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Estimates of Fekete-Szegö Functional of a Subclass of Analytic and Bi-Univalent Functions by Means of Chebyshev Polynomials

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Abstract

In this work, a new subclass $C_\Phi(\gamma, \sigma, n, t)$ of analytic and bi-univalent functions is defined by subordination principle and investigated. The initial coefficient bounds and the upper estimates of the Fekete-Szegö functional were obtained using Chebyshev polynomials.

Keywords

Analytic functions, Salagean differential operator and Chebyshev polynomial.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and satisfy the condition $f(0) = 0$ and $f^{'}(0) = 1$. Let S denote the subclass of A consisting of functions univalent in E. A funtion $f(z) \in S$ is said to be starlike in the unit disk if and only if

$$Re\frac{zf'(z)}{f(z)} > 0, \qquad z \in E \tag{1.2}$$

Also, a function $f(z) \in S$ is said to be convex in the unit disk if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in E \tag{1.3}$$

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A function $f(z)$ is subordinate to $g(z)$ in E, written as

$$f(z) \prec g(z), \quad z \in E$$

if there exists a Schwarz function $\omega(z)$, analytic in E with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \ , \quad z \in E$$

such that

$$f(z) = g(\omega(z)), \quad z \in E$$

If the function g is univalent in E,

$$f(z) \prec g(z) \Rightarrow f(0) = g(0)$$

and

$$f(E) \subset g(E)$$

It is well known (see Duren [6]) that every function $f \in S$ has an inverse map f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in E$ and $f(f^{-1}(\omega)) = \omega$, $(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \tag{1.4}$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E.
Let Φ denote the class of bi-univalent functions defined in the unit disk E. Lewin [9] for $f(z)$ of the form (1.1) showed that $|a_2| < 1.51$ for every $f \in \Phi$. Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for bi-starlike functions and $|a_2| \leq 1$ for bi-convex functions.
Also, Brannan and Taha [4] introduced certain subclasses of bi-univalent functions called bi-starlike function of order α denoted by $S_\Phi(\alpha)$ and bi-convex function of order α denoted by $C_\Phi(\alpha)$ corresponding to the classes of functions $S^*(\alpha)$ and $C(\alpha)$ respectively. For a further historical account of functions in the class Φ,(see [2],[5], [13] and [14]).
Let $D^n : A \to A$ be defined by

$$D^0f(z) = f(z)$$
$$D^1f(z) = zf'(z)$$
$$D^nf(z) = z[D^{n-1}f(z)]'$$

This is referred to as Salagean differential operator [12]. Fekete and Szegö [7] introduced the generalised functional $a_3 - \lambda a_2^2$, where λ is a real number. Keogh and Merkes [8] studied the Fekete-Szegö problem for the classes S^* and K. Ma and Minda [10] bring together various subclasses of starlike and convex functions for which either the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ has positive real part in the unit disk. The results of Ali et al. [1] generalized the results of Brannan and Taha [4] using subordination.
Chebyshev polynomials plays a considerable role in numerical analysis and mathematical physics. It is well-known that the Chebyshev polynomials are of four kinds, but in this paper the second kind shall be considered.

The second kind of Chebyshev polynomials is defined as

$$U_n(t) = \frac{\sin(n+1)\lambda}{\sin\lambda} \tag{1.5}$$

$U_n(t)$ satisfies the recurrence relation

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \quad n \geq 2$$

Few examples of Chebyshev polynomials of the second kind are

$$U_0(t) = 1, \quad U_1(t) = 2t, U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \cdots [11] \tag{1.6}$$

The generating function for $U_n(t)$ is given by

$$H(z,t) = \frac{1}{1-2tz+z^2} = \sum_{n=0}^{\infty} U_n(t) z^n, \quad z \in E \tag{1.7}$$

For $\gamma \in [0,1]$, $\sigma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2}, 1)$, a function $f(z) \in \Phi$ given by (1.1) is said to be in the class $C_\Phi(\gamma, \sigma, n, t)$ if the following subordination holds for all $z, \omega \in E$

$$\left(1 - e^{-2i\sigma}\gamma^2 z^2\right) \frac{D^{n+1} f(z)}{z} \prec H(z,t) = \frac{1}{1-2tz+z^2} \tag{1.8}$$

and

$$\left(1 - e^{-2i\sigma}\gamma^2 \omega^2\right) \frac{D^{n+1} g(\omega)}{\omega} \prec H(\omega,t) = \frac{1}{1-2t\omega+\omega^2} \tag{1.9}$$

where $g = f^{-1}$ is defined by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots$$

If $t = \cos\lambda$ where $\lambda \in \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ then,

$$H(z,t) = \frac{1}{1-2\cos\lambda z+z^2} = \left(1 - 2\cos\lambda z + z^2\right)^{-1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\lambda}{\sin\lambda} z^n, \quad z \in E$$

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots \tag{1.10}$$

Also,

$$H(\omega,t) = U_1(t)\omega + U_2(t)\omega^2 + U_3(t)\omega^3 + \cdots \tag{1.11}$$

2 Preliminary Lemmas

We need the following lemma to prove our results.

