Journal of Progressive Research in Mathematics
Vol 18,1 - 2021 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

Toeplitz Determinants for a Subclass of Analytic Functions

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Received: March 07, 2021; Accepted: March 29, 2021; Published: May 3, 2021

Abstract

A new subclass of analytic functions that generalizes some known subclasses of analytic functions was defined and investigated.The bounds for Toeplitz determinants of $T_2(2), T_2(3), T_3(1)$ and $T_3(2)$ were obtained.

Keywords

Analytic functions, Salagean differential operator and Toeplitz determinant.

1. Introduction

Let H denote the class of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and satisfy the condition $f(0) = 0$ and $f^{'}(0) = 1$. Let S denote the subclass of H consisting of univalent in E. A funtion $f(z) \in S$ is said to be starlike in the unit disk if and only if

$$Re\frac{zf'(z)}{f(z)} > 0, \qquad z \in E \tag{1.2}$$

Also, a function $f(z) \in S$ is said to be convex in the unit disk if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in E \tag{1.3}$$

Cite this paper: Ayinla, R., Bello, R. (2021). Toeplitz Determinants for a Subclass of Analytic Functions. Journal of Progressive Research in Mathematics, 18(1), 99-106. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2034

Let $D^n : A \to A$ be defined by

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = zf'(z)$$
$$D^n f(z) = z[D^{n-1} f(z)]'$$

which is equivalent to

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n = \{0, 1, 2, ...\}), \quad z \in E$$

D^n is the Salagean differential operator [7].
Hankel determinants play important role in several branches of mathematics such as qantum mechanics, image processing, statistics and probability, queueing networks, signal processing and time series analysis to mention a few [9]. A Toeplitz determinants is an upside down Hankel determinants, that is Hankel determinants have constant entries along the reverse diagonal while Toeplitz determinants have constant entries the diagonal.
Thomas and Halim [8] introduced the symmetric Toeplitz determinant $T_q(n)$ for analytic functions $f(z)$ of the form (1.1) defined as follows

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} \qquad (where\ n, q = 1, 2, 3, \cdots a_1 = 1\ for f(z) \in S)$$

In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}$$

The estimates of the Toeplitz determinant $T_q(n)$ for functions in S^* and K have been studied in [8]. The Toeplitz determinant for some subclasses of analytic function was discussed in [6]. Also, estimates of the Toeplitz determinant for boundary rotation (R) was studied in [5] for small n and q.
Let P denote the class of analytic functions p in E of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.4}$$

such that $Rep(z) > 0$ in E.
A function $f(z)$ of the form (1.1) analytic and univalent in E is said to be in the $R_n(\alpha, \beta)$, $\alpha \in [0, 1]$, $\beta \in \left(\frac{-\pi}{2}\ , \frac{\pi}{2}\right)$ and $n \in \mathbb{N}_0$ if it satisfies the inequality

$$Re\left\{e^{i\beta}\left(1 - e^{-2i\beta}\alpha^2 z^2\right)\frac{D^{n+1}f(z)}{z}\right\} > 0, z \in E. \tag{1.5}$$

Remark 1
(ii) For $n = 0$, $\beta = 0$ gives

$$Re\left\{\left(1 - \alpha^2 z^2\right) f'(z)\right\} > 0, \quad z \in E. \tag{1.6}$$

investigated by kanas and Lecko [2].
For $n = 0$, (1.5) gives

$$Re\left\{e^{i\beta}\left(1 - e^{-2i\beta}\alpha^2 z^2\right) f'(z)\right\} > 0, \quad z \in E. \tag{1.7}$$

studied in[3].

2 Preliminary Lemmas

We need the following lemmas to prove our results.

