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On Some Properties of a Class of Analytic Functions Defined by Salagean Differential Operator

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Abstract

In this work,the upper bounds for Fekete-Szego functional and Second Hankel Determinant are obtained for a class of analytic functions defined by Salagean Differential Operator.The estimates obtained are sharp.

Keywords:

Analytic functions, Salagean Differential Operator, Starlike functions, Univalent functions, Coefficient bounds, Fekete-Szego functional, Second Hankel Determinant and Subordination.

1. Introduction and definitions

Let A denote the class of functions $f(z)$ analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let S be a subclass of A consisting of functions $f(z)$ univalent in U and have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

A function $f(z)$ belonging to the class S is called starlike function if $f(z)$ maps the unit disk U onto a starlike domain. A necessary and sufficient conditions for a $f(z) \in S$ to be starlike with respect to the origin is that;

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in U)) \tag{2}$$

The class of starlike functions is denoted as S^*

A function $f(z) \in S$ is said to be starlike of order α , $0 \leq \alpha < 1$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U)) \tag{3}$$

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The class of starlike functions of order α is denoted as $S^*(\alpha)$

A function $f(z) \in S$ is called a Convex function if $f(z)$ maps the unit dist U onto a convex domain. A function $f(z)$ is a Convex function if and only if

$$\Re\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}>0 \qquad (z \in U)) \tag{4}$$

The class of convex functions is denoted as K.
A function $f(z) \in S$ is said to be Convex of order α, $0 \leq \alpha < 1$ if

$$Re\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}>\alpha \qquad (z \in U)) \tag{5}$$

The class of all Convex functions of order α is denoted as $K(\alpha)$.

Let f and g be analytic in U. Then $f(z)$ is subordinate to $g(z)$ denoted by $f \prec g$ if there exist an analytic function $\omega(z)$ with $\omega(0) = 0$, $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. In particular if $g(z)$ is univalent in U then $f \prec g \Longleftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

The q^{th} Hankel determinant $H_q(n)$,$q \geq 1$,$n \geq 1$ for a function $f(z) \in A$ and have the form (1)is defined as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

In recent years, attention has been given to finding estimates of the determinant $H_q(n)$. The Fekete-Szego functional $|a_3 - \lambda a_2^2|$ is $H_2(1)$.For $f(z) \in S$ it is known that $H_2(1) \leq 1$. see [1], The Second Hankel Determinant $H_2(2) = |a_2a_4 - a_3^2|$ has received more attention from many authors.The sharp upper bound of $H_2(2)$ for starlike and convex functions was studied in [2] and the authors obtained $H_2(2) \leq 1$ and $H_2(2) \leq \frac{1}{8}$ respectively. Many other results have been obtained for $H_2(2)$ for a variety of subclass of S,most of which are subclass of S^*.
Noticing that several subclass of univalent functions are characterized by the quantities $\frac{zf'(z)}{f(z)}$ or $\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}$ lying in a region in the right half plane, Ma and Minda [3] considered the classes.

$$ST(\phi)=\left\{f \in A : \frac{zf'(z)}{f(z)}\right\} \tag{6}$$

$$CV(\phi)=\left\{f \in A : \frac{1+zf''(z)}{f\prime(z)}\right\} \tag{7}$$

For $\phi(z) = \frac{1+Az}{1+Bz}, \quad -1 \leq B < A < 1$,
The class $ST(\phi)$ reduces to the familiar class consisting of Janowski starlike functions denoted by $ST(A, B)$.The corresponding class of convex functions is denoted by $CV(A, B)$.
The special case of $A = 1 - 2\alpha, \quad B = -1, \quad 0 \leq \alpha < 1$,$ST(\phi)$ and $CV(\phi)$gives the classes of starlike functions of orderα and convex functions of order α respectively. Let SC be the class of functions $f \in S$ with the quantity $\frac{zf'(z)}{f(z)}$ lying in the region bounded by the cardiod given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$ Thus a function $f(z) \in SC$ if $\frac{zf'(z)}{f(z)} \in CAR$ where,

$$CAR = \omega \in \mathbb{C} : \omega = 1 + \frac{4}{3}z + \frac{2}{3}z^2, (|z| < 1) \tag{8}$$

