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Uniqueness and existence of an outgoing solution of Helmholtz problem using Green's formula

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Abstract

In this article, first we present a new approach based on Green's formula, to describe the uniqueness and existence of a solution of the Helmholtz equation. By imposing at infinity the outgoing wave condition or also called Sommerfeld radiation condition, we show how it is possible to define in a natural way an outgoing solution of the Helmholtz equation based on physical arguments. Then, we resolve the exterior problem, given by the scattering of time-harmonic acoustic wave by sound-soft obstacle, which leads to find a radiating solution $u \in C^2(\mathbb{R}^3\backslash\overline{D}) \cap C(\mathbb{R}^3\backslash D)$ to the Helmholtz equation.

Keywords: Helmholtz problem; Green formula; outgoing solution; acoustic wave; sound-soft obstacle.

1. Introduction

Time harmonic wave propagations appear in many applications [5, 6, 8, 9, 10, 11], e.g. wave scattering and transmission, noise reduction, fluid-solid interaction and earthquake wave propagation. In many situations time harmonic wave propagations are governed by the following Helmholtz equation in an exterior domain with the so-called Sommerfeld radiation boundary condition

$$(\mathrm{P1}) \quad \begin{cases} -\Delta u(x) - k^2 u(x) = f(x) & \forall\, x \in \mathbb{R}^3, \\ \int_{\{\|x\|=R\}} |u(x)|^2 \; ds = O(1) & R \longrightarrow +\infty, \\ \int_{\{\|x\|=R\}} |(\partial_r - ik)u(x)|^2 \; ds = O\left(\frac{1}{R^2}\right) & R \longrightarrow +\infty. \end{cases}$$

Where $\|x\|$ denotes the Euclidean norm, ∂_r is the radial derivative and $k = \frac{w}{c}$ is the wave number with w is a given frequency and c is the sound speed in the acoustic medium. Formally, the two conditions of the problem (P1) consist to make sure that the solution admits the following behavior at infinity

$$u(x) = \frac{\exp(ik\|x\|)}{\|x\|}\left(u_\infty\left(\frac{x}{\|x\|}\right) + O\left(\frac{1}{\|x\|}\right)\right), \qquad \|x\| \longrightarrow +\infty.$$

The scattering of time-harmonic acoustic waves by sound-soft obstacles leads to the following exterior Dirichlet boundary value problem for the Helmholtz equation

$$(\mathrm{P2}) \quad \begin{cases} -\Delta u(x) - k^2 u(x) = 0 & \forall\, x \in \mathbb{R}^3\backslash\overline{D}, \\ u(x) = g(x) & \forall\, x \in \partial D, \\ \lim_{r\longrightarrow+\infty} r(\partial_r u - iku) = 0, \end{cases}$$

Where g is a given continuous function on ∂D.

Exterior Helmholtz problem present a great challenge to numerical analysts and computational scientists [1,2,3] because the domain is unbounded and the solution is highly oscillatory (when k is large). In this paper, we try to address the problem in the theoretical side, we show the uniqueness and existence of the radiating solution of problems (P1) and (P2) by using Green's representation theorem.

2. The Sommerfeld radiation condition

2.1 Green function and existence of outgoing solution

In this section we show the existence of an outgoing solution of the Helmholtz equation using Green functions and we begin our analysis with a reminder of the first and the second Green's theorem [6]: For any domain D with boundary ∂D of class C^2, we introduce the linear space $\Re(D)$ of all complex-valued functions u and v. Let η denote the unit normal vector to the boundary ∂D directed into the exterior of D. Then, for $u \in C^1(\overline{D})$ and $v \in C^2(\overline{D})$ we have Green's first theorem [6], also called Green's formula

$$\int_D (u\Delta v + \nabla u.\nabla v)\,dx = \int_{\partial D} u\partial_\eta\, v\,ds, \tag{1}$$

and for $u, v \in C^2(\overline{D})$ we have Green's second theorem

$$\int_D (u\Delta v - v\Delta u)\,dx = \int_{\partial D}\left(u\partial_\eta v - v\partial_\eta u\right)\,ds. \tag{2}$$

Lemma 2.1. If u and Δu are in $L^2_{loc}(\mathbb{R}^3)$ then u is in $H^1_{loc}(\mathbb{R}^3)$.