Let P denote the class of Caratheodory functions.
$p(z) = 1 + p_1 z + p_2 z^2 + 3_3 z^3 + ... \quad (z \in E)$
which are analytic and satisfy $p(0) = 1$ and $\Re p(z) > 0$ Let $p \in P$. Then

$$|p_k| \leq 2 \quad (k \in \mathbb{N}) \quad [1] \tag{2.1}$$

. Let $p \in P$ then

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right) \tag{2.2}$$

$$4p_3 = p_1^3 + 2p_1(4 -_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \tag{2.3}$$

for some value of x, z, such that $|x| \leq 1$ and $|z| \leq 1$. [4]

3 Main Results

Let $f(z) \in R_n(\alpha, \beta), \alpha \in [0, 1], \quad \beta \in \left(\frac{-\pi}{2}\ , \frac{\pi}{2}\right)$ and $n \in \mathbb{N}_0$. Then

$$|T_2(2)| = |a_3^2 - a_2^2| \leq \frac{4\cos^2\beta + 4\alpha^2\cos\beta}{3^{2n+2}} + \frac{\cos\beta}{2^{2n}}$$

Proof:
Let $f(z) \in R_n(\alpha, \beta)$, then there exist $p \in P$ such that

$$e^{i\beta}\left\{\left(1 - e^{-2i\beta}\alpha^2 z^2\right)\left(1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1}\right)\right\} = p(z)\cos\beta + i\sin\beta \tag{3.1}$$

Equating coefficients of (3.1) give

$$a_2 = \frac{p_1 e^{-i\beta}\cos\beta}{2^{n+1}} \tag{3.2}$$

$$a_3 = \frac{p_2 e^{-i\beta}\cos\beta + \alpha^2 e^{-2i\beta}}{3^{n+1}} \tag{3.3}$$

$$a_4 = \frac{p_3 e^{-i\beta}\cos\beta + p_1\alpha^2 e^{-3i\beta}\cos\beta}{4^{n+1}} \tag{3.4}$$

Using (3.2) and (3.3) we obtain

$$|T_2(2)| = \left|a_3^2 - a_2^2\right| = \left|\frac{p_2^2 e^{-2i\beta}}{3^{2n+2}}\cos^2\beta + \frac{2p_2\alpha^2 e^{-3i\beta}}{3^{2n+2}}\cos\beta - \frac{p_1^2 e^{-2i\beta}}{2^{2n+2}}\cos^2\beta + \frac{\alpha^4 e^{-4i\beta}}{3^{2n+2}}\right| \tag{3.5}$$

Substituting for p_2 using lemma (2.2) we get

$$\begin{aligned}\left|a_3^2 - a_2^2\right| = \Bigg|&\frac{p_1^4 e^{-2i\beta}}{2^2 \cdot 3^{2n+2}}\cos^2\beta + \frac{p_1^2 x(4 - p_1^2)e^{-2i\beta}}{2 \cdot 3^{2n+2}}\cos^2\beta + \frac{\alpha^2 x(4 - p_1^2)e^{-3i\beta}}{3^{2n+2}}\cos\beta \\ &- \frac{p_1^2 e^{-2i\beta}}{2^{2n+2}}\cos^2\beta + \frac{x^2(4 - p_1^2)^2 e^{-2i\beta}}{2^2 \cdot 3^{2n+2}}\cos^2\beta + \frac{\alpha^2 p_1^2 e^{-3i\beta}}{3^{2n+2}}\cos\beta + \frac{\alpha^4 e^{-4i\beta}}{3^{2n+2}}\Bigg|\end{aligned} \tag{3.6}$$

Let $p_1 = p, |p| \leq 2$ and assuming without loss of generality that $p \in [0, 2]$. Also, applying the triangle inequality with $X = 4 - p_1^2$ gives

$$\begin{aligned}\left|a_3^2 - a_2^2\right| \leq &\frac{p^4}{2^2 \cdot 3^{2n+2}}\cos^2\beta + \frac{p^2}{2^{2n+2}}\cos^2\beta + \frac{p^2|x|X}{2 \cdot 3^{2n+2}}\cos^2\beta + \frac{\alpha^2|x|X}{3^{2n+2}}\cos\beta \\ &+ \frac{|x|^2 X^2}{2^2 \cdot 3^{2n+2}}\cos^2\beta + \frac{p^2\alpha^2}{2n+2}\cos\beta + \frac{\alpha^4}{3^{2n+2}} = \mu(p, |x|)\end{aligned} \tag{3.7}$$

Differentiating $\mu(p,|x|)$ w.r.t $|x|$ and using elementary calculus shows that $\partial\mu(p,|x|/\partial|x|) > 0$ for $|x| \in [0,1]$ and fixed $p \in [0,2]$.This follows that $\mu(p,|x|)$ is an increasing function of $|x|$. So, $\mu(p,|x|) \leq \mu(p,1)$. Therefore,

$$\left|a_3^2 - a_2^2\right| \leq \frac{p^2}{2^{2n+2}}\cos^2\beta + \frac{1}{3^{2n+2}}\left(4\alpha^2\cos\beta + 4\cos^2\beta + \alpha^4\right) \tag{3.8}$$

