$ inside the integral signs is the standard norm on $\mathbb{R}^k$. Let $H^{-1}(G;\mathbb{R}^k)$ be the dual space of $H_0^1(G;\mathbb{R}^k)$, and the normal of $H^{-1}(G;\mathbb{R}^k)$ be denoted as $\|\cdot\|_{H^{-1}}$. Denote $\langle\cdot,\cdot\rangle_{L^2}$ the inner product of $L^2(G;\mathbb{R}^k)$, $\langle\cdot,\cdot\rangle_{H_0^1}$ the inner product of $H_0^1(G;\mathbb{R}^k)$ with respect to the norm $\|\cdot\|$, and $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ the duality pairing between $H^{-1}(G;\mathbb{R}^k)$ and $H_0^1(G;\mathbb{R}^k)$. For any $\mathbf{u} \in L^2(G;\mathbb{R}^k)$, there exists an $\mathbf{u}' \in H^{-1}(G;\mathbb{R}^k)$, such that $\langle \mathbf{u}', \mathbf{v}\rangle_{H^{-1},H_0^1} = \langle \mathbf{u}, \mathbf{v}\rangle_{L^2}$ for all $\mathbf{v} \in H_0^1(G;\mathbb{R}^k)$. The mapping $\mathbf{u} \to \mathbf{u}'$ is linear, injective, compact and continuous, and we can identify $\mathbf{u}'$ with $\mathbf{u}$. In this sense, we identify $(L^2(G;\mathbb{R}^k))^{-1}$ with $L^2(G;\mathbb{R}^k)$. Hence $L^2(G;\mathbb{R}^k)$ is a dense subset of $H^{-1}(G;\mathbb{R}^k)$ and we have evolution triple $H_0^1(G;\mathbb{R}^k) \subset L^2(G;\mathbb{R}^k) \subset H^{-1}(G;\mathbb{R}^k)$. Hence

$$\|\mathbf{u}\|_{H^{-1}} \le k_1 \|\mathbf{u}\|_{L^2} \quad \text{and} \quad \|\mathbf{u}\|_{L^2} \le k_2 \|\mathbf{u}\| \tag{1}$$

for all $\mathbf{u} \in H_0^1(G;\mathbb{R}^k)$ and some constants k_1 and k_2. Because of this inequality, $\|\cdot\|$ and $\|\cdot\|_{H^1}$ are equivalent norms of $H_0^1(G;\mathbb{R}^k)$. We now define some spaces which will be used in the paper. For any k, we denote

- by $\mathcal{C}_0^L(G;\mathbb{R}^k)$ the space $C_{\mathcal{F}}([0,T];L^2(\Omega;L_0^2(G;\mathbb{R}^k)))$;
- by $\mathcal{C}_0^{H^k}(G;\mathbb{R}^k)$ the space $C_{\mathcal{F}}([0,T];L^2(\Omega;H_0^k(G;\mathbb{R}^k)))$;
- by $\mathcal{H}_0^k(G;\mathbb{R}^k)$ the space $L^2_{\mathcal{F}}(\Omega;L^2(0,T;H_0^k(G;\mathbb{R}^k)))$;
- by $\mathcal{L}_0^2(G;\mathbb{R}^k)$ the space $L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_0^2(G;\mathbb{R}^k)))$;
- by $\mathcal{L}^\infty(\mathbb{K};\mathbb{R}^k)$ the space $L^\infty_{\mathcal{F}}([0,T]\times\Omega\times\mathbb{K};\mathbb{R}^k)$, for any space $\mathbb{K}$ other than a Sobolev space.

For notational simplicity, we shall suppress $\mathbb{R}^k$ in above notations when $k=1$.

Let us first introduce the system and make some necessary assumptions. We begin by introducing some operators. We set

$$\begin{aligned} S(t,x,u,v,z) &\triangleq \nabla\cdot(A\nabla u + B\nabla v + Cz) + f(t,x,u,v,\nabla u,\nabla v,z);\\ T(t,x,u,v,z) &\triangleq \nabla\cdot(\Phi\nabla u - \Psi\nabla v - \Upsilon z) + h(t,x,u,v,\nabla u,\nabla v,z);\\ g(x,u) &\triangleq p(x)u + q(x). \end{aligned}$$

We consider the following FBSPDEs:

$$\begin{cases} \mathbf{u}(t,x) = S(t,x,X)dt + \langle\sigma(t,x,Y), dW(t)\rangle \\ \mathbf{v}(t,x) = T(t,x,X)dt + \langle Z(t), dW(t)\rangle \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{v}(T,x) = g(x,\mathbf{u}(T,x)), \end{cases} \tag{2}$$

where the initial condition $\mathbf{u}_0(x)$ is uniformly bounded. We assume the following assumption for Section 3.