The class SC was investigated by Kanika [4]
We say that $f \in S$ belongs to the class S_nC if

$$\frac{D^{n+1}f(z)}{D^nf(z)} \prec \phi(z) \tag{9}$$

where D^n is the Salagean Differential Operator, $n \in N \cup 0$ and

$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \quad z \in U \tag{10}$$

when $n = 0$, the class S_nC reduces to the class SC. In this paper, we obtain the initial coefficient estimates a_2, a_3 and a_4 for functions belonging to the class S_nC.The Upper bounds for the Fekete-Szego functional and the second Hankel Determinant for functions belonging to the class S_nC are also established. Furthermore,when $n = 0$ and $\phi(z) = \sqrt{(}1 + z^2) + z$, the class S_nC becomes e class $S^*(q)$ sudied in [1]

2 Main Result

Let Ω be the class of analytic functions of the form

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \tag{11}$$

such that $|c_k| \leq 1, \quad k = 1, 2, 3, \ldots$
(see[])

Lemma 2.1[5]

If $\omega \in \Omega$, then for any $t \in \mathbb{R}$
$$|c_2 - tc_1^2| \left\{ \begin{array}{c} -t \quad if \quad t < -1 \\ 1 \quad if \quad -1 \leq t \leq 1 \\ t \quad if \quad t > 1 \end{array} \right\}$$

Lemma 2.2[5]

If $\omega \in \Omega$, for any complex number t
$|c_2 - tc_1^2| \leq max\{1 : |t|\}$
The result is sharp for $\omega(z) = z^2$ or $\omega(z) = z$

Lemma 2.3[5]

If $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \therefore \in \Omega$,
then $|c_1^3 + c_3 + 2c_1 c_2| \leq 1$

Theorem 3.1

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S_nC$ then

$$|a_2| \leq \frac{4}{3 \cdot 2^n}, \qquad |a_3| \leq \frac{11}{3^2 \cdot 3^n}, \qquad |a_4| \leq \frac{68}{3^3 \cdot 3 \cdot 4^n}$$

Proof:
Since $f \in S_nC$,
We have that,

$$\frac{D^{n+1}}{D^n f(z)} = \phi(\omega(z))$$

where $\phi(z)$ is given as (10),
Thus

$$\frac{D^{n+1} f(z)}{D^n f(z)} = 1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2 \tag{12}$$

Let $\omega = c_1 z + c_2 z^2 + c_3 z^3 + ... \in \Omega$
Then from (12) we obtain

$$D^{n+1}f(z) = D^n f(z)[1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2]$$

Therefore,

$$D^{n+1}f(z) = z + 2^{n+1}a_2 z^2 + 3^{n+1}a_3 z^3 + 4^{n+1}a_4 z^4 + ... \tag{13}$$

and

$$D^n f(z)[1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2]$$
$$= z+[\frac{4}{3}c_1+2^n a_2]z^2+[\frac{4}{3}c_2+\frac{2}{3}c_1^2+2^n a_2\frac{4}{3}c_1+3^n a_3]z^3+[\frac{4}{3}c_1c_2+2^n a_2\cdot\frac{4}{3}c_2+\frac{2}{3}2^n a_2 c_1^2+3^n a_3\frac{4}{3}c_1+4^n a_4]z^4 \tag{14}$$