Now, $\mu(|x|)$ has a maximum value

$$\frac{\cos^2\beta}{2^{2n}} + \frac{4\cos^2\beta + 4\alpha^2\cos\beta + \alpha^4}{3^{2n+2}}$$

on $[0,2]$ when p=2.
Hence,

$$\left|a_3^2 - a_2^2\right| \leq \frac{4\cos^2\beta + 4\alpha^2\cos\beta + \alpha^4}{3^{2n+2}} + \frac{\cos^2\beta}{2^{2n}}$$

Remark 2
For $n = \alpha = \beta = 0$

$$|T_2(2)| = \left|a_3^2 - a_2^2\right| \leq \frac{13}{9}$$

which is much finer than $|a_3^2 - a_2^2| \leq 5$ that was obtained in [6]. Let $f(z) \in R_n(\alpha,\beta), \alpha \in [0,1], \;\; \beta \in \left(\frac{-\pi}{2}\;,\frac{\pi}{2}\right)$ and $n \in \mathbb{N}_0$. Then

$$|T_2(3)| = \left|a_4^2 - a_3^2\right| \leq \frac{1+\alpha^4}{2^{4n+2}}\cos^2\beta + \frac{\alpha^2}{2^{4n+1}}\cos\beta + \frac{1}{3^{2n+2}}\left(4\cos^2\beta + 4\alpha^2\cos\beta + \alpha^4\right)$$

Proof. Using (3.3), (3.4) and lemma (2.2) to express p_2 and p_3 in terms of p_1, we get

$$\begin{aligned}\left|a_4^2 - a_3^2\right| = \Bigg| & \frac{p_1^6 e^{-2i\beta}}{2^{4n+8}}\cos^2\beta + \frac{p_1^4(4-p_1^2)xe^{-2i\beta}}{2^{4n+6}}\cos^2\beta - \frac{p_1^4(4-p_1^2)x^2e^{-2i\beta}}{2^{4n+7}}\cos^2\beta \\ & + \frac{p_1^3(4-p_1^2)(1-|x|^2)ze^{-2i\beta}}{2^{4n+6}}\cos^2\beta + \frac{p_1^2(4-p_1^2)^2x^2e^{-2i\beta}}{2^{4n+6}}\cos^2\beta \\ & - \frac{p_1^2(4-p_1^2)x^3e^{-2i\beta}}{2^{4n+6}}\cos^2\beta + \frac{p_1^2(4-p_1^2)^2x^4e^{-2i\beta}}{2^{4n+8}}\cos^2\beta + \frac{p_1^4\alpha^2e^{-4i\beta}}{2^{4n+5}}\cos\beta \\ & + \frac{p_1(4-p_1^2)^2(1-|x|^2)xze^{-2i\beta}}{2^{4n+5}}\cos^2\beta - \frac{p_1(4-p_1^2)^2x^2(1-|x|^2)ze^{-2i\beta}}{2^{4n+6}}\cos^2\beta \\ & + \frac{(4-p_1^2)^2(1-|x|^2)^2z^2e^{-2i\beta}}{2^{4n+6}}\cos^2\beta + \frac{p_1^2(4-p_1^2)\alpha^2xe^{-4i\beta}}{2^{4n+4}}\cos\beta \\ & - \frac{p_1^2(4-p_1^2)\alpha^2x^2e^{-4i\beta}}{2^{4n+5}}\cos\beta + \frac{p_1(4-p_1^2)(1-|x|^2)\alpha^2ze^{-41\beta}}{2^{4n+4}}\cos\beta \\ & + \frac{p_1^2\alpha^4e^{-6i\beta}}{2^{4n+4}}\cos^2\beta - \frac{p_1^4e^{-2i\beta}}{2^2\cdot 3^{2n+2}}\cos^2\beta - \frac{p_1^2(4-p_1^2)xe^{-2i\beta}}{2\cdot 3^{2n+2}}\cos^2\beta \\ & - \frac{(4-p_1^2)^2x^2}{2^2\cdot 3^{2n+2}} + \frac{p_1^2\alpha^2e^{-2i\beta}}{3^{2n+2}}\cos\beta + \frac{(4-p_1^2)\alpha^2xe^{-2i\beta}}{3^{2n+2}}\cos\beta - \frac{\alpha^4e^{-4i\beta}}{3^{2n+2}}\Bigg|\end{aligned} \tag{3.9}$$