Proof. Let S be the sphere of radius R and S' the sphere of radius $2R$. By density, it suffices to show that there is a constant $C > 0$ such that

$$\|\nabla u\|_{L^2(S)} \le C\left(\|u\|_{L^2(S')} + \|\Delta u\|_{L^2(S')}\right), \qquad \forall u \in D(\mathbb{R}^3). \tag{3}$$

Let $F \in D(\mathbb{R}^3)$ be a function decreasing along r such as $F(x) = 1$in S and $F(x) = 0$ out of S '. Using Green's first identity, we get

$$\int_{\mathbb{R}^3} \|\nabla Fu\|^2\ dx = -\int_{\mathbb{R}^3} \Delta(Fu)Fu\ dx, \qquad \forall F \in D(\mathbb{R}^3). \tag{4}$$

We can evaluate the term on the right of equality by developing the Laplacian

$$\Delta(Fu) = F\Delta u + 2\nabla F.\nabla u + u\Delta F, \tag{5}$$

Hence

$$F\Delta(Fu) = F^2\Delta u + 2\nabla F.\nabla(\mathrm{Fu}) + (F\Delta F - 2\nabla F.\nabla F)u. \tag{6}$$

Substituting (6) into (4) we obtain

$$\int_{\mathbb{R}^3} \|\nabla Fu\|^2\ dx = -\int_{\mathbb{R}^3} (F^2\Delta u + 2\nabla F.\nabla(\mathrm{Fu}) + u(F\Delta F - 2\nabla F.\nabla F))u\ dx, \qquad \forall F \in D(\mathbb{R}^3), \tag{7}$$

as $\|F\|_{L^\infty(\mathbb{R}^3)} = 1$, we get

$$\|\nabla Fu\|^2_{L^2(\mathbb{R}^3)} \le \|u\|_{L^2(S')}\|\Delta u\|_{L^2(S')} + 2C_1\|\nabla Fu\|_{L^2(S')}\|u\|_{L^2(S')} + C_2\|u\|^2_{L^2(S')}, \tag{8}$$

with $C_1 = \|\nabla F\|_{L^\infty(\mathbb{R}^3)}$ and $C_2 = \|\Delta F\|_{L^\infty(\mathbb{R}^3)}$.

As $ab \le \frac{a^2+b^2}{2}$, we get

$$\|\nabla Fu\|^2_{L^2(\mathbb{R}^3)} \leq \frac{\|\nabla Fu\|^2_{L^2(\mathbb{R}^3)}}{2} + \left(\frac{1}{2} + 2C_1^2 + C_2\right)\|u\|^2_{L^2(S')} + \frac{\|\Delta u\|^2_{L^2(S')}}{2}, \tag{9}$$

it follows that

$$\frac{\|\nabla Fu\|^2_{L^2(\mathbb{R}^3)}}{2} \leq \left(\frac{1}{2} + 2C_1^2 + C_2\right)\|u\|^2_{L^2(S')} + \frac{\|\Delta u\|^2_{L^2(S')}}{2}. \tag{10}$$

As $F = 1$ in S, it follows

$$\|\nabla u\|_{L^2(S)} \leq \|\nabla \mathrm{F}u\|_{L^2(\mathbb{R}^3)} \tag{11}$$

hence we deduce (3). □

Definition 2.2. We call Green function $\psi(x)$ any solution of the linear partial derivatives equation

$$\mathfrak{H}\,\psi(x) = \delta(x) \tag{12}$$

with $\mathfrak{H}$ presents a linear differential operator and $\delta(x)$ presents a Dirac distribution.

In the following, we denote for all $k \in \mathbb{C}$ the function $\psi: \mathbb{R}^3\backslash\{0\} \longrightarrow \mathbb{C}$, with $\psi(x) = \frac{e^{(ik\|x\|)}}{4\pi\|x\|}$ called a fundamental solution to the Helmholtz equation [5].