Comparing coefficiens in (13)and (14) we have

$$2^{n+1}a_2 = \frac{4}{3}c_1 + 2^n a_2$$

i.e

$$a_2 = \frac{4}{3\cdot 2^n}c_1$$

$$3^{n+1}a_3 = \frac{4}{3}c_2 + \frac{2}{3}c_1^2 + \frac{4}{3}2^n a_2 c_1 + 3^n a_3 \tag{15}$$

which gives

$$a_3 = \frac{4}{3\cdot 2\cdot 3^n}c_2 + \frac{11}{3^2\cdot 3^n}c_1^2 \tag{16}$$

and

$$4^{n+1}a_4 = \frac{4}{3}c_3 + \frac{4}{3}c_1c_2 + \frac{4}{3}2^n a_2 c_2 + \frac{2}{3}2^n a_2 c_1^2 + \frac{4}{3}3^n a_3 c_1 + 4^n a_4$$

Thus,

$$a_4 = \frac{4}{3\cdot 3\cdot 4^n}c_3 + \frac{36}{3^2\cdot 3\cdot 4^n}c_1c_2 + \frac{68}{3^3\cdot 3\cdot 4^n}c_1^3 \tag{17}$$

From equation (15)and using $|c_k| \leq 1$ we get

$$\begin{aligned}|a_3| &= \left|\frac{4}{3\cdot 2\cdot 3^n}\right|\left|\left(c_1 + \frac{22}{3^2\cdot 2\cdot 3^n}\cdot\frac{3\cdot 2\cdot 3^n c_1^2}{4}\right)\right| \\ &= \frac{4}{3\cdot 2\cdot 3^n}\left|\left(c_1 + \frac{22}{3^2\cdot 2\cdot 3^n}\cdot\frac{3\cdot 2\cdot 3^n c_1^2}{4}\right)\right| \\ &= \frac{4}{3\cdot 2\cdot 3^n}\left|(c_2 + \frac{22}{12}c_1^2\right| \\ &= \frac{4}{3\cdot 2\cdot 3^n}\left|(c_2 - (-\frac{22}{12})c_1^2\right|\end{aligned}$$

By Applying Lemma (2.1),we have that

$$|a_3| \leq \frac{4}{3\cdot 2\cdot 3^n}\left[(-\frac{22}{12})\right] = \frac{11}{9\cdot 3^n}$$
$$|a_3| \leq \frac{11}{9\cdot 3^n}$$

Also from equation (17) we have

$$\begin{aligned} a_4 &= \frac{4}{3\cdot 3\cdot 4^n}c_1c_2 + \frac{68}{81\cdot 4^n}c_1^3 + \frac{4}{9\cdot 4^n}c_3 \\ &= \frac{68}{3^3\cdot 3\cdot 4^n}\left(c_1^3 + \frac{3^3\cdot 3\cdot 4^n}{68}\cdot\frac{4}{3\cdot 3\cdot 4^n}c_3 + \frac{3^3\cdot 3\cdot 4^n}{68}\cdot\frac{36}{3^2\cdot 3\cdot 4^n}c_1c_2\right) \\ &= \frac{68}{3^3\cdot 3\cdot 4^n}\left(c_1^3 + \frac{36}{68}c_3 + \frac{27}{17}c_1c_2\right) \\ &\leq \frac{68}{3^3\cdot 3\cdot 4^n}\left(c_1^3 + c_3 + 2c_1c_2\right) \end{aligned}$$

and lemma (2.3)we obtain

$$\begin{aligned} |a_4| &\leq \frac{68}{3^3\cdot 3\cdot 4^n}|c_1^3 + c_3 + 2c_1c_2| \\ &\leq \frac{68}{81\cdot 4^n} \end{aligned}$$

which complete the proof.

Theorem 3.2

Let $\sigma_1 = \frac{5\cdot 2^{2n}}{16\cdot 3^{n,}}$ $\quad\sigma_2 = \frac{17\cdot 2^{2n}}{16\cdot 3^n}$, If $f(z) \in S_nC$, then for any real number λ

$$|a_3 - \lambda a_2^2| \leq \left\{\begin{array}{ccc} \frac{1}{9}\left(\frac{11}{3^n} - \frac{16}{2^{2n}}\lambda\right) & if & \lambda < \sigma_1 \\ \frac{4}{6\cdot 3^n} & if & \sigma_1 \leq \lambda \leq \sigma_2 \\ -\frac{1}{9}\left(\frac{11}{3^n} - \frac{16}{2^{2n}}\lambda\right) & if & \lambda > \sigma_2 \end{array}\right\} \tag{19}$$