Letting $p_1 = p, |p_1| \leq 2$, assuming without loss of generality that $p \in [0,2]$ and applying triangle inequality with $X = 4 - p_1^2$ and $Y = (1-|x|^2)$ we obtain

$$\begin{aligned}\left|a_4^2 - a_3^2\right| \leq & \frac{p^6}{2^{4n+8}}\cos^2\beta + \frac{p^4|x|X}{2^{4n+6}}\cos^2\beta + \frac{p^4|x|^2X}{2^{4n+7}}\cos^2\beta + \frac{p^3XY}{2^{4n+6}}\cos^2\beta \\ & + \frac{p^2|x|^2X^2}{2^{4n+6}}\cos^2\beta + \frac{p^2|x|^3X^2}{2^{4n+6}}\cos\beta + \frac{p^2|x|^4X^2}{2^{4n+8}} + \frac{p|x|X^2Y}{2^{4n+5}}\cos^2\beta \\ & + \frac{p|x|^2X^2Y}{2^{4n+6}}\cos^2\beta + \frac{X^2Y^2}{2^{4n+6}}\cos^2\beta + \frac{p^4\alpha^2}{2^{4n+5}}\cos\beta + \frac{p^2|x|X\alpha^2}{2^{4n+4}}\cos\beta \\ & + \frac{p^2|x|^2X\alpha^2}{2^{4n+5}}\cos\beta + \frac{pXY\alpha^2}{2^{4n+4}}\cos\beta + \frac{p^2\alpha^4}{2^{4n+4}}\cos^2\beta + \frac{p^4}{2^2\cdot 3^{2n+2}}\cos^2\beta \\ & + \frac{p^2|x|X}{2\cdot 3^{2n+2}}\cos^2\beta + \frac{|x|^2X^2}{2^2\cdot 3^{2n+2}}\cos^2\beta + \frac{p^2\alpha^2}{3^{2n+2}}\cos\beta + \frac{\alpha^2|x|X}{3^{2n+2}}\cos\beta \\ & + \frac{\alpha^4}{3^{2n+2}} = \phi(p,|x|) \end{aligned} \tag{3.10}$$

Differentiating $\phi(p,|x|)$ w.r.t $|x|$ and using elementary calculus shows that $\partial\phi(p,|x|/\partial|x|) > 0$ for $|x| \in [0,1]$ and fixed $p \in [0,2]$.This follows that $\phi(p,|x|)$ is an increasing function of $|x|$. So, $\phi(p,|x|) \leq \phi(p,1)$.

$$\begin{aligned}\left|a_4^2 - a_3^2\right| \leq & \frac{p^6}{2^{4n+8}}\cos^2\beta + \frac{3p^4(4-p^2)}{2^{4n+7}}\cos^2\beta + \frac{5p^2(4-p^2)^2}{2^{4n+8}}\cos^2\beta + \frac{p^4\alpha^2}{2^{4n+5}}\cos\beta \\ & + \frac{3p^2(4-p^2)\alpha^2}{2^{4n+5}}\cos\beta + \frac{p^2\alpha^4}{2^{4n+4}}\cos^2\beta + \frac{p^4}{2^2\cdot 3^{2n+2}}\cos^2\beta + \frac{p^2(4-p^2)}{2\cdot 3^{2n+2}}\cos^2\beta \\ & + \frac{(4-p^2)^2}{2^2\cdot 3^{2n+2}}\cos^2\beta + \frac{p^2\alpha^2}{3^{2n+2}}\cos\beta + \frac{(4-p^2)\alpha^2}{3^{2n+2}}\cos\beta + \frac{\alpha^4}{3^{2n+2}} \end{aligned} \tag{3.11}$$

