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Variational Iteration Method for Partial Differential Equations with Piecewise Constant Arguments

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Abstract

In this paper, the variational iteration method is applied to solve the partial differential equations with piecewise constant arguments. This technique provides a sequence of functions which converges to the exact solutions of the problem and is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. Employing this technique, we obtain the approximate solutions of the above mentioned equation in every interval $[n, n+1)$ $(n = 0, 1, \cdots)$. Illustrative examples are given to show the efficiency of the method.

Keywords

Variational iteration method; Partial differential equations; Piecewise constant arguments; Approximate solutions.

1. Introduction

The variational iteration method (VIM) was proposed originally by Ji-Huan He [1]. In recent years, the VIM method has been used quite effectively for obtaining exact or approximate solutions for a wide spectrum of linear and nonlinear equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations besides, its interactions are direct and straightforward. The method provides an approximate analytical solution of differential equations in the form of an infinite series [1]. The terms of the series are determined using correction functional that involves the Lagrange multiplier [2], as a key element, identified using the calculus of variations theory. Generally speaking, one or two iterations lead to high accurate solutions. Applications of the method have been increased due to its flexibility, convenience and efficiency. The convergence of the method is systematically discussed in [3, 4, 5]. There are several modifications of He's VIM method [6].

This method has been advantageously employed for solving various kinds of mathematical problems. For example, it has been successfully applied to delay differential equation [7], Hilfer advection-diffusion equation

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[8], quadratic optimal control problem [9], time-fractional Fornberg-Whitham equation [10], differential-algebraic equations [11], and other problems [12, 13].

Though a great deal of attention has been devoted to the study of the VIM, however, we did not witness or observe enough exploration and analysis of the VIM to find the solution of differential equations with piecewise constant arguments (EPCA). EPCA arises frequently in various applied areas [14, 15]. Up to now, research for many properties including existence, uniqueness, stability and oscillation of EPCA becomes a hot issue in the field of differential equations. Moreover, lots of numerical methods have been developed to find the numerical solutions [16, 17, 18, 19, 20, 21]. The general theory and basic results for EPCA have been thoroughly developed in the book of Wiener [22].

The main objective in this work is to effectively employ VIM to establish approximate solutions of partial differential equations with piecewise constant arguments (PEPCA). Several examples are used to illustrate this purpose.

The remaining of this paper is organized as follows. In Section 2, the basic theory of the VIM is presented. In Section 3, we obtain the approximate solutions by applying VIM to PEPCA. In Section 4, we give some examples and Section 5 includes a conclusion that briefly summarizes the results.

2. The VIM method

In this section, we briefly review the main points of the VIM method. Consider the differential equation

$$L[u(t)] + N[u(t)] = g(t), \tag{1}$$

where L and N are linear and nonlinear operators, respectively and $g(t)$ is an inhomogeneous term. According to the method, the correction functional is considered

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(L[u_n(s)] + N[\tilde{u}_n(s)] - g(s))ds, \tag{2}$$

where λ is a general Lagrange multiplier, u_n is the nth-order approximate solution and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$ [23, 24, 25].

In this method, first we determine the Lagrange multiplier λ that can be identified via variational theory, i.e. the multiplier should be chosen such that the correction functional is stationary, i.e. $\delta u_{n+1}(u_n(t), t) = 0$. Then the successive approximation u_n of the solution u will be obtained by using any selective initial function u_0 and the calculated Lagrange multiplier λ. Consequently $u = \lim\limits_{n\to\infty} u_n$. It means that, by the correction functional (2) several approximations will be obtained. Therefore, the exact solution emerges at the limit of the resulting successive approximations.

In the next section, this method is successfully applied for solving the PEPCA.