Proof:
If $f(z) \in S_nC$,then from equation(15)and (16) we have that

$$a_3 - \lambda a_2^2 = \frac{4}{6\cdot 3^n}c_2 + \frac{22}{18\cdot 3^n}c_1^2 - \lambda\frac{16}{9\cdot 2^{2n}}c_1^2$$

$$= \frac{4}{6\cdot 3^n}\left[c_2 - \left(\frac{2^{3-2n}}{3^{1-n}}\lambda - \frac{11}{6}\right)c_1^2\right] \tag{20}$$

By applying lemma (2.1),equation (20) yields

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n}\left(\frac{11}{6} - \frac{2^{3-2n}}{3^{1-n}}\lambda\right)$$

For

$$\frac{2^{3-2n}}{3^{1-n}}\lambda - \frac{11}{6} < -1$$

i.e

$$\lambda < \frac{5 \cdot 2^{2n}}{16 \cdot 3^n}$$

and taking$\sigma_1 = \frac{5 \cdot 2^{2n}}{16 \cdot 3^n}$
we obtain that

$$|a_3 - \lambda a_2^2| \leq \frac{1}{9}\left(\frac{11}{3^n} - \frac{16}{2^{2n}}\lambda\right) \tag{21}$$

when $\lambda < \sigma_1$
Also, using Lemma 2.1, inequality (20) yields

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n}$$

for

$$-1 \leq \frac{2^{3-2n}}{3^{1-n}}\lambda - \frac{11}{6} \leq 1$$

that is, for

$$\frac{5 \cdot 2^{2n}}{16 \cdot 3^n} \leq \lambda \leq \frac{17 \cdot 2^{2n}}{16 \cdot 3^n}$$

Taking $\sigma_2 = \frac{17 \cdot 2^{2n}}{16 \cdot 3^n}$
we obtain that

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n} \quad \text{if} \quad \sigma_1 \leq \lambda \leq \sigma_2 \tag{22}$$

Applying Lemma (2.1) again to inequality (20) we have,

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n}\left(\frac{2^{3-2n}}{3^{1-n}}\lambda - \frac{11}{6}\right)$$

for

$$\frac{2^{3-2n}}{3^{1-n}}\lambda - \frac{11}{6} > 1$$

that is,

$$|a_3 - \lambda a_2^2| \leq \frac{1}{9}\left(\frac{16}{2^n\lambda} - \frac{11}{3^n}\right) \tag{23}$$

for $\lambda > \sigma_2$
Combining inequality (21)(22)and (23)we obtain the result of the theorem.

Theorem 3.3

If $f(z) \in S_nC$ then for any complex number λ

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n} \max\{1; \left|\frac{11}{6} - \frac{8 \cdot 3^n}{3 \cdot 2^{2n}}\right|\}$$

Proof:
From inequality (15)and(16)we have that

$$|a_3 - \lambda a_2^2| = \left|\frac{4}{6 \cdot 3^n}\left[c_2 - \left(\frac{2^{3-2n}\lambda}{3^{1-n}} - \frac{11}{6}\right)c_1^2\right]\right|$$

i.e

$$|a_3 - \lambda a_2^2| = \frac{4}{6 \cdot 3^n}\left|c_2 - \left(\frac{2^{3-2n}\lambda}{3^{1-n}} - \frac{11}{6}\right)c_1^2\right|$$

