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Distributed Control for Non-Cooperative Systems Under Conjugation Conditions

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Abstract

In this paper, the distributed control for non-cooperative elliptic systems under conjugation conditions is established. First, the existence and uniqueness of the state for these systems with Dirichlet and conjugation conditions is proved, then the set of equations and inequalities that characterizes the distributed control of these systems is found. The non-cooperative Neumann systems with conjugation conditions is also discussed.

Keywords

Non cooperative elliptic systems - Conjugation conditions - Dirichlet and Neumann conditions - Existence and uniqueness of solutions - Distributed control

1. Introduction

The necessary and sufficient conditions of optimality for systems governed by partial differential equations have been studied by Lions [11]. The control problems described by either infinite order operators or operators with an infinite number of variables were established by Gali et. al. [3, 4, 5]. These results have been extended to cooperative systems [1, 2, 6, 13, 16] or non-cooperative systems [10, 17].

Serag et. al. discussed the optimal control for systems involving schrodinger operators [12, 14]. The existence results have been proved for some non linear systems in [8, 9, 14, 15]. Some applications for control problems have been introduced for example in [7, 10].

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New optimal control problems of distributed systems described by an elliptic, parabolic and hyperbolic operators with conjugation conditions and by a quadratic cost functional have been studied by Sergienko and Deineka [18-20].

In the present work, using the theory of Lions [11], Sergienko and Deineka [18-20], the distributed control for $n \times n$ non cooperative Dirichlet elliptic systems is discussed . First the existence and uniqueness of the state for these systems is proved, then the set of equations and inequalities that characterizes the distributed control of these systems is found. The optimal control of distributed type for non-cooperative Neumann problems under conjugation conditions is also studied.

2 Distributed control for non-cooperative Dirichlet elliptic systems under conjugation conditions

In this section, we study the distributed control for the following $n \times n$ non cooperative Dirichlet elliptic systems:

$$\begin{cases} -\Delta h_i + \sum_{j=1}^{n} a_{ij} h_j = f_i & \text{in} \quad \Omega, \\ h_i = 0 & \text{on} \quad \Gamma, \quad i = 1, 2, ..., n, \end{cases} \tag{1}$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^+ = [h_i] + \delta & \text{on} \quad \gamma, \\ \left[\frac{\partial h_i}{\partial v_A} \right] = \left[\sum_{i,j=1}^{n} \frac{\partial h_i}{\partial x_j} \cos(v, x_i) \right] = w_i & \text{on} \quad \gamma, \quad i = 1, 2, ...n, \end{cases} \tag{2}$$

where Ω_1 and Ω_2, with boundary $\partial\Omega_1$ and $\partial\Omega_2$ respectively, are bounded, continuous and strictly Lipchitz domains from R^n such that :

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \phi,$$

$$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)/\gamma, \text{is boundary of } \Omega, \gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi, \gamma = \gamma^+ \cup \gamma^-$$

$$\partial\Omega_1 \cap \gamma = \gamma^+, \partial\Omega_2 \cap \gamma = \gamma^-, \quad f_i \in L^2(\Omega), (i = 1, 2, ..., n),$$
$$R_1, R_2, w, \delta \in C(\gamma) \ , R_1, R_2 \geq 0, R_1 + R_2 \geq R_0 > 0, \quad R_0 = \text{constant}, \tag{3}$$

$\vec{n}$ is an ort of an outer normal to γ, $[\varphi] = \varphi^+ - \varphi^-$,

$$\varphi^+ = \{\varphi\}^+ = \varphi(x) \quad \text{for } x \in \gamma^+$$
$$\varphi^- = \{\varphi\}^- = \varphi(x) \quad \text{for } x \in \gamma^-.$$

System (1) is called cooperative system if $a_{ij} > 0 \quad \forall \ i \neq j$, otherwise is called non-cooperative system.

In our work, we assume

$$a_{ij} = \begin{cases} 1 & if \quad i \geq j, \\ -1 & if \quad i < j, \qquad i, j = 1, 2, ..., n. \end{cases}$$

(i.e. non-cooperative systems).

We first prove the existence of the state for system (1) under conjugation conditions (2). Then, we discus the existence of distributed control for this system; and we find the set of equations and inequalities that characterizes this distributed control.

