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Some identities involving the λ -Daehee numbers and polynomials

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Abstract. In this paper, we introduce the λ -Daehee numbers and polynomials of higher order. And we obtain some properties of this numbers and polynomials. In addition, we explore some new equalities and relations involving λ -Daehee numbers and polynomials.

Keywords: λ -Daehee polynomials; Various Daehee polynomials; Generating functions; The Bernoulli numbers and polynomials; Cauchy numbers; Four kinds of stirling numbers.

1. Introduction

Let $k \in \mathbb{N}$, $x \in \mathbb{R}$, the High-Daehee polynomials are defined by the following generating function

$$\left(\frac{ln(1+t)}{t}\right)^k (1+t)^x = \sum_{n=0}^{\infty} D_n^{(k)}(x)\frac{t^n}{n!}. \quad (see[1,3]) \tag{1}$$

When $x=0$ in (1), $D_n^{(k)} = D_n^{(k)}(0)$ are called the High-Daehee numbers.

When $k=1$, we can obtain Daehee polynomials.

The degenerate Daehee polynomials are given by the generating function to be

$$\frac{ln(1+t)}{ln(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x = \sum_{n=0}^{\infty} \mathrm{d}_n(x|\lambda)\frac{t^n}{n!}. \quad (see[2]) \tag{2}$$

When $x=0$, $d_n(\lambda) = d_n(0|\lambda)$ are called the degenerate Daehee numbers.

The partially degenerate Daehee polynomials are defined by

$$\frac{(1+t)^{\lambda}-1}{\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \tilde{d}_n(x|\lambda)\frac{t^n}{n!}. \quad (see[2]) \tag{3}$$

When $x=0$, $\tilde{d}_n(\lambda) = \tilde{d}_n(0|\lambda)$ are called the partially degenerate Daehee numbers.

The totally degenerate Daehee polynomials are defined by the following generating function

$$\frac{(1+t)^{\lambda}-1}{ln(1+\lambda t)}(1+t)^x = \sum_{n=0}^{\infty} d_n^*(x|\lambda)\frac{t^n}{n!}. \quad (see[2]) \tag{4}$$

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When $x = 0$, $d_n^*(\lambda) = d_n^*(0|\lambda)$ are called the totally degenerate Daehee numbers.

The high λ-Daehee polynomials of the second kind are defined by the following generating function

$$\left(\frac{\lambda ln(1+t)}{(1+t)^\lambda - 1}\right)^k (1+t)^{\lambda k + x} = \sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \quad (see[1]) \tag{5}$$

The high cauchy polynomials of the first kind are given by the generating function

$$\left(\frac{t}{ln(1+t)}\right)^k (1+t)^x = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!}. \quad (see[3,4]) \tag{6}$$

when $x = 0$, $k = 1$, $C_n = C_n(0)$ are called the cauchy numbers.

The high cauchy polynomials of the second kind are given by the generating function

$$\left(\frac{t}{(1+t)ln(1+t)}\right)^k (1+t)^x = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x) \frac{t^n}{n!}. \quad ([4]) \tag{7}$$

when $x = 0$, $k = 1$, $\hat{C}_n = \hat{C}_n(0)$ are called the cauchy numbers of the second kind.

The Bernoulli polynomils are given by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad ([5,6,7]) \tag{8}$$

The high degenerate Bernoulli numbers of the second are given by the generating function

$$\left(\frac{\lambda t}{(1+t)^\lambda - 1}\right)^k (1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \quad ([6]) \tag{9}$$

The partially degenerate Bernoulli polynomils of the first kind which are given by the generating function to be

$$\frac{ln(1+\lambda t)^{\frac{1}{\lambda}}}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}. \quad ([7]) \tag{10}$$

The λ-Changhee Genocchi polynomials are given by the generating function to be

$$\frac{2ln(1+t)}{(1+t)^\lambda + 1} (1+t)^{\lambda x} = \sum_{n=0}^{\infty} CG_{n,\lambda}(x) \frac{t^n}{n!}. \quad ([8]) \tag{11}$$

The classcial Harmonic numbers are given by the generating function

$$\frac{-ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n t^n. \quad ([9]) \tag{12}$$

The Lah numbers are given by the generating function

$$\frac{\left(\frac{-t}{1+t}\right)^k}{k!} = \sum_{n \geq k} L(n,k) \frac{t^n}{n!}. \quad ([10]) \tag{13}$$

The stirling numbers of first kind and two kind are defined by

$$\frac{ln^k(1+t)}{k!}\sum_{n\geq k}s(n,k)\frac{t^n}{n!}. \quad ([11]) \tag{14}$$

$$\frac{(e^t-1)^k}{k!}=\sum_{n\geq k}S(n,k)\frac{t^n}{n!}. \quad ([11]) \tag{15}$$