(**A.1**) (non-degeneracy) Assume the following regularity: $A,\ \Psi\ \in L^\infty_{\mathcal{F}}(0,T;C^1_b(\mathbb{R}^n;\mathbb{R}^{n\times n}))$, $B,\ \Phi\ \in L^\infty_{\mathcal{F}}(0,T;C^1_b(\mathbb{R}^n;S^n))$, and $C,\ \Upsilon\ \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times d})$. Suppose that there exist positive constants c_0, c_1, c_2 and c_3, such that

$$\begin{pmatrix} B-CC^T & 0 \\ 0 & \Phi-\Upsilon\Upsilon^T \end{pmatrix} \geq c_0 I,$$

and

$$\begin{pmatrix} AA^T & 0 \\ 0 & \Psi\Psi^T \end{pmatrix} \leq c_1 I,\quad \begin{pmatrix} BB^T & 0 \\ 0 & \Phi\Phi^T \end{pmatrix} \leq c_2 I,\quad \begin{pmatrix} CC^T & 0 \\ 0 & \Upsilon\Upsilon^T \end{pmatrix} \leq c_3 I,$$

where I is an identity matrix with appropriate dimension. Suppose that $p \in L^\infty_{\mathcal{F}}(\Omega;L^\infty(G))$ and $q \in L^2_{\mathcal{F}}(\Omega;L^2_0(G))$, and there exist positive constants c_4 and c_5, such that $c_4 \leq p \leq c_5$ and $c_4 \leq p \leq c_5$.

(**A.2**) (linear growth) Let $\mathbf{y} \triangleq (y_1,y_2,y_3,y_4,y_5)$ and $\mathbf{z} \triangleq (z_1,z_2,z_3,z_4,z_5)$. The functions f, h are in $L^\infty_{\mathcal{F}}(0,T;C_b(G\times\mathfrak{R}))$, and σ is in $L^\infty_{\mathcal{F}}(0,T;C_b(G\times\mathfrak{R};\mathbb{R}^d))$, such that there exist positive constants l_1,

$$|b(t,x,\mathbf{y})| \leq |b(t,x,\mathbf{0})| + l_1|\mathbf{y}|, \qquad b = f,h,\sigma,$$

for all $t\in[0,T]$, $x\in G$ and $\mathbf{y}\in\mathfrak{R}$.

(**A.3**) (monotonicity) There exist positive constants $\frac{1}{2} < l_2$, such that

$$\begin{aligned} &\Big(f(t,x,\mathbf{y})-f(t,x,\mathbf{z})\Big)(y_2-z_2)+\Big(h(t,x,\mathbf{y})-h(t,x,\mathbf{z})\Big)(y_1-z_1) \\ &+\langle\sigma(t,x,\mathbf{y})-\sigma(t,x,\mathbf{z}),y_5-z_5\rangle \\ &\leq -l_2(|y_1-z_1|^2+|y_2-z_2|^2+|y_5-z_5|^2), \end{aligned}$$

for all $t\in[0,T]$, $x\in G$, $\mathbf{y}$, $\mathbf{z}\in\mathfrak{R}$.

(**A.4**) (compatibility) The constants satisfies the inequality $c_0^2 > c_1$.

Definition 2.1. *A triple $(\mathbf{u},\mathbf{v},Z)$ of adapted processes is said to be a solution of (2) if it satisfies (2) P-a.s. and it is in the space $\Big\{\mathcal{C}^L_0(G)\cap\mathcal{H}^1_0(G)\Big\}^2 \times \mathcal{L}^2_0(G;\mathbb{R}^d)$.*

The following form of Itô formula can be found in [21] and [23]. We are going to use this version of Itô formula throughout this paper.

Lemma 2.2. *Suppose that $u\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^1_0(\mathbb{R}^n)))$, $\alpha\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^{-1}(\mathbb{R}^n)))$ and $\beta\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;L^2(\mathbb{R}^n:\mathbb{R}^d)))$, such that $du=\alpha ds+\langle\beta,dW\rangle$, $s\in[0,T]$. Then for $s\in[0,T]$ it holds that*

$$\begin{aligned} &\|u(t)\|^2_{L^2}-\|u(0)\|^2_{L^2} \\ &=\int_0^t\Big\{2\langle\alpha,u\rangle_{H^{-1},H^1}+\|\beta\|^2_{L^2}\Big\}ds+2\int_0^t\int_{\mathbb{R}^n}\langle u\beta dx,dW(s)\rangle. \end{aligned} \tag{3}$$

Now let us recall some well-known results on Yosida-approximations of monotone and continuous functions. The proof of these results can be found in [16] and [17]. These results are key to the existence of an adapted solution to system (2).