Lemma 2.3. [4] For all $k \in \mathbb{C}$, the function ψ is a fundamental solution of the operator $(-\Delta - k^2)$ i.e.

$$-\Delta\,\psi - k^2\psi = \delta(x) \qquad in\ D'(\mathbb{R}^3). \tag{13}$$

Proposition 2.4. Let $k \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^3)$ with compact support. The function $u(x) = \psi * f(x)$ verifies

$$\begin{cases} u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3), \\ -\Delta u(x) - k^2 u(x) = f(x) \qquad \forall\, x \in \mathbb{R}^3, \\ \int_{\{\|x\|=R\}} |u(x)|^2 \ ds = O(1) \qquad R \longrightarrow +\infty, \\ \int_{\{\|x\|=R\}} |(\partial_r - ik)u(x)|^2 \ ds = O\left(\frac{1}{R^2}\right) \qquad R \longrightarrow +\infty. \end{cases} \tag{14}$$

Proof. By definition, function u: $\mathbb{R}^3 \longrightarrow \mathbb{C}$ is given by

$$u(x) = \psi * f(x) = \int_{\mathbb{R}^3} \frac{e^{(ik\|x-y\|)}}{4\pi\|x-y\|} f(y)\, dy. \tag{15}$$

The convolution theory in $D'(\mathbb{R}^3)$ allows us to obtain that u is a solution of the inhomogeneous Helmholtz equation

$$-(\Delta + k^2)(\psi * f)(x) = -(\Delta\psi + k^2\psi) * f(x) = f(x). \tag{16}$$

According lemma 2.1., we note that $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$. As the support of f is bounded, there is a positive real λ such that, the ball of radius λ contains the support of f. For all y in the support of f and for all x such that $\|x\| > 2\lambda$, we have

$$\|x - y\| \geq \|x\| - \|y\| \geq \frac{\|x\|}{2} + \frac{\|x\|}{2} - \|y\| \geq \frac{\|x\|}{2} + \frac{2\lambda}{2} - \lambda \geq \frac{\|x\|}{2}. \tag{17}$$

By increasing the expression (15), it follows

$$|u(x)| \leq \frac{1}{2\pi\|x\|} \int_{\mathbb{R}^3} |f(y)|\ dy, \tag{18}$$

we integrate this expression on the sphere of radius R, we get

$$\int_{\{\|x\|=R\}} |u(x)|^2 \ ds \ \leq \ \frac{1}{\pi}\left(\int_{\mathbb{R}^3} |f(y)| \ dy\right)^2 = O(1) \qquad R \longrightarrow +\infty. \tag{19}$$

Similarly, we have for $\|x\| > 2\lambda$

$$\nabla u(x) = \int_{\mathbb{R}^3} \frac{e^{(ik\|x-y\|)}}{4\pi} \frac{x-y}{\|x-y\|^2}\left(ik - \frac{1}{\|x-y\|}\right) f(y) \, dy, \tag{20}$$

$$\partial_r u(x) = \int_{\mathbb{R}^3} \frac{e^{(ik\|x-y\|)}}{4\pi} \frac{x-y}{\|x-y\|^2} \frac{x}{\|x\|}\left(ik - \frac{1}{\|x-y\|}\right) f(y) \, dy, \tag{21}$$

hence

$$(\partial_r - ik)u(x) = \int_{\mathbb{R}^3} \frac{e^{(ik\|x-y\|)}}{4\pi} \frac{x-y}{\|x-y\|^3} \cdot \frac{x}{\|x\|} \ f(y) \, dy + \int_{\mathbb{R}^3} \frac{e^{(ik\|x-y\|)}}{4\pi\|x-y\|}\left(1 - \frac{x-y}{\|x-y\|} \cdot \frac{x}{\|x\|}\right) ik \ f(y) \, dy, \tag{22}$$