The degenerate stirling numbers of first kind and two kind are defined by

$$\frac{(\frac{1}{\lambda}((1+t)^\lambda-1))^k}{k!}=\sum_{n\geq k}s_{1,\lambda}(n,k)\frac{t^n}{n!}. \quad ([12]) \tag{16}$$

$$\frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^k}{k!}=\sum_{n\geq k}S_{2,\lambda}(n,k)\frac{t^n}{n!}. \quad ([12]) \tag{17}$$

2. Properties of λ-Daehee numbers and polynomials

In this section, we give the definition of λ-Daehee numbers and polynomials and some properties of them.

The λ-Daehee polynomials are given by

$$\frac{\lambda ln(1+t)}{(1+t)^\lambda-1}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{18}$$

when $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the λ-Daehee numbers.

Theorem 2.1 Suppose $n_i\geq 0$, $k_i\geq 0$, $i\in[m]$, $m\geq 1$, there are properties about λ-Daehee polynomials $D_{n,\lambda}(x)$ as follows

$$\sum_{n_1+n_2+\cdots n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}D_{n_1,\lambda}^{(k_1)}(x_1)D_{n_2,\lambda}^{(k_2)}(x_2)\cdots D_{n_m,\lambda}^{(k_m)}(x_m)=D_{n,\lambda}^{(k_1+\cdots+k_m)}(x_1+\cdots+x_m). \tag{19}$$

Proof By (18), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}D_{n,\lambda}^{(k_1+\cdots+k_m)}(x_1+\cdots+x_m)\frac{t^n}{n!}\\
&=\left(\frac{\lambda ln(1+t)}{(1+t)^\lambda-1})\right)^{(k_1+k_2+\cdots+k_m)}(1+t)^{x_1+x_2+\cdots+x_m}\\
&=\sum_{n_1=0}^{\infty}D_{n_1}^{(k_1)}(x_1)\frac{t^{n_1}}{n_1!}\cdots\sum_{n_m=0}^{\infty}D_{n_m}^{(k_m)}(x_m)\frac{t^{n_m}}{n_m!}\\
&=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}D_{n_1,\lambda}^{(k_1)}(x_1)\cdots D_{n_m,\lambda}^{(k_m)}(x_m)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get the identities (19).

Corollary 2.1 For $x_1 = \cdots = x_m = 0$ in (19), we obtain the following identities

$$\sum_{n_1+n_2+\cdots+n_m=n} \binom{n}{n_1, n_2, \cdots, n_m} D_{n_1,\lambda}^{(k_1)} D_{n_2,\lambda}^{(k_2)} \cdots D_{n_m,\lambda}^{(k_m)} = D_{n,\lambda}^{(k_1+k_2\cdots+k_m)}. \tag{20}$$

Theorem 2.2 For $n \geq 0$, $\lambda \geq 1$, we have

$$\sum_{m=0}^{n} D_{m,\lambda}(x)(y)_{n-m} = D_{n,\lambda}(x+y). \tag{21}$$

Proof By the method of generating function, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} D_{m,\lambda}(x)(y)_{n-m}\frac{t^n}{n!} = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!}$$
$$= \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x(1+t)^y = \sum_{n=0}^{\infty} D_{n,\lambda}(x+y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.3 For $n \geq 0$, $\lambda \geq 1$, we have

$$D_{n,\lambda}(\lambda) - D_{n,\lambda} = \begin{cases} 0, & n = 0, \\ \lambda(-1)^{n-1}(n-1)!, & n \geqslant 1. \end{cases} \tag{22}$$

Proof On the one hand, we get

$$\lambda ln(1+t) = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}((1+t)^{\lambda}-1)$$
$$= \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^{\lambda} - \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}$$
$$= \sum_{n=0}^{\infty}(D_{n,\lambda}(\lambda) - D_{n,\lambda})\frac{t^n}{n!}.$$

On the other hand

$$\lambda ln(1+t) = \lambda\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\frac{t^n}{n!}.$$

Which completes the proof.