Lemma 2.3. *Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a continuous function satisfying*

$$\langle F(x_1)-F(x_2),x_1-x_2\rangle\leq 0,\quad \forall\ x_1,x_2\in\mathbb{R}^n,$$

where $\langle,\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Then *(i)* *for any* $\alpha > 0$ *and any* $y \in \mathbb{R}^n$*, there exists a unique* $x = J^\alpha(y)$*, such that* $x - \alpha F(x) = y$*,*

(ii) *for any* $x \in \mathbb{R}^n$*,* $\lim_{\alpha \to 0} J^\alpha(x) = x$*.*

We then define the Yosida approximations as follows.

Definition 2.4. *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function. The Yosida approximations* F^α*,* $\alpha > 0$*, of* F*, is defined to be*

$$F^\alpha(x) = \frac{1}{\alpha}(J^\alpha(x) - x), \quad \forall\, x \in \mathbb{R}^n.$$

Lemma 2.5. *Let* F *be continuous, and suppose there exists a constant* $c > 0$*, such that* $\langle F(x_1) - F(x_2), x_1 - x_2\rangle \leq -c|x_1 - x_2|^2$*. Then*

(i) $\langle F^\alpha(x_1) - F^\alpha(x_2), x_1 - x_2\rangle \leq -c|x_1 - x_2|^2, \quad \forall\, x_1, x_2 \in \mathbb{R}^n$*,*

(ii) $|F^\alpha(x_1) - F^\alpha(x_2)| \leq (\frac{2}{\alpha} + c)|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}^n$*,*

(iii) $|F^\alpha(x)| \leq |F(x)| + 2c|x|, \quad \forall\, x \in \mathbb{R}^n$*,*

(iv) *For any* $\alpha > 0$*,* $\beta > 0$*, and* x_1*,* $x_2 \in \mathbb{R}^n$*, we have*
$\langle F^\alpha(x_1) - F^\beta(x_2), x_1 - x_2\rangle \leq (\alpha + \beta)(|F(x_1)| + |F(x_2)| + c|x_1| + c|x_2|)^2 - c|x_1 - x_2|^2$*,*

(v) *For any* $\{x^\alpha\}_{\alpha>0} \subset \mathbb{R}^n$*,* $x \in \mathbb{R}^n$*, if* $\lim_{\alpha\to 0} x^\alpha = x$*, then* $\lim_{\alpha \to 0} F^\alpha(x^\alpha) = F(x)$*.*

3 Main Result

When the Lipschitz condition is satisfied by the coefficients, the existence and uniqueness of a solution to (2) has been shown. The following result can be found in [24].

Theorem 3.1. *Let* $\mathbf{y} \triangleq (y_1, y_2, y_3, y_4, y_5)$ *and* $\mathbf{z} \triangleq (z_1, z_2, z_3, z_4, z_5)$*. Suppose there exists a positive constant* C*, such that*

$$|f(t,x,\mathbf{y}) - f(t,x,\mathbf{z})|^2 + |h(t,x,\mathbf{y}) - h(t,x,\mathbf{z})|^2 + |\sigma(t,x,\mathbf{y}) - \sigma(t,x,\mathbf{z})|^2 \leq C|\mathbf{y} - \mathbf{z}|^2,$$

for all $t \in [0,T]$*,* $x \in G$*,* $\mathbf{y}$*,* $\mathbf{z} \in \mathfrak{R}$*. Then under the assumptions (A.1), (A.3) and (A.4), the system* (2) *admits a unique adapted solution in the space* $\left\{\mathcal{C}_0^L(G) \cap \mathcal{H}_0^1(G)\right\}^2 \times \mathcal{L}_0^2(G; \mathbb{R}^d)$*.*

Note that $X = (\mathbf{u}, \mathbf{v}, Z)$ and $Y = (\mathbf{u}, \mathbf{v}, \nabla\mathbf{u}, \nabla\mathbf{v}, Z)$. We denote $X^\alpha = (\mathbf{u}^\alpha, \mathbf{v}^\alpha, Z^\alpha)$ and $Y^\alpha = (\mathbf{u}^\alpha, \mathbf{v}^\alpha, \nabla\mathbf{u}^\alpha, \nabla\mathbf{v}^\alpha, Z^\alpha)$. For simplicity, sometimes we suppress variables (t, x). To utilize Yosida Approximation scheme, we consider the following FBSPDEs for any $\alpha > 0$.

$$\begin{cases} \mathbf{u}^\alpha(t,x) = \mathbf{u}_0(x) + \int_0^t S^\alpha(s,x,X^\alpha)ds + \int_0^t \langle \sigma^\alpha(s,x,Y^\alpha), dW(s)\rangle \\ \mathbf{v}^\alpha(t,x) = g(x, \mathbf{u}^\alpha(T,x)) - \int_t^T T^\alpha(s,x,X^\alpha)ds - \int_t^T \langle Z^\alpha(s), dW(s)\rangle. \end{cases} \tag{1}$$