So we get the inequality

$$|(\partial_r - ik)u(x)| \ \leq \ \int_{\mathbb{R}^3} \frac{|f(y)|}{4\pi\|x-y\|^2} \ dy + k\int_{\mathbb{R}^3} \frac{|f(y)|}{4\pi \ \|x-y\|} \left(1 - \frac{x-y}{\|x-y\|} \cdot \frac{x}{\|x\|}\right) dy. \tag{23}$$

We note that

$$1 - \frac{x-y}{\|x-y\|} \cdot \frac{x}{\|x\|} = \left(\frac{x}{\|x\|} - \frac{x-y}{\|x-y\|}\right) \cdot \frac{x}{\|x\|}, \tag{24}$$

from which and after the inequality of Cauchy-Schwartz , it follows

$$\left|1 - \frac{x-y}{\|x-y\|} \cdot \frac{x}{\|x\|}\right| \ \leq \ \left\|\frac{x}{\|x\|} - \frac{x-y}{\|x-y\|}\right\|. \tag{25}$$

As

$$\frac{x}{\|x\|} - \frac{x-y}{\|x-y\|} = \left(\frac{\|x-y\|-\|x\|}{\|x-y\|}\right)\frac{x}{\|x\|} + \frac{y}{\|x-y\|}, \tag{26}$$

we get

$$\left|1 - \frac{x-y}{\|x-y\|} \cdot \frac{x-y}{\|x\|}\right| \leq \left|\frac{\|x-y\|-\|x\|}{\|x-y\|}\right| \frac{\|x\|}{\|x\|} + \frac{\|y\|}{\|x-y\|}, \tag{27}$$

so according to the triangle inequality, we obtain

$$\left|1 - \frac{x-y}{\|x-y\|} \cdot \frac{x}{\|x\|}\right| \leq \frac{\|y\|}{\|x-y\|} \frac{\|x\|}{\|x\|} + \frac{\|y\|}{\|x-y\|} \ \leq 2\frac{\|y\|}{\|x-y\|}. \tag{28}$$

We deduce from (17) and (23) that

$$|(\partial_r - ik)u(x)| \ \leq \ \int_{\mathbb{R}^3} \frac{(1+2k\|y\|)|f(y)|}{4\pi\|x-y\|^2} \, dy, \tag{29}$$

$$|(\partial_r - ik)u(x)| \ \leq \left(\int_{\mathbb{R}^3} \frac{(1+2k\|y\|)|f(y)|}{\pi} \, dy\right) \frac{1}{\|x\|^2}. \tag{30}$$

Integrating over the sphere of radius R, we get

$$\int_{\{\|x\|=R\}} |(\partial_r - ik)u(x)|^2 \ ds \ \leq \frac{4}{\pi R^2}\left(\int_{\mathbb{R}^3} |f(y)| \ dy\right)^2 \left(1 + \frac{k\lambda}{2}\right)^2 = O\left(\frac{1}{R^2}\right) \qquad R \longrightarrow +\infty. \tag{31}$$

This concludes the proof of Proposition 2.4. □

2.2 Uniqueness of the outgoing solution of the Helmholtz equation

The uniqueness of the outgoing solution of the Helmholtz equation is based on the following theorem due to Rellich [11].

Theorem 2.5. [11] Let k > 0 and $u \in L^2_{loc}(\mathbb{R}^3)$ verifies

$$\begin{cases} -\Delta u(x) - k^2 u(x) = 0 \qquad \forall\, x \in \mathbb{R}^3, \\ \int_{\{\|x\|=R\}} |u(x)|^2 \; ds = O(1) \qquad R \longrightarrow +\infty, \\ \int_{\{\|x\|=R\}} |(\partial_r - ik)u(x)|^2 \; ds = O\left(\frac{1}{R^2}\right) \qquad R \longrightarrow +\infty. \end{cases} \tag{32}$$

Then $u = 0$ in $\mathbb{R}^3$.