Theorem 2.4 For $n \geq 0$, $\lambda \geq 1$, we have

$$\int_0^1 D_{n,\lambda}(x)dx = b_{n,\lambda}. \tag{23}$$

Proof By the method of generating function, we get

$$\sum_{n=0}^{\infty}\int_0^1 D_{n,\lambda}(x)dx\frac{t^n}{n!} = \int_0^1\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}dx$$
$$= \int_0^1 \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x dx = \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}\frac{(1+t)^x|_0^1}{ln(1+t)}$$
$$= \frac{\lambda t}{(1+t)^{\lambda}-1} = \sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.5 For $n \geq k$, $\lambda \geq 1$, we have

$$\frac{1}{k!}(D_{n,\lambda}(x))^{(k)} = \sum_{m=0}^{n-k} D_{m,\lambda}(x) s(n-m,k). \tag{24}$$

Proof By the method of generating function, we get

$$\begin{aligned}
&\frac{1}{k!}\sum_{n=0}^{\infty}(D_{n,\lambda}(x))^{(k)}\frac{t^n}{n!} = \frac{1}{k!}(\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!})^{(k)}\\
=&\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x \frac{ln^k(1+t)}{k!}\\
=&\sum_{n=0}^{\infty} D_{n,\lambda}(x)\sum_{n\geq k} s(n,k)\frac{t^n}{n!}\\
=&\sum_{n\geq k}\sum_{m=0}^{n-k} D_{m,\lambda}(x)s(n-m,k)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

3. Identities about λ-Daehee numbers and polynomials

In this part, we derive some new equalities involving λ-Daehee numbers and polynomials

.

Theorem 3.1 For $n \geq 1$, we have

$$\sum_{m=0}^{n}\binom{n}{m} CG_{m,\lambda} b_{n-m,\lambda} + \frac{n\lambda}{2} CG_{n-1,\lambda} = nD_{n-1,\lambda}. \tag{25}$$

Proof On the one hand, we have

$$t\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1} = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} nD_{n-1,\lambda}\frac{t^n}{n!}.$$

On the other hand, we hane

$$\begin{aligned}
&t\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1} = \frac{1}{2}\frac{2ln(1+t)}{(1+t)^{\lambda}+1}\frac{\lambda t((1+t)^{\lambda}+1)}{(1+t)^{\lambda}-1}\\
=&\frac{1}{2}\sum_{n=0}^{\infty} CG_{n,\lambda}\frac{t^n}{n!}\lambda t(1+\frac{2}{(1+t)^{\lambda}-1})\\
=&\frac{1}{2}\sum_{n=0}^{\infty} CG_{n,\lambda}\frac{t^n}{n!}(\lambda t + 2\sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!})\\
=&\frac{\lambda}{2}\sum_{n=1}^{\infty} nCG_{n-1,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} CG_{m,\lambda} b_{n-m,\lambda}\frac{t^n}{n!}.
\end{aligned}$$

So, we gain

$$\sum_{n=1}^{\infty}(nD_{n-1,\lambda}-\frac{n\lambda}{2}CG_{n-1,\lambda})\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}CG_{m,\lambda}b_{n-m,\lambda}\frac{t^n}{n!}.$$

Which completes the proof.

Theorem 3.2 For $n\geq 0$, we have

$$\sum_{m=0}^{n}\lambda^m B_m^{(k)}(x)s(n,m)=D_{n,\lambda}^{(k)}(\lambda x). \tag{26}$$

$$\sum_{m=0}^{n}D_{m,\lambda}^{(k)}(\lambda x)S(n,m)=\lambda^n B_n^{(k)}(x). \tag{27}$$

Proof By the method of generating function, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^m B_m^{(k)}(x)s(n,m)\frac{t^n}{n!}=\sum_{m=0}^{\infty}\lambda^m B_m^{(k)}(x)\sum_{n\geq m}s(n,m)\frac{t^n}{n!}$$
$$=\sum_{m=0}^{\infty}B_m^{(k)}(x)\frac{\lambda^m ln^m(1+t)}{m!}=\left(\frac{\lambda ln(1+t)}{e^{\lambda ln(1+t)}-1}\right)^k e^{\lambda x ln(1+t)}$$
$$=\left(\frac{\lambda ln(1+t)}{(1+t)^\lambda-1}\right)^k(1+t)^{\lambda x}=\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(\lambda x)\frac{t^n}{n!}$$

.

Which completes the proof of (26), the same reasoning can be proved (27). In addition, we can consider equation (27) as the inversion formula for (26).