3. Applications and analysis

The so-called PEPCA with delay term $[t]$ reads

$$\begin{cases} u_t(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t]), \quad t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin(\pi x), \end{cases} \tag{3}$$

where a, b are two real constants, $[\cdot]$ denotes the greatest integer function. According to the VIM, we construct the correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{(u_n(x,s))_s - a^2(\tilde{u}_n(x,s))_{xx} - b(\tilde{u}_n(x,[s]))_{xx}\}ds, \tag{4}$$

$(\tilde{u}_n(x,s))_{xx}$ and $(\tilde{u}_n(x,[s]))_{xx}$ are both considered as restricted variations [1, 23, 24, 25], that is

$$\delta(\tilde{u}_n(x,s))_{xx}=0, \quad \delta(\tilde{u}_n(x,[s]))_{xx}=0. \tag{5}$$

Making the correction functional (4) stationary, by (5) we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(u_n(x,s))_s ds,$$

with the help of integration by part, we obtain

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\lambda\delta u_n(x,s)|_0^t-\delta\int_0^t \lambda' u_n(x,s)ds,$$

which yields the following stationary conditions

$$1+\lambda(s)|_{s=t}=0, \quad \lambda'(s)=0,$$

the Lagrange multiplier, therefore, can be readily identified

$$\lambda=-1.$$

So we have the following iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\{(u_n(x,s))_s-a^2(u_n(x,s))_{xx}-b(u_n(x,[s]))_{xx}\}ds. \tag{6}$$

Taking into account of the existing of $[\cdot]$, we divide the whole interval $[0,\infty)$ into many little ones $[n,n+1)$, $n=0,1,2,\cdots$.

When $t\in[0,1)$, (6) gives

$$u_{0,n+1}(x,t)=u_{0,n}(x,t)-\int_0^t\{(u_{0,n}(x,s))_s-a^2(u_{0,n}(x,s))_{xx}-b(u_{0,n}(x,0))_{xx}\}ds. \tag{7}$$

Select $u_{0,0}(x,t)=u(x,0)=\sin\pi x$, then $u_{0,0}(x,0)=\sin\pi x$, from (7) we have

$$\begin{aligned}
u_{0,1}(x,t)&=u_{0,0}(x,t)-\int_0^t(a^2+b)\pi^2\sin\pi x ds=-\frac{b}{a^2}\sin\pi x+\left(\sin\pi x+\frac{b}{a^2}\sin\pi x\right)(1-a^2\pi^2 t),\\
u_{0,2}(x,t)&=-\frac{b}{a^2}\sin\pi x+\left(\sin\pi x+\frac{b}{a^2}\sin\pi x\right)\left(1-a^2\pi^2t+\frac{1}{2}(a^2\pi^2t)^2\right),\\
u_{0,3}(x,t)&=-\frac{b}{a^2}\sin\pi x+\left(\sin\pi x+\frac{b}{a^2}\sin\pi x\right)\left(1-a^2\pi^2t+\frac{1}{2}(a^2\pi^2t)^2-\frac{1}{3!}(a^2\pi^2t)^3\right),\\
u_{0,4}(x,t)&=-\frac{b}{a^2}\sin\pi x+\left(\sin\pi x+\frac{b}{a^2}\sin\pi x\right)\left(1-a^2\pi^2t+\frac{1}{2}(a^2\pi^2t)^2-\frac{1}{3!}(a^2\pi^2t)^3+\frac{1}{4!}(a^2\pi^2t)^4\right),
\end{aligned}$$

repeat this process, we have the following general result in the interval $[0,1)$.

Theorem 1. *For $t\in[0,1)$, the $n+1$th-order ($n=0,1,2,\cdots$) approximate solution of (3) can be given as*

$$u_{0,n+1}(x,t)=-\frac{b}{a^2}\sin\pi x+\left(\sin\pi x+\frac{b}{a^2}\sin\pi x\right)\sum_{l=0}^{n+1}(-1)^l\frac{(a^2\pi^2t)^l}{l!}, \tag{8}$$

if $-1/\pi<a<1/\pi$, then the series $\{u_{0,n+1}(x,t)\}_{n=0}^{\infty}$ converges to

$$\left(-\frac{b}{a^2}+(1+\frac{b}{a^2})e^{-a^2\pi^2t}\right)\sin\pi x. \tag{9}$$

Proof. Obviously, when $n = 1$, (8) holds.