Proof. We note, according Lemma 2.1., that $u \in H^1_{loc}(\mathbb{R}^3)$. Moreover, its three partial derivatives verify

$$\partial_i u \in L^2_{loc}(\mathbb{R}^3) \text{ and } \Delta \partial_i u + k^2 \partial_i u = 0. \tag{33}$$

According Lemma 2.1, it follows that $\partial_i u \in H^1_{loc}(\mathbb{R}^3)$. So u is an element of $H^2_{loc}(\mathbb{R}^3)$. Following Green's formula we have

$$\int_{\{\|x\|<R\}} |\nabla u(x)|^2 + \Delta u(x)\overline{u(x)}\, dx = \int_{\{\|x\|=R\}} \overline{u(x)}\, \partial_r u(x)\; ds, \tag{34}$$

hence

$$\int_{\{\|x\|<R\}} |\nabla u(x)|^2 - k^2\, |u(x)|^2\, dx = \int_{\{\|x\|=R\}} \overline{u(x)}\, (\partial_r - ik)u(x) + ik|u(x)|^2 \; ds. \tag{35}$$

We note $\hat{u}: \mathbb{R}^3 \longrightarrow \mathbb{C}$ the function defined by

$$\hat{u}(x) = \frac{u(x)}{e^{(ik\|x\|)}}. \tag{36}$$

It follows from the identity $\nabla \hat{u} + ik\hat{u}\hat{e}_r = exp(-ik\|x\|)\nabla u$ that

$$\int_{\{\|x\|<R\}} |\nabla\, \hat{u} + ik\, \hat{u}\, \hat{e}_r|^2 - k^2 |\, \hat{u}|^2\, dx = \int_{\{\|x\|=R\}} \partial_r\, \hat{u}\overline{\hat{u}} + ik|\, \hat{u}|^2\, ds. \tag{37}$$

Equation (37) can be simplified to

$$\int_{\{\|x\|<R\}} |\nabla\, \hat{u}|^2 + i\, k\big(\, \hat{u}\overline{\partial_r \hat{u}} - \overline{\hat{u}}\partial_r\, \hat{u}\big)\, dx = \int_{\{\|x\|=R\}} \partial_r\, \hat{u}\overline{\hat{u}} + ik|\hat{u}|^2\, ds. \tag{38}$$

If we take the imaginary part, we get

$$\int_{\{\|x\|=R\}} \partial_r\, \hat{u}\overline{\hat{u}} - \hat{u}\overline{\partial_r \hat{u}} + 2\, ik|\, \hat{u}|^2 \; ds = 0, \tag{39}$$

then, we deduce

$$\int_{\{\|x\|<R\}} ik(\partial_r\, \hat{u}\overline{\hat{u}} - \hat{u}\overline{\partial_r \hat{u}})\, dx = 2\, k^2 \int_{\{\|x\|<R\}} |\hat{u}|^2 \;\; dx, \tag{40}$$

Consequently

$$\int_{\{\|x\|<R\}} |\nabla \hat{u}|^2 + 2k^2 |\hat{u}|^2 \, dx = \int_{\{\|x\|=R\}} \partial_r \hat{u}\overline{\hat{u}} + ik|\hat{u}|^2 \, ds. \tag{41}$$

Taking the imaginary part of this expression, we get

$$k \int_{\{\|x\|=R\}} |\hat{u}(x)|^2 \, ds = -Im\left(\int_{\{\|x\|=R\}} \partial_r \hat{u}(x)\overline{\hat{u}(x)} \, ds\right), \tag{42}$$

according to the Cauchy-Schwartz theorem, we get

$$k\|\hat{u}\|^2_{L^2_{(\{\|x\|=R\})}} \leq \|\partial_r \hat{u}\|_{L^2_{(\{\|x\|=R\})}} \|\hat{u}\|_{L^2_{(\{\|x\|=R\})}}, \tag{43}$$

Consequently, we get

$$\|\hat{u}\|_{L^2_{(\{\|x\|=R\})}} \leq \frac{\|\partial_r \hat{u}\|_{L^2_{(\{\|x\|=R\})}}}{k}, \tag{44}$$