Theorem 3.3 For $n\geq 0$, we get

$$\sum_{m=0}^{n}D_m^{(k)}(x)\lambda^m s_{1,\lambda}(n,m)=D_{n,\lambda}^{(k)}(\lambda x). \tag{28}$$

$$\sum_{m=0}^{n}D_{m,\lambda}^{(k)}(\lambda x)S_{2,\lambda}(n,m)=D_n^{(k)}(x)\lambda^n. \tag{29}$$

Proof By the method of generating function, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}D_m^{(k)}(x)\lambda^m s_{1,\lambda}(n,m)\frac{t^n}{n!}$$
$$=\sum_{m=0}^{\infty}D_m^{(k)}(x)\lambda^m\sum_{n\geq m}s_{1,\lambda}(n,m)\frac{t^n}{n!}$$
$$=\sum_{m=0}^{\infty}D_m^{(k)}(x)\lambda^m\frac{(\frac{1}{\lambda}((1+t)^\lambda-1))^m}{m!}$$
$$=\frac{ln(1+((1+t)^\lambda-1))}{(1+t)^\lambda-1}(1+((1+t)^\lambda-1))^x$$
$$=\frac{\lambda ln(1+t)}{(1+t)^\lambda-1}(1+t)^{\lambda x}=\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(\lambda x)\frac{t^n}{n!}.$$

Which completes the proof of (28), the same reasoning can be proved (29). In addition, we can consider equation (29) as the inversion formula for (28).

Theorem 3.4 For $n \geq 0$, we get

$$\sum_{m=0}^{n-k} \binom{n}{m} D_{m,\lambda}^{(k)} s_{1,\lambda}(n-m,k) = s(n,k). \tag{30}$$

Proof By the method of generating function, we have

$$\begin{aligned}
&\sum_{n\geq k} \sum_{m=0}^{n-k} \binom{n}{m} D_{m,\lambda}^{(k)} s_{1,\lambda}(n-m,k) \frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty} D_{n,\lambda}^{(k)} \frac{t^n}{n!} \sum_{n\geq k} s_{1,\lambda}(n,k) \frac{t^n}{n!} \\
=&\left(\frac{ln(1+t)}{\frac{1}{\lambda}((1+t)^\lambda - 1)} \right)^k \frac{(\frac{1}{\lambda}((1+t)^\lambda - 1))^k}{k!} \\
=&\frac{ln^k(1+t)}{k!} = \sum_{n\geq k} s(n,k) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof.

Theorem 3.5 For $n \geq 0$, we get

$$\sum_{m=0}^{n-1} b_{m,\lambda} (-1)^{n-m-1} (n-m-1)! = nD_{n-1,\lambda}. \tag{31}$$

Proof By the method of generating function, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} b_{m,\lambda} (-1)^{n-m-1} (n-m-1)! \frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty} b_{n,\lambda} \frac{t^n}{n!} \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \frac{t^n}{n!} \\
=&\frac{\lambda t}{(1+t)^\lambda - 1} ln(1+t) = \frac{\lambda ln(1+t)}{(1+t)^\lambda - 1} t \\
=&\sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} nD_{n-1,\lambda} \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof.

Theorem 3.6 For $n \geq 0$, $d \in \mathbb{N}^+$, we have

$$\frac{1}{d} \sum_{\alpha=0}^{d-1} D_{n,\lambda d}(\alpha\lambda + x) = D_{n,\lambda}(x). \tag{32}$$

Proof By the method of generating function, we have

$$\begin{aligned}
&\frac{1}{d} \sum_{n=0}^{\infty} \sum_{\alpha=0}^{d-1} D_{n,\lambda d}(\alpha\lambda + x) \frac{t^n}{n!} = \frac{1}{d} \sum_{\alpha=0}^{d-1} \sum_{n=0}^{\infty} D_{n,\lambda d}(\alpha\lambda + x) \frac{t^n}{n!} \\
=&\frac{1}{d} \sum_{\alpha=0}^{d-1} \frac{\lambda d ln(1+t)}{(1+t)^{\lambda d} - 1} (1+t)^{\alpha\lambda + x} = \frac{\lambda ln(1+t)}{(1+t)^{\lambda d} - 1} \sum_{\alpha=0}^{d-1} (1+t)^{\alpha\lambda + x} \\
=&\frac{\lambda ln(1+t)}{(1+t)^{\lambda d} - 1} (1+t)^x \frac{1-(1+t)^{\lambda d}}{1-(1+t)^\lambda} \\
=&\frac{\lambda ln(1+t)}{(1+t)^\lambda - 1} (1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.7 For $n \geq 0$, we have

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(\lambda-1)_{m+1}}{(m+1)} D_{n-m,\lambda}(x) = D_n(x). \tag{33}$$

Proof By the method of generating function, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{(\lambda-1)_{m+1}}{(m+1)} D_{n-m,\lambda}(x) \frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty} \frac{(\lambda)_{n+1}}{\lambda(n+1)} \frac{t^n}{n!} \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!} \\
=&\frac{1}{\lambda t} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!} \\
=&\frac{(1+t)^{\lambda}-1}{\lambda t} \frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1} (1+t)^x \\
=&\frac{ln(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.8 For $n \geq 1$, we have