Here

$$\begin{aligned} S^\alpha(t,x,u,v,z) &\triangleq \nabla \cdot (A\nabla u + B\nabla v + Cz) + f^\alpha(t,x,u,v,\nabla u,\nabla v,z); \\ T^\alpha(t,x,u,v,z) &\triangleq \nabla \cdot (\Phi\nabla u - \Psi\nabla v - \Upsilon z) + h^\alpha(t,x,u,v,\nabla u,\nabla v,z); \end{aligned}$$

and f^α, h^α and σ^α are Yosida approximations of f, h and σ. We will also use the following notations for the rest of the paper:

- $\|X\|_{L^2}^2 \triangleq \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2 + \|Z\|_{L^2}^2$.

- $\|X\|_{H^1}^2 \triangleq \|\mathbf{u}\|_{H^1}^2 + \|\mathbf{v}\|_{H^1}^2 + \|Z\|_{L^2}^2$.
- $\|X^\alpha\|_{H^1}^2 \triangleq \|\mathbf{u}^\alpha\|_{H^1}^2 + \|\mathbf{v}^\alpha\|_{H^1}^2 + \|Z^\alpha\|_{L^2}^2$ for any $\alpha > 0$.
- $\|\hat{X}^{\alpha,\beta}\|_{H^1}^2 \triangleq \|\mathbf{u}^\alpha - \mathbf{u}^\beta\|_{H^1}^2 + \|\mathbf{v}^\alpha - \mathbf{v}^\beta\|_{H^1}^2 + \|Z^\alpha - Z^\beta\|_{L^2}^2$ for any $\alpha > 0, \beta > 0$.

One can easily check that the coefficients in system (1) satisfy the assumptions in Theorem 3.1. Hence there exists a unique adapted solution $(\mathbf{u}^\alpha, \mathbf{v}^\alpha, Z^\alpha)$ in $\Big\{\mathcal{C}_0^L(G) \cap \mathcal{H}_0^1(G)\Big\}^2 \times \mathcal{L}_0^2(G;\mathbb{R}^d)$, and

$$\sup_{t\in[0,T]} E\|\mathbf{u}^\alpha\|_{L^2}^2 + \sup_{t\in[0,T]} E\|\mathbf{v}^\alpha\|_{L^2}^2 + E\int_0^T \|Z^\alpha\|_{L^2} ds < \infty. \tag{2}$$

Proposition 3.2. *Under the standing assumptions of this section, and let $(\mathbf{u}^\alpha, \mathbf{v}^\alpha, Z^\alpha)$ be an adapted solution to system (1). Then for some positive constant k, we have the following uniform a priori estimate*

$$\begin{aligned}
&E\|\mathbf{u}^\alpha(T,x)\|_{L^2}^2 + E\int_0^T \|X_s^\alpha\|_{H^1}^2 ds \\
\le& k\bigg(E\int_0^T \Big\{\|S^\alpha(s,x,0)\|_{L^2}^2 + \|T^\alpha(s,x,0)\|_{L^2}^2 + \|\sigma^\alpha(s,x,0)\|_{L^2}^2\Big\} ds + E\|\mathbf{u}_0(x)\|_{L^2}^2 + 1\bigg).
\end{aligned}$$

Proof. Applying Itô formula to $\langle \mathbf{u}^\alpha, \mathbf{v}^\alpha\rangle_{L^2}$, we get

$$\begin{aligned}
&E\langle \mathbf{u}^\alpha(T,x), g(x,\mathbf{u}^\alpha(T,x))\rangle_{L^2} - E\langle \mathbf{u}_0(x), \mathbf{v}^\alpha(0,x)\rangle_{L^2} \\
=& E\int_0^T \Big\{\langle S^\alpha(s,x,X^\alpha), \mathbf{v}^\alpha\rangle_{H^{-1},H^1} + \langle T^\alpha(s,x,X^\alpha), \mathbf{u}^\alpha\rangle_{H^{-1},H^1} + \langle \sigma^\alpha(s,x,Y^\alpha), Z^\alpha\rangle_{L^2}\Big\} ds
\end{aligned}$$