As $\hat{u}(x) = exp(-ik\|x\|)\, u(x),$ then we deduce from the Sommerfeld radiation conditions that

$$\|\partial_r \hat{u}\|_{L^2_{(\{\|x\|=R\})}} = \|\partial_r u - iku\|_{L^2_{(\{\|x\|=R\})}} = O\left(\frac{1}{R}\right) \quad R \longrightarrow +\infty, \tag{45}$$

according to (44), we have

$$\|\hat{u}\|_{L^2_{(\{\|x\|=R\})}} = O\left(\frac{1}{R}\right) \quad R \longrightarrow +\infty. \tag{46}$$

According to the Cauchy-Schwartz theorem, (45) and (46), we note that

$$\int_{\{\|x\|=R\}} \partial_r \hat{u}(x)\overline{\hat{u}(x)} + ik\,|\hat{u}|^2 \, ds = O\left(\frac{1}{R^2}\right), \quad R \longrightarrow +\infty. \tag{47}$$

We can then pass to the limit in (41)

$$\int_{\mathbb{R}^3} |\nabla \hat{u}|^2 + 2k^2\, |\hat{u}|^2 \, dx = 0. \tag{48}$$

Finally, it follows that $\hat{u} = 0$ and therefore $u = 0$ in $\mathbb{R}^3$. □

3. Scattering problem from a Sound-Soft obstacle

Definition 3.1. A solution u to the Helmholtz equation whose domain of definition contains the exterior of some spheres is called radiating if it satisfies the Sommerfeld radiation condition

$$\lim_{r \longrightarrow +\infty} r(\partial_r u - iku) = 0 \tag{49}$$

Where $r = \|x\|$ and the limit is assumed to hold uniformly in all direction $\frac{x}{\|x\|}$.

The scattering of time-harmonic acoustic waves by sound-soft obstacles leads to the following exterior Dirichlet boundary value problem for the Helmholtz equation

$$\begin{cases} -\Delta u(x) - k^2 u(x) = 0 & \forall\, x \in \mathbb{R}^3 \backslash \overline{D}, \\ u(x) = g(x) & \forall\, x \in \partial D, \\ \lim_{r \longrightarrow +\infty} r(\partial_r u - iku) = 0, & \end{cases}$$

where g is a given continuous function on ∂D.

Theorem 3.2. [6] The exterior Dirichlet problem has at most one solution.

Lemma 3.3. Let $u \in C^2(\mathbb{R}^3\backslash\overline{D}) \cap C(\mathbb{R}^3\backslash D)$ be a solution to the Helmholtz equation in $\mathbb{R}^3\backslash\overline{D}$ which satisfies the homogeneous boundary condition $u = 0$ on ∂D. Define $D_R = \{y \in \mathbb{R}^3\backslash\overline{D}: \|y\| < R\}$ and $S_R = \{y \in \mathbb{R}^3\backslash\overline{D}: \|y\| = R\}$ for sufficiently large R. Then $\nabla u \in L^2(D_R)$ and

$$\int_{D_R} |\nabla u|^2\, dx - k^2 \int_{D_R} |u|^2\, dx = \int_{S_R} u\partial_\eta \overline{u}\; ds. \tag{50}$$

Proof. We first assume that u is real valued. We choose an odd function $\varphi \in C^1(\mathbb{R})$ such that $\varphi(t) = 0$ for $0 \le t \le 1$, $\varphi(t) = t$ for $t \ge 2$ and $\varphi'(t) \ge 0$ for all t, and set $u_n = \frac{\varphi(nu)}{n}$. we then have uniform convergence $\|u - u_n\|_\infty \longrightarrow 0,\ n \longrightarrow \infty$. Since $u = 0$ on the boundary ∂D, the function u_n vanish in a neighborhood of ∂D and we apply Green's theorem (1) to obtain

$$\int_{D_R} \nabla u_n . \nabla u\, dx = k^2 \int_{D_R} u_n\, u\, dx + \int_{S_R} u_n\, \partial_\eta u\, ds. \tag{51}$$

It can be easily seen that

$$0 \le \nabla u_n(x).\nabla u(x) = \varphi'\big(nu(x)\big)|\nabla u(x)|^2 \to |\nabla u(x)|^2, \quad n \to +\infty, \tag{52}$$

for all x not contained in $\{x \in D_R : u(x) = 0,\ \nabla \mathrm{u}(\mathrm{x}) \ne 0\}$. Since as a consequence of the implicit function theorem the latter set has Lebesgue measure zeros, Fatou's lemma tells us that $\nabla u \in L^2(D_R)$.