Assume that (8) is true when $n = k$, that is

$$u_{0,k+1}(x,t) = -\frac{b}{a^2}\sin\pi x + \left(\sin\pi x + \frac{b}{a^2}\sin\pi x\right)\sum_{l=0}^{k+1}(-1)^l\frac{(a^2\pi^2 t)^l}{l!},$$

further we compute that

$$\begin{aligned}(u_{0,k+1}(x,s))_s &= -a^2\pi^2\left(\sin\pi x + \frac{b}{a^2}\sin\pi x\right)\sum_{l=0}^{k}(-1)^l\frac{(a^2\pi^2 s)^l}{l!},\\ -a^2(u_{0,k+1}(x,s))_{xx} &= -b\pi^2\sin\pi x + (-a^2\pi^2\sin\pi x + b\pi^2\sin\pi x)\sum_{l=0}^{k+1}(-1)^l\frac{(a^2\pi^2 s)^l}{l!},\\ -b(u_{0,k+1}(x,0))_{xx} &= b\pi^2\sin\pi x,\end{aligned}$$

so (7) gives

$$u_{0,k+2}(x,t) = -\frac{b}{a^2}\sin\pi x + \left(\sin\pi x + \frac{b}{a^2}\sin\pi x\right)\sum_{l=0}^{k+2}(-1)^l\frac{(a^2\pi^2 t)^l}{l!},$$

which means that (8) holds when $n = k+1$.

On the other hand, (9) can be obtained directly from (8) under the condition $-1/\pi < a < 1/\pi$. The proof is completed. □

Next, we will consider the second interval.

When $t \in [1,2)$, (6) gives

$$u_{1,n+1}(x,t) = u_{1,n}(x,t) - \int_1^t\{(u_{1,n}(x,s))_s - a^2(u_{1,n}(x,s))_{xx} - b(u_{1,n}(x,1))_{xx}\}ds. \tag{10}$$

Select

$$u_{1,0}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2 t}\right)\sin\pi x,$$

then

$$u_{1,0}(x,1) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)\sin\pi x,$$

from (10) we have

$$\begin{aligned}u_{1,1}(x,t) &= u_{1,0}(x,t) - (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})\textstyle\int_1^t(a^2+b)\pi^2\sin\pi x ds\\ &= (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})(-\tfrac{b}{a^2}\sin\pi x + (\sin\pi x + \tfrac{b}{a^2}\sin\pi x)(1 - a^2\pi^2(t-1))),\end{aligned}$$

$$\begin{aligned}u_{1,2}(x,t) &= (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})(-\tfrac{b}{a^2}\sin\pi x + (\sin\pi x + \tfrac{b}{a^2}\sin\pi x)\\ &\quad(1 - a^2\pi^2(t-1) + \tfrac{1}{2}(a^2\pi^2(t-1))^2)),\end{aligned}$$

$$\begin{aligned}u_{1,3}(x,t) &= (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})(-\tfrac{b}{a^2}\sin\pi x + (\sin\pi x + \tfrac{b}{a^2}\sin\pi x)\\ &\quad(1 - a^2\pi^2(t-1) + \tfrac{1}{2}(a^2\pi^2(t-1))^2 - \tfrac{1}{3!}(a^2\pi^2(t-1))^3)),\end{aligned}$$

continue this process we have

$$u_{1,n+1}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)\left(-\frac{b}{a^2}\sin\pi x + (\sin\pi x + \frac{b}{a^2}\sin\pi x)\sum_{l=0}^{n+1}(-1)^l\frac{(a^2\pi^2(t-1))^l}{l!}\right). \tag{11}$$

Using the same method in Theorem 1, we can prove that (11) is true.

Therefore, we can get the following result in general sense.

Theorem 2. *For $t \in [i, i+1)$ $(i = 0, 1, 2, \cdots)$, the $n+1$th-order $(n = 0, 1, 2, \cdots)$ approximate solution of (3) can be given as*

$$u_{i,n+1}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)^i \left(-\frac{b}{a^2}\sin\pi x + (\sin\pi x + \frac{b}{a^2}\sin\pi x)\sum_{l=0}^{n+1}(-1)^l\frac{(a^2\pi^2(t-i))^l}{l!}\right), \tag{12}$$

if $-1/\pi < a < 1/\pi$, then the series $\{u_{i,n+1}(x,t)\}_{n=0}^{\infty}$ converges to

$$\left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2(t-i)}\right)^{i+1}\sin\pi x.$$

Proof. Obviously, when $i = 0, 1$, (12) holds.