(3.11) has maximum value

$$\frac{1+\alpha^4}{2^{4n+2}}\cos^2\beta + \frac{\alpha^2}{2^{4n+1}}\cos\beta + \frac{1}{3^{2n+2}}\left(4\cos^2\beta + 4\alpha^2\cos\beta + \alpha^4\right)$$

on $[0,,2]$ when $p = 2$.
Remark 3: When $n = \alpha = \beta = 0$,

$$|T_2(3)| = \left|a_4^2 - a_3^2\right| \leq \frac{25}{36}$$

. which is much finer than $|a_4^2 - a_3^2| \leq 7$ that was obtained in [6] for the class of function with real positive part. Let $f(z) \in R_n(\alpha,\beta), \alpha \in [0,1], \quad \beta \in \left(\frac{-\pi}{2}\ ,\frac{\pi}{2}\right)$ and $n \in \mathbb{N}_0$. Then

$$\begin{aligned}|T_3(2)| = & \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right| \\ & \leq \left[\frac{1+\alpha^2}{2^{2n+1}}\cos\beta + \frac{1}{2^n}\cos\beta\right]\left[\frac{1}{2^{2n}}\cos^2\beta + \frac{(\alpha^2-1)}{2^{3n+1}}\cos^2\beta\right. \\ & \qquad \left. + \frac{8}{3^{2n+2}}\cos^2\beta - \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{2\alpha^4}{3^{2n+2}}\right]\end{aligned}$$

Proof.

$$|T_3(2)| = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right| \leq |a_2 - a_4|\left|a_2^2 - 2a_3^2 + a_2a_4\right|$$

Now, using (3.2), (3.4) and lemma (2.2)

$$\begin{aligned}|a_2 - a_4| = & \left|\frac{p_1e^{-i\beta}}{2^{n+1}}\cos\beta - \frac{p_1^3e^{-i\beta}}{2^{2n+4}}\cos\beta - \frac{p_1(4-p_1^2)xe^{-i\beta}}{2^{2n+3}}\cos\beta\right. \\ & \left. + \frac{p_1(4-p_1^2)x^2e^{-i\beta}}{2^{2n+4}}\cos\beta - \frac{(4-p_1^2)(1-|x|^2)ze^{-i\beta}}{2^{2n+3}}\cos\beta - \frac{p_1\alpha^2e^{-3i\beta}}{2^{2n+2}}\cos\beta\right|\end{aligned}$$

Using triangle inequality with $p_1 = p$

$$|a_2 - a_4| \leq \frac{p^3}{2^{2n+4}}\cos\beta + \frac{p}{2^{n+1}}\cos\beta + \frac{p|x|X}{2^{2n+3}}\cos\beta + \frac{p|x|^2X}{2^{2n+4}}\cos\beta + \frac{XY}{2^{2n+3}}\cos\beta + \frac{p\alpha^2}{2^{2n+2}}\cos\beta$$

Using the same techniques as in theorem (3.1) and (3.2)

$$|a_2 - a_4| \leq \frac{1+\alpha^2}{2^{2n+1}}\cos\beta + \frac{1}{2^n}\cos\beta$$

From (3.2),(3.3) and (3.4) we get

$$\left|(a_2^2 - 2a_3^2 + a_2a_4)\right| = \left|\frac{p_1^2e^{-2i\beta}}{2^{2n+2}}\cos^2\beta - \frac{2p_2^2e^{-2i\beta}}{3^{2n+2}}\cos^2\beta - \frac{4p_2\alpha^2e^{-3i\beta}}{3^{2n+2}}\cos\beta - \frac{2\alpha^4e^{-4i\beta}}{3^{2n+2}} + \frac{p_1p_3e^{-2i\beta}}{2^{3n+3}}\cos^2\beta + \frac{p_1^2\alpha^2e^{-4i\beta}}{2^{3n+3}}\cos^2\beta\right|$$

Letting $p_1 = p \in [0,2]$ and applying triangle inequality and simplifying, we obtain

$$\begin{aligned}\left|(a_2^2 - 2a_3^2 + a_2a_4)\right| \leq{}& \left[\frac{\cos^2\beta}{2^{2n+2}} - \frac{-2\alpha^2\cos\beta}{3^{2n+2}} + \frac{\alpha^2\cos^2\beta}{2^{3n+3}}\right]p^2 + \left[\frac{\cos^2\beta}{2\cdot 3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+5}}\right]p^4\\ &+ \left[\frac{\cos^2\beta}{3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+4}}\right](4-p^2)|x|p^2 + \frac{2\alpha^2(4-p^2)|x|}{3^{2n+2}}\cos\beta\\ &+ \frac{p^2(4-p^2)|x|^2}{2^{3n+5}}\cos^2\beta + \frac{(4-p^2)^2|x|^2}{2\cdot 3^{2n+2}}\cos^2\beta\\ &+ \frac{p(4-p^2)(1-|x|^2)}{2^{3n+4}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}} = \gamma(p,|x|)\end{aligned}$$