Existence and uniqueness of the state

Since

$$H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega),$$

then by Cartesian product, we have chain of the form

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H^{-1}(\Omega))^n.$$

On $(H_0^1(\Omega))^n$, we introduce the bilinear form:

$$\begin{aligned} a(h,\psi) =& \sum_{i=1}^{n} \int_{\Omega} \nabla h_i \nabla \psi_i dx + \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} h_i \psi_i dx \\ &+ \sum_{i=1}^{n} \int_{\gamma} \frac{[h_i][\psi_i]}{R_1 + R_2} d\gamma. \end{aligned} \tag{4}$$

It is easy to check that

$$|a(h,\psi)| \leq k_1 \|h\| \|\psi\|. \tag{5}$$

The bilinear form (4) is coercive on $(H_0^1(\Omega))^n$; that is, there exists a positive constant C such that

$$a(h,h) \geq C \|h\|^2_{(H_0^1(\Omega))^n} \qquad \forall h = \{h_1, h_2, ..., h_n\} \in (H_0^1(\Omega))^n \tag{6}$$

Proof.

$$\begin{aligned} a(h,h) =& \sum_{i=1}^{n} \int_{\Omega} |\nabla h_i|^2 \, dx + \sum_{i=1}^{n} \int_{\Omega} |h_i|^2 \, dx + \sum_{i=1}^{n} \int_{\gamma} \frac{[h_i]^2}{R_1 + R_2} d\gamma \\ =& \sum_{i=1}^{n} \int_{\Omega} (|\nabla h_i|^2 + |h_i|^2) dx + \sum_{i=1}^{n} \int_{\gamma} \frac{[h_i]^2}{R_1 + R_2} d\gamma \end{aligned}$$

(3), implies

$$a(h,h) \geq C \sum_{i=1}^{n} \int_{\Omega} (|\nabla h_i|^2 + |h_i|^2) dx,$$

therefore

$$\begin{aligned} a(h,h) \geq& C \sum_{i=1}^{n} \|h_i\|^2_{H_0^1(\Omega)} \\ =& C \|h\|^2_{(H_0^1(\Omega))^n}, \end{aligned}$$

□

which proves the coerciveness condition of the bilinear form (4) on $(H_0^1(\Omega))^n$.
Now, let

$$L(\psi) = \sum_{i=1}^{n} \int_{\Omega} f_i(x)\psi_i(x)dx + \sum_{i=1}^{n} \int_{\gamma} \frac{(R_2 w - \delta)\psi_i}{R_1 + R_2} d\gamma - \sum_{i=1}^{n} \int_{\gamma} w\psi_i^+ d\gamma \tag{7}$$

be a linear form on $(H_0^1(\Omega))^n$, this linear form is continuous, since :

$$|L(\psi)| \leq K\|\psi\|_{(H_0^1(\Omega))^n} \qquad \forall \psi \in (H_0^1(\Omega))^n, \text{K is a constant.} \tag{8}$$

Then using Lax Milgram lemma, there exists a unique solution $h \in (H_0^1(\Omega))^n$ such that:

$$a(h, \psi) = L(\psi) \qquad \forall \psi = (\psi_i)_{i=1}^n \in (H_0^1(\Omega))^n. \tag{9}$$

Then, we have proved the following theorem

For a given $f = \{\mathrm{f}_i\}_{i=1}^n \in (L^2(\Omega))^n$ there exists a unique solution $h = \{h_i\}_{i=1}^n \in (H_0^1(\Omega))^n$ for non-cooperative Dirichlet system (1) with conjugation conditions (2)

Formulation of the control problem

The space $U = (L^2(\Omega))^n$ is the space of controls. For a control $u = \{u_1, u_2, ..., u_n\} \in (L^2(\Omega))^n$, the state $h(u) = \{h_1(u), h_2(u), ..., h_n(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^n a_{ij} h_j(u) = f_i(u) + u_i & \text{in} \quad \Omega, \\ h_i(u) = 0 & \text{on} \quad \Gamma, \quad i = 1, 2, ..., n, \end{cases} \tag{10}$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{\frac{\partial h_i(u)}{\partial v_A}\right\}^- + R_2 \left\{\frac{\partial h_i(u)}{\partial v_A}\right\}^+ = [h_i(u)] + \delta & \text{on} \quad \gamma, \\ \left[\frac{\partial h_i(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^n \frac{\partial h_i(u)}{\partial x_j} \cos(v, x_i)\right] = w_i & \text{on} \quad \gamma, \quad i = 1, 2, ...n. \end{cases} \tag{11}$$

Specify the observation equation by

$$z(u) = \{z_1(u), z_2(u), ..., z_n(u)\} = Ch(u) = C\{h_1(u), h_2(u), ..., h_n(u)\}$$

$$= \{h_1(u), h_2(u), ..., h_n(u)\}.$$

For a given $z_d = \{z_{1d}, z_{2d}, ..., z_{nd}\} \in (L^2(\Omega))^n$,the cost functional is given by

$$J(v) = \sum_{i=1}^{n} \|h_i(v) - z_{id}\|^2_{L^2(\Omega)} + (Nv, v)_{(L^2(\Omega))^n}, \tag{12}$$

where N is a hermitian positive definite operator such that :

$$(Nv, v)_{(L^2(\Omega))^n} \geq M\|v\|^2_{(L^2(\Omega))^n}, \qquad M > 0, \quad \forall\, v \in U. \tag{13}$$