The above equation is equivalent to

$$\begin{aligned}
&E\langle \mathbf{u}^\alpha(T,x), g(x,\mathbf{u}^\alpha(T,x))\rangle_{L^2} - E\langle \mathbf{u}_0(x), \mathbf{v}^\alpha(0,x)\rangle_{L^2} \\
=& E\int_0^T \Big\{\langle S^\alpha(s,x,X^\alpha) - S^\alpha(s,x,0), \mathbf{v}^\alpha\rangle_{H^{-1},H^1} + \langle T^\alpha(s,x,X^\alpha) - T^\alpha(s,x,0), \mathbf{u}^\alpha\rangle_{H^{-1},H^1} \\
&\qquad + \langle \sigma^\alpha(s,x,Y^\alpha) - \sigma^\alpha(s,x,0), Z^\alpha\rangle_{L^2}\Big\} ds \\
&+ E\int_0^T \Big\{\langle S^\alpha(s,x,0), \mathbf{v}^\alpha\rangle_{H^{-1},H^1} + \langle T^\alpha(s,x,0), \mathbf{u}^\alpha\rangle_{H^{-1},H^1} + \langle \sigma^\alpha(s,x,0), Z^\alpha\rangle_{L^2}\Big\} ds
\end{aligned}$$

Notice the structure of g, and by Lemma 2.5 (i), one can show that for some constants $k_1 > 0$ and $\epsilon > 0$,

$$\begin{aligned}
&\Big\{(k_1 + \frac{1}{4\epsilon}) E\|\mathbf{u}^\alpha(T,x)\|_{L^2}^2 + \epsilon\Big\} + (c-\epsilon) E\int_0^T \|X_s^\alpha\|_{H^1}^2 ds \\
\le& \frac{1}{4\epsilon} E\int_0^T \Big\{\|S^\alpha(s,x,0)\|_{L^2}^2 + \|T^\alpha(s,x,0)\|_{L^2}^2 + \|\sigma^\alpha(s,x,0)\|_{L^2}^2\Big\} ds + E\langle \mathbf{u}_0(x), \mathbf{v}^\alpha(0,x)\rangle_{L^2}
\end{aligned}$$

Here ϵ comes from the inequality $ab \le \frac{a^2}{4\epsilon} + \epsilon b$ and can be choosing according to our propose.

On the other hand, by Lemma 2.5 (iii) and the linear growth assumption, one can easily deduce that there exists a constant $k_2 > 0$, such that

$$E\|\mathbf{v}^\alpha(0,x)\|_{L^2}^2 \le k_2 \Big\{E\|\mathbf{v}^\alpha(T,x)\|_{L^2}^2 + E\int_0^T \|T(s,x,0)\|_{L^2}^2 ds + E\int_0^T \|X_s^\alpha\|_{H^1}^2 ds + 1\Big\}$$

Combining the above two inequalities and by choosing an appropriate ϵ, it is clear that for some constant $k_3 > 0$

$$\begin{aligned}&E\|\mathbf{u}^\alpha(T,x)\|_{L^2}^2 + E\int_0^T \|X_s^\alpha\|_{H^1}^2 ds\\ \le & k_3\Big(E\int_0^T \Big\{\|S^\alpha(s,x,0)\|_{L^2}^2 + \|T^\alpha(s,x,0)\|_{L^2}^2 + \|\sigma^\alpha(s,x,0)\|_{L^2}^2\Big\}ds + E\|\mathbf{u}_0(x)\|_{L^2}^2 + 1\Big).\end{aligned}$$

■

Now we are ready to present the main result of this paper.

Theorem 3.3. *Under the assumptions (A.1)-(A.4), the system* (2) *admits a unique adapted solution in the space* $\Big\{\mathcal{C}_0^L(G)\cap\mathcal{H}_0^1(G)\Big\}^2\times\mathcal{L}_0^2(G;\mathbb{R}^d)$.

Proof. Let us first prove the existence. For any $\alpha > 0$ and $\beta > 0$, applying Itô formula to $\langle \mathbf{v}^\alpha-\mathbf{v}^\beta, \mathbf{u}^\alpha-\mathbf{u}^\beta\rangle_{L^2}$, we get

$$\begin{aligned}&E\langle \mathbf{u}^\alpha(T,x)-\mathbf{u}^\beta(T,x), g(x,\mathbf{u}^\alpha(T,x))-g(x,\mathbf{u}^\beta(T,x))\rangle_{L^2}\\ =&E\int_0^T\Big\{\langle S^\alpha(s,x,X^\alpha)-S^\beta(s,x,X^\beta), \mathbf{v}^\alpha-\mathbf{v}^\beta\rangle_{H^{-1},H^1}\\ &+\langle T^\alpha(s,x,X^\alpha)-T^\beta(s,x,X^\beta), \mathbf{u}^\alpha-\mathbf{u}^\beta\rangle_{H^{-1},H^1}\\ &+\langle \sigma^\alpha(s,x,Y^\alpha)-\sigma^\beta(s,x,Y^\beta), Z^\alpha-Z^\beta\rangle_{L^2}\Big\}ds.\end{aligned}$$

