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# On Curvatures of the Torus Hypersurface in 4-Space

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**Abstract.**

We study curvatures $\mathfrak{C}_{i=1,2,3}$ of torus hypersurface in the four dimensional Euclidean space. We also give some relations on $\mathfrak{C}_i$ of torus hypersurface.

**Keywords:** four space, torus hypersurface, curvatures.

## 1. Introduction

Surfaces and hypersurfaces have been worked by the mathematicians for over three centuries. We see some recent papers about torus surfaces and torus hypersurfaces in the literature such as [2–15].

In a way analogous to the construction of an ordinary torus in $\mathbb{E}^3$, Aminov [1] obtained the three dimensional submanifold $M^3$ in $\mathbb{E}^4$which is homeomorphic to $S^1 \times S^2$.

Assume $\gamma$ be acircle of radius $R$ with the center at the origin $O$ in a coordinate plane $\mathbb{E}^2$, and $P$ be a point of $\gamma$. Spanning $\mathbb{E}^3$ on vectors $OP$, $e_3$, $e_4$, we consider the sphere $S^2(P)$ of radius $r$ with the center at $P$. While $P$ moves along $\gamma$, then all points of $S^2(P)$ form the submanifold $M^3$ in $\mathbb{E}^4$.

Finally, a torus hypersurface in the four dimensional Euclidean space$\mathbb{E}^4$can be parametrized by the following form:

$$\mathbf{x}(u,v,w) = \begin{pmatrix} (R + r\cos u \cos v)\cos w \\ (R + r\cos u \cos v)\sin w \\ \mathrm{r}\cos u \sin v \\ \mathrm{r}\sin u \end{pmatrix} = \begin{pmatrix} \mathrm{x}_1(u,v,w) \\ \mathrm{x}_2(u,v,w) \\ \mathrm{x}_3(u,v,w) \\ \mathrm{x}_4(u,v,w) \end{pmatrix}, \tag{1.1}$$

where $u, v, w \in I \subset \mathbb{R}$.

In this paper, we compute curvatures of hypersurfaces in $\mathbb{E}^4$. We present fundamental elements of the four dimensional Euclidean geometry. In addition, we compute curvatures $\mathfrak{C}_{i=1,2,3}$ of torus hypersurface.

## 2. Preliminaries

For $i$-th curvature formulas $\mathfrak{C}_{i=0,1,\dots n}$ in $\mathbb{E}^{n+1}$, we get characteristic polynomial of shape operator **S**:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}. \tag{2.1}$$

Here, $I_n$ shows identity matrix. Then, we get curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$, where $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ by definition. $k$-th fundamental form of hypersurface $M^n$ is given by $\mathrm{I}\left(\mathbf{S}^{k-1}(X), Y\right) = \langle \mathbf{S}^{k-1}(X), Y \rangle$. Then, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}\left(\mathbf{S}^{k-1}(X), Y\right) = 0. \tag{2.2}$$

Throughout the work, we will identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose.

One can assume $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface $M^3$ in $\mathbb{E}^4$. Dot product of $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is defined by $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{4} x_i y_i$. Vector product in $\mathbb{E}^4$ is given by

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface **M** is defined by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. We obtain following matrices for a hypersurface **M** in $\mathbb{E}^4$,

$$\mathrm{I}=\begin{pmatrix}E & F & A\\ F & G & B\\ A & B & C\end{pmatrix},\ \mathrm{II}=\begin{pmatrix}L & M & P\\ M & N & T\\ P & T & V\end{pmatrix},\ \mathrm{III}=\begin{pmatrix}X & Y & O\\ Y & Z & R\\ O & R & S\end{pmatrix},$$

where the coefficients are given by

$$E=\langle \mathbf{M}_u,\mathbf{M}_u\rangle,\ F=\langle \mathbf{M}_u,\mathbf{M}_{\boldsymbol{v}}\rangle,\ G=\langle \mathbf{M}_v,\mathbf{M}_v\rangle,\ A=\langle \mathbf{M}_u,\mathbf{M}_w\rangle,\ B=\langle \mathbf{M}_v,\mathbf{M}_{\boldsymbol{w}}\rangle,\ C=\langle \mathbf{M}_w,\mathbf{M}_w\rangle,$$

$$L=\langle \mathbf{M}_{uu},e\rangle,\ M=\langle \mathbf{M}_{uv},e\rangle,\ N=\langle \mathbf{M}_{vv},e\rangle,\ P=\langle \mathbf{M}_{uw},e\rangle,\ T=\langle \mathbf{M}_{vw},e\rangle,\ V=\langle \mathbf{M}_{ww},e\rangle,$$

$$X=\langle e_u,e_u\rangle,\ Y=\langle e_u,e_{\boldsymbol{v}}\rangle,\ Z=\langle e_v,e_{\boldsymbol{v}}\rangle,\ O=\langle e_u,e_{\boldsymbol{w}}\rangle,\ R=\langle e_v,e_{\boldsymbol{w}}\rangle,\ S=\langle e_w,e_{\boldsymbol{w}}\rangle.$$