Let P denote the class of Caratheodory functions. $p(z) = 1 + p_1z + p_2z^2 + 3_3z^3 + ...$ $\quad (z \in E)$ which are analytic and satisfy $p(0) = 1$ and $\Re p(z) > 0$ Let $p \in P$. Then

$$|p_k| \leq 2 \quad (k \in \mathbb{N}) \quad [6] \tag{2.1}$$

3 Main Results

Let $f(z) \in C_\Phi(\gamma, \sigma, n, t)$, $\gamma \in [0,1]$, $\sigma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2}, 1)$. Then

$$|a_2| \leq \sqrt{\frac{t^2(2t+\gamma^2)}{|3^{n+1} \cdot t^2 - 2^{2n}(4t^2-1)|}} \tag{3.1}$$

$$|a_3| \leq \frac{t^2}{2^{2n}} + \frac{2t}{3^{n+1}} \tag{3.2}$$

Proof
Let the function $f(z) \in \Phi$ given by (1.1) be in the class $C_\Phi(\gamma, \sigma, n, t)$.
From (1.8) and (1.9)

$$\left(1 - e^{-2i\sigma}\gamma^2 z^2\right) \frac{D^{n+1}f(z)}{z} = 1 + U_1(t)r(z) + U_2(t)r^2(z) + \cdots \tag{3.3}$$

and

$$\left(1 - e^{-2i\sigma}\gamma^2 \omega^2\right) \frac{D^{n+1}g(\omega)}{\omega} = 1 + U_1(t)s(z) + U_2(t)s^2(\omega) + \cdots \tag{3.4}$$

For some analytic functions $r(z)$ and $s(\omega)$

$$r(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad z \in E \tag{3.5}$$

$$s(\omega) = d_1 \omega + d_2 \omega^2 + d_3 \omega^3 + \cdots \qquad z \in E \tag{3.6}$$

such that $r(0) = s(0) = 0, \quad |r(z)| < 1$ and $|s(\omega)| < 1$. It is well known that if $|r(z)| < 1$ and $|s(\omega)| < 1$ then

$$|c_j| \leq 1 \tag{3.7}$$

and

$$|d_j| \leq 1 \tag{3.8}$$

But,

$$\left(1 - e^{-2i\sigma}\gamma^2 z^2\right) \frac{D^{n+1}f(z)}{z} = 1 + 2^{n+1}a_2 z + \left(3^{n+1}a_3 - e^{-2i\sigma}\gamma^2\right) z^2 + \left(4^{n+1}a_4 - 2^{n+1}e^{-2i\sigma}\gamma^2 a_2\right) z^3 + \cdots \tag{3.9}$$

From (3.3),3.4),(3.5) and (3.6)

$$\left(1 - e^{-2i\sigma}\gamma^2 z^2\right) \frac{D^{n+1}f(z)}{z} = 1 + U_1(t)c_1 z + \left[U_1(t)c_2 + U_2(t)c_1^2\right] z^2 + \left[U_1(t)c_3 + U_2(t)2c_1c_2 + c_1^3\right] z^3 + \cdots \tag{3.10}$$

$$\left(1 - e^{-2i\sigma}\gamma^2 \omega^2\right) \frac{D^{n+1}g(\omega)}{\omega} = 1 + U_1(t)d_1 \omega + \left[U_1(t)d_2 + U_2(t)d_1^2\right] \omega^2 + \left[U_1(t)d_3 + U_2(t)2d_1d_2 + d_1^3\right] \omega^3 + \cdots \tag{3.11}$$

Equating coefficients in (3.9) and (3.10) give

$$2^{n+1}a_2 z = U_1(t)c_1 \tag{3.12}$$

$$3^{n+1}a_3 - e^{-2i\sigma}\gamma^2 = U_1(t)c_2 + U_2(t)c_1^2 \tag{3.13}$$

and

$$-2^{n+1}a_2 = U_1(t)d_1 \tag{3.14}$$

$$3^{n+1}(2a_2^2 - a_3) - e^{-2i\sigma}\gamma^2 = U_1(t)d_2 + U_2(t)d_1^2 \tag{3.15}$$