Assume that (12) is true when $i = k$, then

$$u_{k,n+1}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)^k \left(-\frac{b}{a^2}\sin\pi x + (\sin\pi x + \frac{b}{a^2}\sin\pi x)\sum_{l=0}^{n+1}(-1)^l\frac{(a^2\pi^2(t-k))^l}{l!}\right). \tag{13}$$

When $t \in [k+1, k+2)$, we have

$$u_{k+1,n+1}(x,t) = u_{k+1,n}(x,t) - \int_{k+1}^{t}((u_{k+1,n}(x,s))_s - a^2(u_{k+1,n}(x,s))_{xx} - b(u_{k+1,n}(x,k+1))_{xx})ds. \tag{14}$$

Let

$$u_{k+1,0}(x,t) = \lim_{n\to\infty} u_{k,n+1}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2(t-k)}\right)^{k+1}\sin\pi x,$$

then

$$u_{k+1,0}(x,k+1) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)^{k+1}\sin\pi x,$$

so (14) gives

$$u_{k+1,1}(x,t) = (-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2})^{k+1}(-\frac{b}{a^2}\sin\pi x + (\sin\pi x + \frac{b}{a^2}\sin\pi x)(1 - a^2\pi^2(t-(k+1)))),$$

$$\begin{aligned} u_{k+1,2}(x,t) &= (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})^{k+1}(-\tfrac{b}{a^2}\sin\pi x + (\sin\pi x + \tfrac{b}{a^2}\sin\pi x) \\ &\quad (1 - a^2\pi^2(t-(k+1)) + \tfrac{1}{2!}(a^2\pi^2(t-(k+1)))^2)), \\ u_{k+1,3}(x,t) &= (-\tfrac{b}{a^2} + (1+\tfrac{b}{a^2})e^{-a^2\pi^2})^{k+1}(-\tfrac{b}{a^2}\sin\pi x + (\sin\pi x + \tfrac{b}{a^2}\sin\pi x) \\ &\quad (1 - a^2\pi^2(t-(k+1)) + \tfrac{1}{2!}(a^2\pi^2(t-(k+1)))^2 \\ &\quad -\tfrac{1}{3!}(a^2\pi^2(t-(k+1)))^3)), \end{aligned}$$

in the same way, continue this proceed we have

$$u_{k+1,n+1}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)^{k+1} \left(-\frac{b}{a^2}\sin\pi x + (\sin\pi x + \frac{b}{a^2}\sin\pi x)\sum_{l=0}^{n+1}(-1)^l\frac{(a^2\pi^2(t-k-1))^l}{l!}\right). \tag{15}$$

By Theorem 1 we know that (12) holds when $i = k+1$. The proof is finished. □

4. Test examples

To demonstrate our theoretical result, some test examples are adopted in this section.

Firstly, in order to describe the error, we introduce the following concept. For $t \in [i, i+1)$, we define the error function which means the difference between the nth-order and $n+1$th-order approximations as follows

$$g_{i,(n+1)n}(x,t) = u_{i,n+1}(x,t) - u_{i,n}(x,t),$$

then

$$g_{i,(n+1)n}(x,t) = \left(-\frac{b}{a^2} + (1+\frac{b}{a^2})e^{-a^2\pi^2}\right)^i \left(1+\frac{b}{a^2}\right)\frac{(-1)^{n+1}}{(n+1)!}(a^2\pi^2(t-i))^{n+1}\sin \pi x. \qquad (16)$$

Consider the following problem

$$\begin{cases} u_t(x,t) = \frac{1}{16}u_{xx}(x,t) + \frac{1}{16}u_{xx}(x,[t]), \quad t>0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin(\pi x). \end{cases} \qquad (17)$$

According to Theorem 2, when $t \in [i, i+1)$ $(i = 0, 1, 2, \cdots)$, the $n+1$th-order $(n = 0, 1, 2, \cdots)$ approximate solution of (17) can be given as

$$u_{i,n+1}(x,t) = \left(-1+2e^{-\frac{\pi^2}{16}}\right)^i \left(-1+2\sum_{l=0}^{n+1}\frac{(-1)^l}{l!}\left(\frac{\pi^2}{16}(t-i)\right)^l\right)\sin \pi x, \qquad (18)$$

and the series $\{u_{i,n+1}(x,t)\}_{n=0}^{\infty}$ converges to $(-1+2e^{-\frac{\pi^2}{16}(t-i)})^{i+1}\sin \pi x$.