The control problem then is to find :

$$\begin{cases} u = \{u_1, u_2, ..., u_n\} \in U_{ad} & \text{such that:} \\ J(u) = inf\ J(v) \qquad \forall v \in U_{ad}, \end{cases} \tag{14}$$

where U_{ad} is a closed convex subset of $(L^2(\Omega))^n$.
The cost functional (12)can be written as

$$J(v) = \pi(v,v) - 2H(v) + \sum_{i=1}^{n} \|z_{id} - h_i(0))\|^2_{L^2(\Omega)},$$

where

$$\pi(u,v) = \sum_{i=1}^{n} (h_i(u) - h_i(0), h_i(v) - h_i(0))_{L^2(\Omega)} + (Nv, v)_{(L^2(\Omega))^n}, \tag{15}$$

is a continuous bilinear form and from (13), it is coercive, that is:

$$\pi(v,v) \geq N\|v\|^2_{(L^2(\Omega))^n} \; and$$

$$H(v) = \sum_{i=1}^{n} (z_{id} - h_i(0), h_i(v) - h_i(0))_{L^2(\Omega)}, \tag{16}$$

is a continuous linear form on $(L^2(\Omega))^n$. Then, using the theory of Lions [11], there exists a unique optimal control of problem (14); Moreover it is characterized by Let us suppose that (6) holds and the cost functional is given by (12), then the distributed control u is characterized by

$$\begin{cases} -\Delta p_i(u) + \sum_{j=1}^{n} a_{ij} p_i(u) = h_i(u) - z_{id} & \text{in} \quad \Omega, \\ p_i(u) = 0 & \text{on} \quad \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_{A^*}}\right] = 0 & \text{on} \quad \gamma, \\ \left\{\frac{\partial p_i(u)}{\partial v_{A^*}}\right\}^{\pm} = \frac{1}{R_1+R_2}[p_i(u)] & \text{on} \quad \gamma, \\ \sum_{i=1}^{n} (p_i(u), v_i - u_i) + (Nu, v-u)_{(L^2(\Omega))^n} \geq 0 & \\ & , i = 1, 2, ..., n, \end{cases} \tag{17}$$

together with(10),where $p(u) = \{p_1(u), p_2(u), ..., p_n(u)\}$ is the adjoint state.

Proof. The optimal control $u = \{u_i\}_{i=1}^n \in (L^2(\Omega))^n$ is characterized by [11] :

$$\pi(u, v-u) \geq H(v-u) \qquad \forall \; v = \{v_1, v_2, ..., v_n\} \in U_{ad}.$$

From (15), and (16):

$$\begin{aligned} \pi(u, v-u) - H(v-u) &= \sum_{i=1}^{n} (h_i(u) - z_{id}, h_i(v) - h_i(u))_{L^2(\Omega)} \\ &+ \sum_{i=1}^{n} (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0. \end{aligned} \tag{18}$$

Since the model A of the system is given by

$$Ah(u) = A(h_1(u), h_2(u), ..., h_n(u)) = \sum_{i=1}^{n} (-\Delta h_i(u) + \sum_{j=1}^{n} a_{ij} h_j(u)),$$

and since

$$(p(u), Ah(u)) = (A^* p(u), h(u)),$$

then

$$(p(u), Ah(u))_{(L^2(\Omega))^n} = \sum_{i=1}^{n}(p_i(u), -\Delta h_i(u) + \sum_{j=1}^{n} a_{ij}h_j(u))$$
$$= \sum_{i=1}^{n}(-\Delta p_i(u) + \sum_{j=1}^{n} a_{ji}p_j(u), h_i(u))_{L^2(\Omega)},$$

hence (18) is equivalent to

$$\sum_{i=1}^{n}(-\Delta p_i(u) - \sum_{j=1}^{n} a_{ji}p_j(u), h_i(v-u) - h_i(0))_{L^2(\Omega)}$$
$$+ \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.$$

Therefore

$$\sum_{i=1}^{n}(A^*p_i(u), h_i(v-u) - h_i(0))_{L^2(\Omega)}$$
$$+ \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.$$

Applying Green's formula, we obtain

$$\sum_{i=1}^{n}(p_i(u), Ah_i(v-u) - Ah_i(0))_{L^2(\Omega)}$$
$$+ \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.$$