By Lemma 2.5 (iv) and the standing assumptions of this paper, it is easy to show that there exists a constant $k_1 > 0$, such that

$$\begin{aligned}&E\|\mathbf{u}^\alpha(T,x)-\mathbf{u}^\beta(T,x)\|_{L^2}^2 + k_1E\int_0^T\|\hat{X}_s^{\alpha,\beta}\|_{H^1}^2ds\\ \le&(\alpha+\beta)E\int_0^T\Big\{\|S(s,x,X^\alpha)\|_{L^2}^2+\|S(s,x,X^\beta)\|_{L^2}^2+\|T(s,x,X^\alpha)\|_{L^2}^2+\|T(s,x,X^\beta)\|_{L^2}^2\\ &+\|\sigma(s,x,X^\alpha)\|_{L^2}^2+\|\sigma(s,x,X^\beta)\|_{L^2}^2+k_1\|X^\alpha\|_{H^1}^2+k_1\|X^\beta\|_{H^1}^2\Big\}ds.\end{aligned}$$

By Proposition 3.2, we deduce that there exists a constant $k_2 > 0$, such that

$$E\|\mathbf{u}^\alpha(T,x)-\mathbf{u}^\beta(T,x)\|_{L^2}^2 + E\int_0^T\|\hat{X}_s^{\alpha,\beta}\|_{H^1}^2ds \le k_2(\alpha+\beta).$$

Hence we know the $\{X^\alpha, \alpha > 0\}$ is a Cauchy sequence in $\mathcal{L}_0^2(G;\mathbb{R}^d)^3$and let $X = (\mathbf{u},\mathbf{v},Z)$ be the limit. Again by the standing assumptions and applying Lemma 2.5 (iii), we see that there exists a constant $k_3 > 0$,

$$|S^\alpha(s,x,X^\alpha)| \le |S(s,x,X^\alpha)| + 2c|X^\alpha| \le |S(s,x,0)| + k_3|X^\alpha|.$$

Similar results apply to T^α and σ^α. Thus there exists a constant $k_4 > 0$, such that

$$E\int_0^T\Big\{\|S^\alpha(s,x,X^\alpha)\|_{L^2}^2+\|T^\alpha(s,x,X^\alpha)\|_{L^2}^2+\|\sigma^\alpha(s,x,X^\alpha)\|_{L^2}^2\Big\}ds \le k_4.$$

This lets to the weak limits of $T^\alpha(s,x,X^\alpha)$, $S^\alpha(s,x,X^\alpha)$ and $\sigma^\alpha(s,x,X^\alpha)$ in $\mathcal{L}_0^2(G;\mathbb{R}^d)$. Let the corresponding limits be $\mathcal{S}$, $\mathcal{T}$ and Σ. Consequently, taking the limit of system (1), we get

$$\begin{cases}\mathbf{u}(t,x) = \mathbf{u}_0(x) + \int_0^t\mathcal{S}(s)ds + \int_0^t\langle\Sigma(s), dW(s)\rangle\\ \mathbf{v}(t,x) = g(x,\mathbf{u}(T,x)) - \int_t^T\mathcal{T}(s)ds - \int_t^T\langle Z(s), dW(s)\rangle.\end{cases}\tag{3}$$

Let $\mathcal{X} \in \left\{\mathcal{H}_0^1(G)\right\}^2 \times \mathcal{L}_0^2(G;\mathbb{R}^d)$. Then by Lemma 2.5 (i),

$$\begin{aligned}
&E\int_0^T \langle S^\alpha(s,x,X^\alpha) - S(s,x,\mathcal{X}), X^\alpha - \mathcal{X}\rangle_{H^{-1},H^1} ds \\
=&E\int_0^T \langle\Big(S^\alpha(s,x,X^\alpha) - S^\alpha(s,x,\mathcal{X})) + (S^\alpha(s,x,\mathcal{X}) - S(s,x,\mathcal{X})\Big), X^\alpha - \mathcal{X}\rangle_{H^{-1},H^1} ds \\
\leq& -cE\int_0^T \|X^\alpha - \mathcal{X}\|_{H^1}^2 ds + E\int_0^T \langle S^\alpha(s,x,\mathcal{X}) - S(s,x,\mathcal{X}), X^\alpha - \mathcal{X}\rangle_{H^{-1},H^1} ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&E\int_0^T \langle \mathcal{S} - S(s,x,\mathcal{X}), X - \mathcal{X}\rangle_{H^{-1},H^1} ds \\
=&\varlimsup_{\alpha\to 0} E\int_0^T \langle S^\alpha(s,x,X^\alpha) - S(s,x,\mathcal{X}), X^\alpha - \mathcal{X}\rangle_{H^{-1},H^1} ds \leq -cE\int_0^T \|X - \mathcal{X}\|_{H^1}^2 ds.
\end{aligned}$$