Now, we need to obtain the maximum value of $\gamma(p,|x|)$ on the closed region $[0,2]X[0,1]$. Let us assume that there is a maximum at an interior point $(p_0,|x|)$ of $[0,2]X[0,1]$. Then, differentiating $\gamma(p_0,|x|)$ w.r.t $|x|$ we get

$$\begin{aligned}\frac{\partial\gamma}{\partial|x|} ={}& \left[\frac{\cos^2\beta}{3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+4}}\right](4-p^2)p^2 + \frac{2\alpha^2(4-p^2)}{3^{2n+2}} + \frac{p^2(4-p^2)|x|}{2^{3n+4}}\cos^2\beta\\ &+ \frac{(4-p^2)^2|x|}{3^{2n+2}}\cos^2\beta - \frac{p(4-p^2)|x|}{2^{3n+3}}\end{aligned} \tag{3.12}$$

equating (3.12) to zero imply that $p = 2$ which is a contradiction. Thus, we need to consider only the endpoints of $[0,2]X[0,1]$ in other to obtain the maximum of $\gamma(p,|x|)$.

When $p = 0$,

$$\gamma(0,|x|) = \frac{8\alpha^2|x|}{3^{2n+2}}\cos\beta + \frac{16|x|^2}{2\cdot 3^{2n+2}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}} \leq \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{8}{3^{2n+2}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}$$

When $p = 2$,

$$\gamma(2,|x|) = \frac{4}{2^{2n+2}}\cos^2\beta - \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{4\alpha^2}{2^{3n+3}}\cos^2\beta + \frac{8}{3^{2n+2}}\cos^2\beta - \frac{1}{2^{3n+1}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}$$

When $|x| = 0$,

$$\begin{aligned}\gamma(p,0) ={}& \left[\frac{\cos^2\beta}{2^{2n+2}} - \frac{2\alpha^2\cos\beta}{3^{2n+2}} + \frac{\alpha^2\cos^2\beta}{2^{3n+3}}\right]p^2 + \left[\frac{\cos^2\beta}{2\cdot 3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+5}}\right]p^4\\ &+ \frac{p(4-p^2)}{2^{3n+4}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}\end{aligned}$$

which has a maximum value

$$\frac{1}{2^{2n}}\cos^2\beta - \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{\alpha^2}{2^{3n+1}}\cos^2\beta - \frac{1}{2^{3n+1}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}$$

on $[0,2]$. Also, when $|x| = 1$

$$\begin{aligned}\gamma(p,1) = &\left[\frac{\cos^2\beta}{2^{2n+2}} - \frac{2\alpha^2\cos\beta}{3^{2n+2}} + \frac{\alpha^2\cos^2\beta}{2^{3n+3}}\right]p^2 + \left[\frac{\cos^2\beta}{2\cdot 3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+5}}\right]p^4\\ &+ \left[\frac{\cos^2\beta}{3^{2n+2}} - \frac{\cos^2\beta}{2^{3n+4}}\right]p^2(4-p^2) + \frac{2\alpha^2(4-p^2)}{3^{2n+2}}\cos\beta + \frac{p^2(4-p^2)}{2^{3n+5}}\cos^2\beta\\ &+ \frac{(4-p^2)^2}{2\cdot 3^{2n+2}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}\end{aligned}$$

which has a maximum value

$$\frac{1}{2^{2n}}\cos^2\beta - \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{\alpha^2}{2^{3n+1}} + \frac{8}{3^{2n+2}}\cos^2\beta - \frac{1}{2^{3n+1}}\cos^2\beta + \frac{2\alpha^4}{3^{2n+2}}$$

on $[0,2]$. Hence,

$$\begin{aligned}|T_3(2)| &= \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right|\\ &\leq \left[\frac{1+\alpha^2}{2^{2n+1}}\cos\beta + \frac{1}{2^n}\cos\beta\right]\left[\frac{1}{2^{2n}}\cos^2\beta + \frac{(\alpha^2-1)}{2^{3n+1}}\cos^2\beta\right.\\ &\qquad\left. + \frac{8}{3^{2n+2}}\cos^2\beta - \frac{8\alpha^2}{3^{2n+2}}\cos\beta + \frac{2\alpha^4}{3^{2n+2}}\right]\end{aligned}$$