Adding (3.12) and (3.14) give

$$2^{n+1}a_2 + \left(-2^{n+1}a_2\right) = U_1(t)c_1 + U_1(t)d_1 \tag{3.16}$$

$$\Rightarrow c_1 = -d_1 \tag{3.17}$$

$$c_1^2 = d_1^2 \tag{3.18}$$

Squaring (3.12), (3.14) and adding the new equations and simplifying give

$$a_2^2 = \frac{U_1(t)(c_1^2 + d_1^2)}{2^{2n+3}} \tag{3.19}$$

Also, adding (3.13),(3.15) and simplifying give

$$a_2^2 = \frac{U_1^3(t)(c_2 + d_2) + 2U_1^2(t)e^{-2i\sigma}\gamma^2}{2 \cdot 3^{n+1}U_1^2(t) - 2^{2n+3}U_2(t)} \tag{3.20}$$

Applying (1.6) in (3.20) and simplifying gives

$$|a_2| \leq \sqrt{\frac{t^2(2t + \gamma^2)}{|3^{n+1} \cdot t^2 - 2^{2n}(4t^2 - 1)|}} \tag{3.21}$$

Subtracting (3.15) from (3.13) gives

$$a_3 = a_2^2 + \frac{U_1(t)(c_2 - d_2)}{2 \cdot 3^{n+1}} \tag{3.22}$$

putting (3.19) in (3.22) gives

$$a_3 = \frac{U_1^2(t)(c_1^2 + d_1^2)}{2^{2n+3}} + \frac{U_1(t)(c_2 - d_2)}{2 \cdot 3^{n+1}} \tag{3.23}$$

Applying (1.6),(3.6),(3.7) in (3.23) gives

$$|a_3| \leq \frac{t^2}{2^{2n}} + \frac{2t}{3^{n+1}} \tag{3.24}$$

which coincides with the result in [5].
Remark. For $n = 0$ and $\gamma = 1$, the following consequence is given as follow
Corollary 1. For $t \in (\frac{1}{2}, 1)$. Let the function $f \in \Phi$ given by (1.1) be in the class $C_\Phi(\gamma, \sigma, n, t)$.Then

$$|a_2| \leq \frac{t\sqrt{2t + 1}}{\sqrt{|1 - t^2|}} \tag{3.25}$$

Corollary 2. For $n = 0$

$$|a_3| \leq t^2 + \frac{2t}{3} \tag{3.26}$$

which coincides with the result in [5]. Let $f(z) \in C_\Phi(\gamma, \sigma, n, t)$, $\gamma \in [0, 1]$, $\sigma\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2}, 1)$. Then for any real number μ

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 2|1-\mu|\frac{(2t^3+t^2\gamma^2)}{|3^{n+1}\cdot t^2-2^{2n}(4t^2-1)|} & if \quad |1-u|\frac{(2t^3+t^2\gamma^2)}{|3^{n+1}t^2-2^{2n}(4t^2-1)|} \geq \frac{2t}{3^{n+1}} \\ \frac{4t}{3^{n+1}} & if \quad |1-\mu|\frac{(2t^3+t^2\gamma^2)}{|3^{n+1}t^2-2^{2n}(4t^2)|} \leq \frac{2t}{3^{n+1}} \end{cases}$$

Proof. Substituting (3.22)

$$a_3 - \mu a_2^2 = a_2^2 + \frac{U_1(t)(c_2 - d_2)}{2 \cdot 3^{n+1}} - \mu a_2^2$$

Applying (3.20) and simplifying further give

$$a_3 - \mu a_2^2 = \frac{U_1(t)(c_2 - d_2)}{2 \cdot 3^{n+1}} + (1 - \mu)\frac{\left[U_1^3(t)(c_2 + d_2) + 2U_1^2(t)e^{-2i\sigma}\gamma^2\right]}{2 \cdot 3^{n+1}U_1^2(t) - 2^{2n+3}U_2(t)} \tag{3.27}$$

Applying (1.6) in (3.27) gives

$$|a_3 - \mu a_2^2| \leq \frac{2t}{3^{n+1}} + |1 - \mu|\frac{t^2(2t + \gamma^2)}{|3^{n+1} \cdot t^2 - 2^{2n}(4t^2 - 1)|} \tag{3.28}$$

If

$$|1-\mu|\frac{t^2(2t+\gamma^2)}{|3^{n+1}\cdot t^2-2^{2n}(4t^2-1)|}\geq\frac{2t}{3^{n+1}}$$

then

$$|a_3-\mu a_2^2|\leq 2|1-\mu|\frac{t^2(2t+\gamma^2)}{|3^{n+1}\cdot t^2-2^{2n}(4t^2-1)|} \tag{3.29}$$

Also, if

$$|1-\mu|\frac{t^2(2t+\gamma^2)}{|3^{n+1}\cdot t^2-2^{2n}(4t^2-1)|}\leq\frac{2t}{3^{n+1}}$$

then

$$|a_3-\mu a_2^2|\leq\frac{4t}{3^{n+1}} \tag{3.30}$$

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