Using equation (10), we get

$$\sum_{i=1}^{n}(p_i(u), v_i - u_i) + \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0,$$

i.e

$$\sum_{i=1}^{n}\int_{\Omega}(p_i(u) + Nu_i)(v_i - u_i)dx \geq 0.$$

□

3 Distributed control for non-cooperative Neumann elliptic systems with conjugation conditions

In this section, we consider the following non-cooperative Neumann elliptic system

$$\begin{cases} -\Delta h_i + \sum_{j=1}^{n} a_{ij}h_j = f_i & \text{in} \quad \Omega, \\ \frac{\partial h_i}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \tag{19}$$

with conjugation conditions (2), where $g = \{g_1, g_2, ..., g_n\} \in (L^2(\Gamma))^n$ is given function. We introduce again the bilinear form (4) which is coercive on $(H^1(\Omega))^n$, since

$$((H_0^1(\Omega))^n) \subseteq ((H^1(\Omega))^n).$$

Then based on (6), (8) and Lax-Milgram lemma, there exists a unique solution h for system (19) such that :

$$a(h, \psi) = L_N(\psi), \quad \forall \psi \in (H^1(\Omega))^n,$$

where

$$\begin{aligned} L_N(\psi) = & \sum_{i=1}^{n} \int_\Omega f_i(x)\psi_i(x)dx + \sum_{i=1}^{n} \int_\Gamma g_i(x)\psi_i(x)d\Gamma \\ & + \sum_{i=1}^{n} \int_\gamma \frac{(R_2 w - \delta)\psi_i}{R_1 + R_2}\, d\gamma - \sum_{i=1}^{n} \int_\gamma w\psi_i^+ \, d\gamma, \end{aligned}$$

is a continuous linear form defined on $(H^1(\Omega))^n$.
Let us multiply both sides of first equation of (19) by $\psi \in (H^1(\Omega))^n$ and integrate over Ω , we obtain

$$\sum_{i=1}^{n} \int_\Omega (-\Delta h_i + \sum_{j=1}^{n} a_{ij} h_j)\psi_i(x)dx = \sum_{i=1}^{n} \int_\Omega f_i \psi_i dx.$$

Applying Green's formula,

$$\begin{aligned} & \sum_{i=1}^{n} \int_\Omega (-\Delta h_i + \sum_{j=1}^{n} a_{ij} h_j)\psi_i(x)dx + \sum_{i=1}^{n} \int_\Gamma (\frac{\partial h_i}{\partial \nu_A})\psi_i(x)d\Gamma + \\ & \sum_{i=1}^{n} \int_\gamma (\frac{\partial h_i}{\partial \nu_A})\psi_i(x)d\gamma + a(h, \psi) = \sum_{i=1}^{n} \int_\Omega f_i \psi_i dx. \end{aligned}$$

Then, from

$$a(h, \psi) = L_N(\psi) \ ,$$

we deduce the Neumann conditions

$$\frac{\partial h_i}{\partial \nu_A} = g_i, \qquad on\ \Gamma.$$

So we can formulate the corresponding control problem:
The space $U = (L^2(\Omega))^n$ is the space of controls. For a control $u = \{u_1, u_2, ..., u_n\} \in (L^2(\Omega))^n$, the state $h(u) = \{h_1(u), h_2(u), ..., h_n(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^{n} a_{ij} h_j(u) = f_i(u) + u_i & \text{in} \quad \Omega, \\ \frac{\partial h_i(u)}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \tag{20}$$

under conjugation conditions (11). For a given $z_d = \{z_{1d}, z_{2d}, ..., z_{nd}\} \in (L^2(\Omega))^n$, the cost functional is again given by (12), then there exists a unique optimal control $u \in U_{ad}$ such that:

$$\begin{cases} u = \{u_1, u_2, ..., u_n\} \in U_{ad} & \text{such that:} \\ J(u) = inf\ J(v) & \forall v \in U_{ad}. \end{cases}$$

Moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_i(u) + \sum_{j=1}^{n} a_{ij} p_i(u) = h_i(u) - z_{id} & \text{in} \quad \Omega, \\ \frac{\partial p_i(u)}{\partial v_A*} = 0 & \text{on} \quad \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_A*}\right] = 0 & \text{on} \quad \gamma, \\ \left\{\frac{\partial p_i(u)}{\partial v_A*}\right\}^{\pm} = \frac{1}{R_1+R_2}[p_i(u)] & \text{on} \quad \gamma \\ \sum_{i=1}^{n}(p_i(u), v_i - u_i) + (Nu, v-u)_{(L^2(\Omega))^n} \geq 0 & \\ & , i = 1, 2, ..., n, \end{cases}$$

together with(20), where $p(u) = \{p_1(u), p_2(u), ..., p_n(u)\}$ is the adjoint state.

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