Remark 4 For $n = \alpha = \beta = 0$,

$$|T_3(2)| = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right| \leq \frac{25}{12}$$

which is also finer than $\left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right| \leq 12$ obtained in [6] for functions with real positive part. Let $f(z) \in R_n(\alpha,\beta), \alpha \in [0,1], \quad \beta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $n \in \mathbb{N}_0$. Then

$$|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \leq 1 + \frac{(4-2\alpha^2)}{2^{2n}\cdot 3^{n+1}}\cos^2\beta + \frac{2}{2^{2n}}\cos^2\beta + \frac{(4\alpha^2\cos\beta - 4\cos^2\beta + \alpha^4)}{3^{2n+2}}$$

Proof. By using (3.2), (3.3) , lemma (2.2) and simplifying, we get

$$\begin{aligned}T_3(1) = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| = &\left|1 + \left[\frac{e^{-3i\beta}\cos^2\beta}{2^{2n+2}\cdot 3^{n+1}} - \frac{e^{-2i\beta}\cos^2\beta}{2^2\cdot 3^{2n+2}}\right]p_1^4\right.\\ &- \left[\frac{e^{-2i\beta}\cos^2\beta}{2^{2n+1}} + \frac{\alpha^2e^{-i\beta}\cos\beta}{3^{2n+2}} - \frac{\alpha^2e^{-4i\beta}\cos^2\beta}{2^{2n+1}\cdot 3^{n+1}}\right]p_1^2\\ &\left.- \left[\frac{e^{-i\beta}\cos^2\beta}{2\cdot 3^{2n+2}} - \frac{e^{-3i\beta}\cos^2\beta}{2^{2n+2}\cdot 3^{n+1}}\right]xXp_1^2 + \frac{e^{-3i\beta}\cos^2\beta}{2^2\cdot 3^{2n+2}}x^2X^2 - \frac{\alpha^2e^{-i\beta}\cos\beta}{3^{2n+2}}xX - \frac{\alpha^4e^{-4i\beta}}{3^{2n+2}}\right|\end{aligned}$$

Assuming without loss of generality that $p_1 = p \in [0,2]$ and $X = (4-p^2)$. Also, using the triangle inequality and the fact that $|x| = 1$, we get

$$\begin{aligned}|T_3(1)| \leq 1 &+ \left[\frac{\cos^2\beta}{2^{2n+2}\cdot 3^{n+1}} - \frac{\cos^2\beta}{2^2\cdot 3^{2n+2}}\right]p^4 + \left[\frac{\cos^2\beta}{2^{2n+1}} + \frac{\alpha^2\cos\beta}{3^{2n+2}} - \frac{\alpha^2\cos^2\beta}{2^{2n+1}\cdot 3^{n+1}}\right]p^2\\ &+ \left[\frac{\cos^2\beta}{2\cdot 3^{n+2}} - \frac{1}{2^{2n+2}\cdot 3^{n+1}}\right](4-p^2)p^2 + \frac{\cos^2\beta(4-p^2)^2}{2^2\cdot 3^{2n+2}}\\ &+ \frac{\alpha^2\cos\beta}{3^{2n+2}}(4-p^2) + \frac{\alpha^4}{3^{2n+2}}\end{aligned} \tag{3.13}$$

The RHS of (3.13) has a maximum at $p = 2$, therefore,

$$|T_3(1)| \leq 1 + \frac{(4-2\alpha^2)}{2^{2n} \cdot 3^{n+1}} \cos^2\beta + \frac{2}{2^{2n}} \cos^2\beta + \frac{(4\alpha^2 \cos\beta - 4\cos^2\beta + \alpha^4)}{3^{2n+2}}$$

Remark 5 For $n = \alpha = \beta = 0$

$$T_3(1) = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \leq \frac{35}{9}$$

which is finer than the result in [6].

4 Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

5 Acknowledgements

The authors wish to thank the referees for their valuable suggestions that lead to improvement of the quality of the work in this paper.

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