Letting $\mathcal{X}$ to be $X - \epsilon(\mathcal{S}(s) - S(s,x,X))$ in the above inequality for some $\epsilon > 0$, and we get

$$E\int_0^T \|\mathcal{S} - S(s,x,X)\|_{H^1}^2 ds = 0$$

when letting ϵ go to 0. Hence $\mathcal{S} = S(s,x,X)$, P-a.s. for every $s \geq 0$. Similarly, one can show that $\mathcal{T} = T(s,x,X)$ and $\Sigma = \sigma(s,x,X)$, P-a.s. for every $s \geq 0$. Plugging these into system (3), we have completed the existence of a solution to system (2). The regularity of the solution can be obtained by applying the Itô formula to $\|\mathbf{u}\|_{L^2}^2$ and $\|\mathbf{v}\|_{L^2}^2$, respectively and the fact that $(\mathbf{u},\mathbf{v})$ is in $\left\{\mathcal{H}_0^1(G)\right\}^2$.

Let us now turn to the uniqueness. Suppose $(\mathbf{u},\mathbf{v},Z)$ and $(\mathbf{u}',\mathbf{v}',Z')$ are two solutions. Let $\hat{\mathbf{u}} = \mathbf{u} - \mathbf{u}'$, $\hat{\mathbf{v}} = \mathbf{v} - \mathbf{v}'$, and $\hat{Z} = Z - Z'$. Applying the Itô formula to $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle_{L^2}$ we get

$$\begin{aligned}
&E\langle \hat{\mathbf{u}}(T,x), \hat{\mathbf{v}}(T,x)\rangle_{L^2} \\
=&E\int_0^T \Big\{\langle S(X) - S(X'), \hat{\mathbf{v}}\rangle_{H^{-1},H^1} + \langle T(X) - T(X'), \hat{\mathbf{u}}\rangle_{H^{-1},H^1} + \langle \sigma(Y) - \sigma(Y'), \hat{Z}\rangle_{L^2}\Big\} ds \\
=&E\int_0^T \Big\{\langle \nabla\cdot(A\nabla\hat{\mathbf{u}} + B\nabla\hat{\mathbf{v}} + C\hat{Z}) - B\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle_{H^{-1},H^1} \\
&\qquad + \langle \nabla\cdot(\Phi\nabla\hat{\mathbf{u}} - \Psi\nabla\hat{\mathbf{v}} - \Upsilon\hat{Z}) - \Phi\hat{\mathbf{u}}, \hat{\mathbf{u}}\rangle_{H^{-1},H^1}\Big\} ds \\
&+ E\int_0^T \Big\{\langle f(\mathbf{u},\nabla\mathbf{u},\mathbf{v},\nabla\mathbf{v},Z) - f(\mathbf{u}',\nabla\mathbf{u}',\mathbf{v}',\nabla\mathbf{v}',Z'), \hat{\mathbf{v}}\rangle_{H^{-1},H^1}\Big\} ds \\
&+ E\int_0^T \Big\{\langle h(\mathbf{u},\nabla\mathbf{u},\mathbf{v},\nabla\mathbf{v},Z) - h(\mathbf{u}',\nabla\mathbf{u}',\mathbf{v}',\nabla\mathbf{v}',Z'), \hat{\mathbf{u}}\rangle_{H^{-1},H^1}\Big\} ds \\
&+ E\int_0^T \Big\{\langle \sigma(\mathbf{u},\nabla\mathbf{u},\mathbf{v},\nabla\mathbf{v},Z) - \sigma(\mathbf{u}',\nabla\mathbf{u}',\mathbf{v}',\nabla\mathbf{v}',Z'), \hat{Z}\rangle_{L^2}\Big\} ds \\
\leq&(-c_0 + \sqrt{c_1})E\int_0^T \Big\{\|\hat{\mathbf{u}}\|_{H^1}^2 + \|\hat{\mathbf{v}}\|_{H^1}^2\Big\} ds + \frac{1}{2}E\int_0^T \|\hat{Z}\|_{L^2}^2 ds \\
&- l_2 E\int_0^T \Big\{\|\hat{\mathbf{u}}\|_{L^2}^2 + \|\hat{\mathbf{v}}\|_{L^2}^2 + \|\hat{Z}\|_{L^2}^2\Big\} ds \\
\leq& -(c_0 - \sqrt{c_1})E\int_0^T \Big\{\|\hat{\mathbf{u}}\|_{H^1}^2 + \|\hat{\mathbf{v}}\|_{H^1}^2\Big\} ds - (l_2 - \frac{1}{2})E\int_0^T \|\hat{Z}\|_{L^2}^2 ds.
\end{aligned}$$