Now assume $u = v + iw$ with real functions v and w. Then, since v and w also satisfy the assumptions of our lemma, we have $\nabla v, \nabla w \in L^2(D_R)$. From

$$\nabla v_n + i\nabla w_n = \varphi'(nv)\nabla v + i\varphi'(nw)\,\nabla w \tag{53}$$

we can estimate

$$|(\nabla v_n + i\nabla w_n).\nabla\overline{u}| \le 2\|\varphi'\|_\infty \{|\nabla \mathrm{v}|^2 + |\nabla \mathrm{w}|^2\}. \tag{54}$$

Hence, by the Lebesgue dominated convergence theorem, we can pass to the limit $n \to +\infty$ in Green's theorem

$$\int_{D_R} \{(\nabla v_n + i\nabla w_n).\nabla\overline{u} + (v_n + iw_n)\Delta\overline{u}\}\, dx = \int_{S_R} (v_n + iw_n)\, \partial_\eta \overline{u}\, ds \tag{55}$$

to obtain (50). □

Theorem 3.4. [6] Let $0 < \alpha < \beta \le 1$ and let G be compact. Then the imbedding operators

$$I^\beta : C^{0,\beta}(G) \to C(G), \qquad I^{\alpha,\beta} : C^{0,\beta}(G) \to C^{0,\alpha}(G)$$

are compact.

Theorem 3.5. [6] Let ∂D be of class C^2. Then the single- and double-layer operators M and K, given by

$$(M\mu)(x) = 2\int_{\partial D} \psi(x,y)\,\mu(y)\,ds(y), \qquad x \in \partial D,$$

$$(K\mu)(x) = 2\int_{\partial D} \partial_{\eta(y)}\psi(x,y)\,\mu(y)\,ds(y), \quad x \in \partial D,$$

are bounded operators from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$, the operators M and K are also bounded from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$. The normal derivative operators K', T given by

$$(K'\mu)(x) = 2\int_{\partial D} \partial_{\eta(x)}\psi(x,y)\,\mu(y)\,ds(y), \quad x \in \partial D,$$

$$(T\mu)(x) = 2\partial_{\eta(x)}\left(\int_{\partial D} \partial_{\eta(y)}\psi(x,y)\,\mu(y)\,ds(y)\right), \quad x \in \partial D,$$

where operator T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$.

The existence of a solution to the exterior Dirichlet problem can be based on boundary integral equations. In the so-called layer approach, we seek the solution in the form of acoustic surface potentials. Here, we choose an approach in the form of a combined acoustic double- and single-layer potential

$$u(x) = \int_{\partial D} \left\{\frac{\partial\psi(x,y)}{\partial\eta(y)} - i\beta\psi(x,y)\right\}\mu(y)ds(y), \quad x \in \mathbb{R}^3\backslash\partial D, \tag{56}$$

With a density $\mu \in C(\partial D)$ and a real coupling parameter $\beta \neq 0$. Then from the jump relations of theorem 3.1 given by Colton and Kress [6]. We see that the potential u given by (56) in $\mathbb{R}^3\backslash\overline{D}$ solves the exterior Dirichlet problem provided the density is a solution of the integral equation

$$(I + K - i\beta M)\mu = 2g. \tag{57}$$

Combining Theorems 3.4. and 3.5. given by Colton and Kress [6], the operators $M, K: C(\partial D) \to C(\partial D)$ are seen to be compact. Therefore, the existence of a solution to (57) can be established by the Riesz-Fredholm theory for equations of the second kind with a compact operator.