By applying lemma 2.2 we have

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot 3^n} \max\left|\{1; \frac{8 \cdot 3^n}{3 \cdot 2^{2n}} - \frac{11}{6}\right|\}$$

$$\begin{aligned}|a_2a_4 - a_3^2| &= \left|\frac{272}{3^5 \cdot 2^{3n}}c_1\left(c_1^3 + 2c_1c_2 + c_3\right) - \frac{4}{9 \cdot 3^{2n}}\left(c_2 + \frac{22}{12}c_1^2\right)^2\right| \\ &= \frac{272}{3^5 \cdot 2^{3n}}|c_1|\left|c_1^3 + 2c_1c_2 + c_3\right| + \frac{4}{9 \cdot 3^{2n}}\left|c_2 + \frac{22}{12}c_1^2\right|^2 \\ &\leq \frac{272}{3^5 \cdot 2^{3n}} + \frac{4}{9 \cdot 3^{2n}}\left(\frac{22}{12}\right)^2 \\ &= \frac{272}{3^5 \cdot 2^{3n}} + \frac{121}{3^4 \cdot 3^{2n}}\end{aligned}$$

Theorem 3.4

Let $f(z)$ belong to the class of functions in S_nC, then

$$H_2(2) = |a_2a_4 - a_3^2| \leq \frac{272}{3^5 \cdot 2^{3n}} + \frac{121}{3^4 \cdot 3^{2n}}$$

Proof:
Given $f(z) \in S_nC$,then

$$\begin{aligned}a_2a_4 - a_3^2 &= \frac{4}{3 \cdot 2^n}c_1\left[\frac{4}{3^2 \cdot 2^{2n}}c_3 + \frac{36}{3^3 \cdot 2^{2n}}c_1c_2 + \frac{68}{3^4 2^{2n}}c_1^3\right] - \left[\frac{4}{3^2 3^n}c_2 + \frac{22}{3^3 \cdot 2 \cdot 3^n}c_1^2\right]^2 \\ &= \frac{16}{3^3 \cdot 2^{3n}}c_1c_3 + \frac{144}{3^4 \cdot 2^{3n}}c_1^2c_2 + \frac{272}{3^5 \cdot 2^{3n}}c_1^4 - \frac{4}{9 \cdot 3^{2n}}\left[c_2 + \frac{22}{12}c_1^2\right]^2 \\ &= \frac{272}{3^5 \cdot 2^{3n}}c_1\left[c_1^3 + \frac{432}{272}c_1c_2 + \frac{144}{272}c_3\right] - \frac{4}{9 \cdot 3^{2n}}\left[c_2 + \frac{22}{12}c_1^2\right]^2 \\ &\leq \frac{272}{3^5 \cdot 2^{3n}}c_1\left[c_1^3 + 2c_1c_2 + c_3\right] - \frac{4}{9 \cdot 3^{2n}}\left[c_2 + 2c_1^2\right]^2 \\ |a_2a_4 - a_3^2| &= \left|\frac{272}{3^5 \cdot 2^{3n}}c_1\left[c_1^3 + 2c_1c_2 + c_3\right] - \frac{4}{9 \cdot 3^{2n}}\left[c_2 + \frac{22}{12}c_1^2\right]^2\right| \\ &\leq \frac{272}{3^5 \cdot 2^{3n}}|c_1|\left[c_1^3 + 2c_1c_2 + c_3\right| + \frac{4}{9 \cdot 3^{2n}}\left|c_2 + 2c_1^2\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{272}{3^5 \cdot 2^{3n}} c_1 \left(c_1^3 + 2c_1c_2 + c_3 \right) - \frac{4}{9 \cdot 3^{2n}} \left(c_2 + \frac{22}{12} c_1^2 \right)^2 \right| \\
&= \frac{272}{3^5 \cdot 2^{3n}} |c_1| \left| c_1^3 + 2c_1c_2 + c_3 \right| + \frac{4}{9 \cdot 3^{2n}} \left| c_2 + \frac{22}{12} c_1^2 \right|^2 \\
&\leq \frac{272}{3^5 \cdot 2^{3n}} + \frac{4}{9 \cdot 3^{2n}} \left(\frac{22}{12} \right)^2 \\
&= \frac{272}{3^5 \cdot 2^{3n}} + \frac{121}{3^4 \cdot 3^{2n}}
\end{aligned}
$$

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