On the other hand, one has

$$E\langle \hat{\mathbf{u}}(T,x), \hat{\mathbf{v}}(T,x)\rangle_{L^2} = E\langle \hat{\mathbf{u}}(T,x), p\hat{\mathbf{u}}(T,x)\rangle_{L^2} \geq c_4 E\|\hat{\mathbf{u}}(T)\|^2_{L^2}.$$

Therefore

$$c_4 E\|\hat{\mathbf{u}}(T)\|^2_{L^2} \leq -(c_0 - \sqrt{c_1})E\int_0^T \Big\{\|\hat{\mathbf{u}}\|^2 + \|\hat{\mathbf{v}}\|^2\Big\}ds - (l_2 - \frac{1}{2})E\int_0^T \|\hat{Z}\|^2_{L^2}ds.$$

Hence we have proved that $(\mathbf{u}, \mathbf{v}, Z) = (\mathbf{u}', \mathbf{v}', Z')$, P-a.s., for every $t \geq 0$ and $x \in \mathbb{R}^n$.

■

References

[1] Bismut, J. M. *Conjugate Convex Functions in Optimal Stochastic Control*, J. Math. Anal. Apl., 44 (1973) 384–404.

[2] Bismut, J. M. *Théorie Probabiliste du Contrôle des Diffusions*, Mem. Amer. Math. Soc. 176, Providence, Rhode Island, 1973.

[3] Buckdahn, R. and Hu, Y. *Hedging contingent claims for a large investor in an incomplete market*, Adv. in Appl. Probab. 30 (1998), no. 1, 239–255.

[4] Chow, P. L. *Stochastic Partial Differential Equations*, Taylor & Francis Group, Boca Raton, 2007.

[5] Confortola, F. *Dissipative backward stochastic differential equations in infinite dimensions*, Ph.D. thesis, Politecnico di Milano, Milano, Italy, 2005.

[6] Cvitanic, J. and Ma, J. *Hedging options for a large investor and forward-backward SDE's*, Ann. Appl. Probab. **6** (1996), no. 2, 370–398.

[7] Cvitanic, J. and Zhang, J. *Contract Theory in Continuous-Time Models*, Springer, 2012. Ann. Appl. Probab. **6** (1996), no. 2, 370–398.

[8] Da Prato, G. and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and Its Applications. **44**, Cambridge University Press, 1992.

[9] Da Prato, G. and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, London Mathematical Society Lecture Note Series. **229**, Cambridge University Press, 1996.

[10] Hu, Y. *On the solution of forward-backward SDEs with monotone and continuous coefficients*, Nonlinear Anal., 42 (2000) 1–12.

[11] Hu, Y., Ma, J., and Yong, J. *On semi-linear degenerate backward stochastic partial differential equations*, Probab. Theory Related Fields **123** (2002), no. 3, 381–411.

[12] Hu, Y. and Peng, S. *Adapted solution of a backward semilinear stochastic evolution equation*, Stochastic Analysis and Applications, vol. 9, no. 4, pp. 445-459, 1991.

[13] Kallianpur, G. and Xiong, J. *Stochastic Differential Equations in Infinite Dimensional Spaces*, IMS Lecture Notes-Monograph Series, 26, 1995.

[14] Liu, Y. and Ma, J. *Optimal Reinsurance/Investment for General Insurance Models*, Ann. Appl. Probab. 19 (2009), no. 4, 1495–1528

[15] Ma, J., Protter, P. and Yong, J. *Solving Forward-Backward Stochastic Differential Equations Explicitly–A Four Step Scheme*, Prob. Th. & Rel. Fields, 98 (1994), 339–359.

[16] Da Prato, G. and Zabczyk, J. *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[17] Da Prato, G. and Zabczyk, J. *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[18] Ma, J., Yin, H., and Zhang, J., *On non-Markovian forward-backward SDEs and Backward SPDEs*, Stochastic Processes and their Applications, 122 (2012) 3980–4004.

[19] Ma, J. and Yong, J., *Adapted solution of a degenerate backward spde, with applications*, Stochastic Processes and their Applications 70 (1997), 59–84.

[20] Ma, J. and Yong, J., *On linear, degenerate backward stochastic partial differential equations*, Probab. Theory Related Fields **113** (1999), no. 2, 135–170.

[21] Pardoux, E., *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics, 3, 127-167 (1979).

[22] Pardoux, E. and Peng, S. *Adapted Solution of a Backward Stochastic Differential Equation*, Systems and Control Letters, 14 (1990) 55–66.

[23] Prévôt, C. and Röckner, M. *A concise Course on Stochastic Partial Differential Equations*, Springer-Verlag, Berlin, 2007.

[24] Yin, H. *Solvability of Forward-Backward Stochastic Partial Differential Equations*, Stochastic Processes and their Applications, 124 (2014) 2583-2604.