Let μ be a continuous solution to the homogeneous form of (57). Then the potential u given by (56) satisfies the homogeneous boundary condition $u_+ = 0$ on ∂D hence by the uniqueness for the exterior Dirichlet problem $u = 0$ in $\mathbb{R}^3\backslash\overline{D}$ follows. The jump relations given by Colton and Kress [6] in theorem 3.1 yield

$$-u_- = \mu, \qquad -\partial_\eta u_- = i\beta\mu \quad on\ \partial D. \tag{58}$$

Hence, using Green's theorem (1), we obtain

$$i\beta\int_{\partial D}|\mu|^2\ ds = \int_{\partial D}\overline{u}_-\,\partial_\eta u_-\ ds = \int_D \{|\nabla u|^2 - k^2|u|^2\}\ dx. \tag{59}$$

Taking the imaginary part of the last equation shows that $\mu = 0$. Thus, we have established uniqueness for the integral equation (57), i.e., injectivity of the operator $I + K - i\beta M: C(\partial D) \to C(\partial D)$. Therefore, by the Riesz-Fredholm theory, $I + K - i\beta M$ is bijective and the inverse $(I + K - i\beta M)^{-1}: C(\partial D) \to C(\partial D)$ is bounded. Hence, the inhomogeneous equation (57) possesses a solution and this solution depends continuously of g in the maximum norm. From the representation (56) of the solution as a combined double- and single-layer potential, with the aid of the regularity estimates in theorem 3.1 given by Colton and Kress [6], the continuous dependence of the density μ on the boundary data g shows that the exterior Dirichlet problem is well-posed, i.e., small deviations in g in the maximum norm ensure small deviations in u in the maximum norm on $\mathbb{R}^3\backslash D$ and small deviations of all its derivatives in the maximum norm on closed subsets of $\mathbb{R}^3\backslash\overline{D}$.

We summarize these results in the following theorem

Theorem 3.6. The exterior Dirichlet problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3\backslash D$ and all its derivatives on closed subsets of $\mathbb{R}^3\backslash\overline{D}$.

Note that for $\beta = 0$ the integral equation (57) becomes non-unique if k is a so-called irregular wave number or internal resonance, i.e., if there exist nontrivial solutions u to the Helmholtz equation in the interior domain Dsatisfying homogeneous Neumann boundary conditions $\partial_\eta u = 0$ on ∂D.

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References

[1] S. Amini and D. T. Wilton, An investigation of boundary element methogs for the exterior acoustic problem, *Comput. Methods Appl. Mech. Engrg*., 54 (1986), pp.49-65.

[2] K. E. Atkinson and G. Chandler, The collocation method for solving the radiosity equation for unoccluded surfaces, *J. Integral Equations App*., 10 (1998), pp. 253-290.

[3] K. E. Atkinson and D. Chien, Piecewise polynomial collocation for boundary integral equations, *SIAM J. Sci. Comput.*, 16 (1995), pp. 651-681.

[4] H. Barucq, J. Diaz and S. Tordeaux, Introduction a l'analyse mathematique de l'equation de Helmholtz, Projet magique 3D, INRIA Bordeaux sud- oust (2012).

[5] D. Colton and R. Kress, Integral Equation Methods in Scattering Theory, John Wiley & Sons, New York, (1983).

[6] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed., Springer-Verlag, New York, (1998).

[7] B. Gieseke, Zum Dirichletschen prinzip fur selbstadjungierte elliptische differentialoperatoren. Math. Z. 68, 54-62 (1964).

[8] P. Hahner, scattering by media. In Scattering (Pike and Sabatier, eds). *Academic Press*, new York, 74-94 (2002).

[9] L. M. Levine, A uniqueness theorem for the reduced wave equation. *Comm. Pure Appl. Math*. 17, 147-176 (1964)

[10] P.A. Martin, Multiple Scattering, Interaction of Time-Harmonic Waves with N obstacles, Encyclopedia of Mathematics and its Appliction 107, Cambridge, (2006).

[11] F. Rellich, Uber das asymptotische verhalten der Losungen van $\Delta u + \lambda u = 0$, in *unendlichen Gebieten. Jber. Deut sch. Math. Verein*. 53, 57-65(1943).