On the other hand, we can get the error function from Definition 4

$$g_{i,(n+1)n}(x,t) = \frac{2(-1)^{n+1}}{(n+1)!}(-1+2e^{-\frac{\pi^2}{16}})^i \left(\frac{\pi^2}{16}(t-i)\right)^{n+1}\sin \pi x.$$

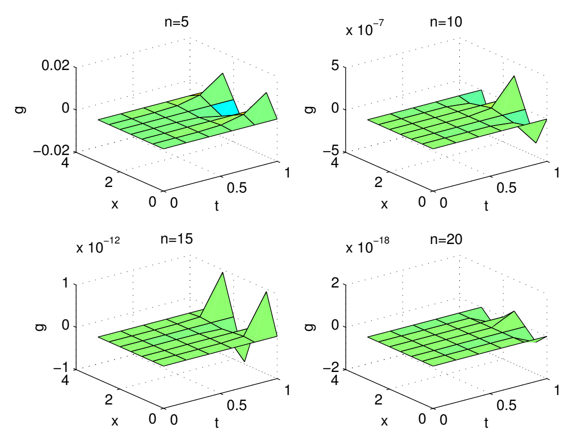

Fig. 1. The error function of (17) in [0, 1).

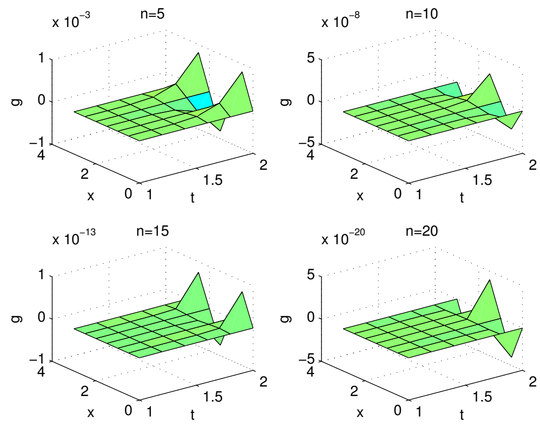

Fig. 2. The error function of (17) in [1, 2).

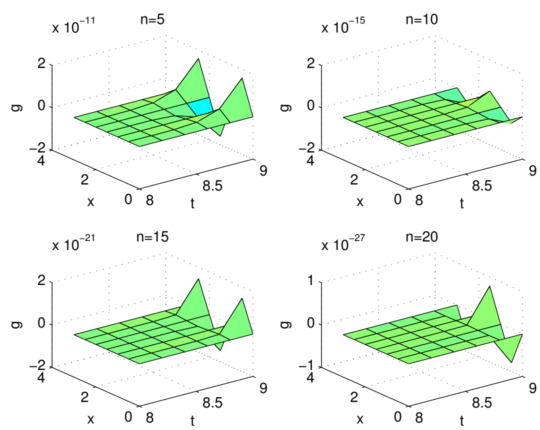

Fig. 3. The error function of (17) in [8, 9).

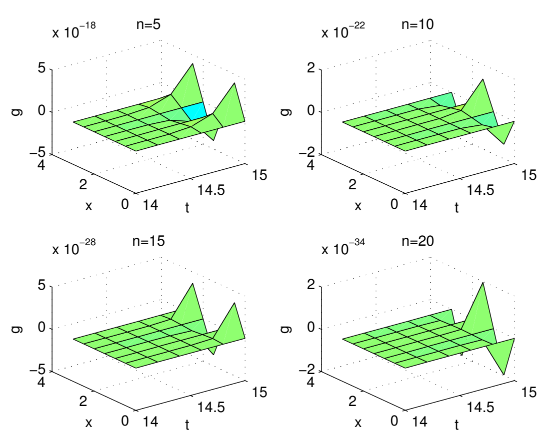

Fig. 4. The error function of (17) in [14, 15).

In Figs.1-4, we draw the error figures for the approximate solutions in different interval. It is easy to see that the values of $g_{i,(n+1)n}(x,t)$ are close to zero with the increasing of n. That is, the approximate solutions are converge to the exact solutions.

5. Conclusions

In this paper, we applied the VIM in finding the approximate solution for the PEPCA. By this method a rapid convergent sequence is produced. The numerical results showed that the VIM performed well for the PEPCA. In the future work, we will consider multi-dimension and stochastic case.

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