## 3. Curvatures

Next, we will obtain curvatures for a hypersurface $\mathbf{M}(u,v,w)$ in $\mathbb{E}^4$. Using characteristic polynomial $P_{\mathbf{S}}(\lambda)=a\lambda^3+b\lambda^2+c\lambda+d=0$, we obtain curvature formulas: $\mathfrak{C}_0=1$(by definition),

$$\binom{3}{1}\mathfrak{C}_1=-\frac{b}{a},\ \binom{3}{2}\mathfrak{C}_2=\frac{c}{a},\ \binom{3}{3}\mathfrak{C}_3=-\frac{d}{a}.$$

Then, we see curvature folmulas, clearly:

**Theorem 3.1.** *Any hypersurface* $\mathrm{M}^3$ *in* $\mathbb{E}^4$ *has following curvatureformulas,* $\mathfrak{C}_0=1$*(by definition),*

$$\mathfrak{C}_1=\frac{(EN+GL-2FM)C+(EG-F^2)V-LB^2-NA^2-2(APG-BPF-ATF+BTE-ABM)}{3[(EG-F^2)C-EB^2+2FAB-GA^2]},\tag{3.1}$$

$$\mathfrak{C}_2=\frac{(EN+GL-2FM)V+(LN-M^2)C-ET^2-GP^2-2(APN-BPM-ATM+BTL-PTF)}{3[(EG-F^2)C-EB^2+2FAB-GA^2]},\tag{3.2}$$

$$\mathfrak{C}_3=\frac{(LN-M^2)V-LT^2+2MPT-NP^2}{(EG-F^2)C-EB^2+2FAB-GA^2}.\tag{3.3}$$

Proof. Solving $\det(\mathbf{S}-\lambda I_3)=0$ with some calculations, we get coefficients of polynomial $P_{\mathbf{S}}(\lambda)$.

**Theorem3.2.** For *any hypersurface* $\mathrm{M}^3$ *in* $\mathbb{E}^4$, *curvatures are related by following formula*

$$\mathfrak{C}_0 IV - 3\mathfrak{C}_1 III + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 I = 0. \tag{3.4}$$

## 4. Curvaturesof Torus Hypersurface

In this section, we compute curvatures of torus hypersurface (1.1).

With the first differentials of(1.1)depends on$u, v, w$, we get the Gauss map of (1.1):

$$e = -\begin{pmatrix} \cos u \cos v \cos w \\ \cos u \cos v \sin w \\ \cos u \sin v \\ \sin u \end{pmatrix}. \tag{4.1}$$

We get the first and the second fundamental form matrices of (1.1), respectively,

$$\mathrm{I} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2 \cos^2 u & 0 \\ 0 & 0 & (R + r\cos u \cos v)^2 \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} r & 0 & 0 \\ 0 & r\cos^2 u & 0 \\ 0 & 0 & (R + r\cos u \cos v)\cos u \cos v \end{pmatrix}.$$

*Using* $\mathrm{I}^{-1}II$, *torus hypersurface* (1.1)*in*$\mathbb{E}^4$ *has following shape operator*

$$\mathbf{S} = \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{\cos u \cos v}{R + r\cos u \cos v} \end{pmatrix}.$$

So, we compute the third fundamental form matrix using (4.1) of (1.1):

$$\mathrm{III} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2 u & 0 \\ 0 & 0 & \cos^2 u \ \cos^2 v \end{pmatrix}.$$

**Theorem 4.1.** *Torus hypersurface*(1.1) *in* $\mathbb{E}^4$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$*(by definition),*

$$\mathfrak{C}_1 = \frac{2R + 3r\cos u\cos v}{3r\,(R + r\cos u\cos v)},$$

$$\mathfrak{C}_2 = \frac{R + 3r\cos u\cos v}{r^2(R + r\cos u\cos v)},$$

$$\mathfrak{C}_3 = \frac{\cos u\cos v}{r^2(R + r\cos u\cos v)}.$$

Proof. Computing(3.1),(3.2), and (3.3)of(1.1), we obtain the curvatures.

**Corollary 4.1.** T*orus hypersurface* (1.1) *has following curvature relations*

$$\frac{3\mathfrak{C}_1}{2R + 3\mathcal{C}r} = \frac{3r\mathfrak{C}_2}{R + 3\mathcal{C}r} = \frac{r\mathfrak{C}_3}{\mathcal{C}},$$

*where* $\mathcal{C} = \cos u\cos v$.

**Corollary 4.2.** $u = v = \pi/2 + k\pi$ *on torus hypersurface* (1.1)*if and onlyif the curvatures of* (1.1) *are as follows*

$$\mathfrak{C}_1 = \frac{2}{3r},\quad \mathfrak{C}_2 = \frac{1}{r^2},\quad \mathfrak{C}_3 = 0,$$

*i.e. hypersurface* (1.1) *is* 3*-minimal torus hypersurface.*

## 5. Conclusion

Torus hypersurfaces have been studied by some authors. We have expanded well-known results of the Torus hypersurfaces by using its curvatures in $\mathbb{E}^4$. In addition, we give 3-minimality condition of it.

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