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(\lambda-1)_{m+1}}{(m+1)} D_{n-m,\lambda}(x) = (-1)^n (H_{n+1} - H_n) n!. \tag{34}$$

Proof By the method of generating function we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{-ln(1+t)}{1+t} \frac{1+t}{-t} \\
=&\sum_{n=1}^{\infty} (-1)^{n+1} H_n t^n (1 + \frac{1}{t}) \\
=&\sum_{n=1}^{\infty} (-1)^{n+1} H_n t^n + \sum_{n=0}^{\infty} (-1)^{n+2} H_{n+1} t^n \\
=&\sum_{n=1}^{\infty} ((-1)^{n+1} H_n + (-1)^{n+2} H_{n+1}) t^n + H_1.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&D_0 = H_1 \\
&D_n = (-1)^n (H_{n+1} - H_n) n!. \quad (n \geq 1)
\end{aligned}$$

Meanwhile, from (33), we know

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(\lambda-1)_{m+1}}{(m+1)} D_{n-m,\lambda} = D_n.$$

So, we can complete the proof.

Theorem 3.9 For $n \geq 0$, we have

$$\sum_{m=0}^{n-k}\binom{n}{m}(-1)^n k! D_{m,\lambda}(k)L(n-m,k)=(-1)^{n-k}(n)_k D_{n-k,\lambda}. \tag{35}$$

Proof By the method of generating function, we get

$$\begin{aligned}
&\sum_{n\geq k}\sum_{m=0}^{n-k}\binom{n}{m}(-1)^n k! D_{m,\lambda}(k)L(n-m,k)\frac{t^n}{n!}\\
=&\sum_{n\geq k}\sum_{m=0}^{n-k}\binom{n}{m}(-1)^m D_{m,\lambda}(k)(-1)^{n-m}L(n-m,k)k!\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}(-1)^n D_{n,\lambda}(k)\frac{t^n}{n!}\sum_{n\geq k}(-1)^n L(n,k)k!\frac{t^n}{n!}\\
=&\frac{\lambda ln(1-t)}{(1-t)^{\lambda}-1}(1-t)^k\left(\frac{t}{1-t}\right)^k=\sum_{n=0}^{\infty}(-1)^n D_{n,\lambda}\frac{t^{n+k}}{n!}\\
=&\sum_{n\geq k}(-1)^{n-k}D_{n-k,\lambda}\frac{t^n}{(n-k)!}=\sum_{n\geq k}(-1)^{n-k}(n)_k D_{n-k,\lambda}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof.

Theorem 3.10 For $n \geq 0$, we have

$$\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}(x)d^*_{n-m}(\lambda)=d_n(x|\lambda). \tag{36}$$

Proof By the method of generating function, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}(x)d^*_{n-m}(\lambda)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}d^*_n(\lambda)\frac{t^n}{n!}\\
=&\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x\frac{(1+t)^{\lambda}-1}{ln(1+\lambda t)}\\
=&\sum_{n=0}^{\infty}d_n(x|\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof.

Theorem 3.11 For $n \geq 0$, we get

$$\sum_{m=0}^{n}\binom{n}{m}\tilde{d}_m(\lambda)D_{n-m,\lambda}(x)=D_n(x). \tag{37}$$

$$\sum_{i+j+k=n}\binom{n}{i,j,k}\tilde{d}_i(\lambda)D_{j,\lambda}(x)_k=D_n(x). \tag{38}$$

Proof By the method of generating function, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\tilde{d}_m(\lambda)D_{n-m,\lambda}(x)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\tilde{d}_n(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}\\
=&\frac{(1+t)^{\lambda}-1}{\lambda t}\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x\\
=&\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof of (37), the same reasoning can be proved (38).

Theorem 3.12 For $n \geq 0$, we get

$$\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}D_{n-m}(x) = D_{n,\lambda}(x). \tag{39}$$

$$\sum_{i+j+k=n}\binom{n}{i,j,k}b_{i,\lambda}D_j(x)_k = D_{n,\lambda}(x). \tag{40}$$

Proof By the method of generating function, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}D_{n-m}(x)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}b_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}\\
=&\frac{\lambda t}{(1+t)^{\lambda}-1}\frac{ln(1+t)}{t}(1+t)^x\\
=&\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof of (39), the same reasoning can be proved (40).

Theorem 3.13 For $n \geq 0$, we get

$$\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}(x)\tilde{d}_j(\lambda)C_k = (x)_n. \tag{41}$$

Proof By the method of generating function, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}(x)\tilde{d}_j(\lambda)C_k\frac{t^n}{n!}\\&=\sum_{i=0}^{\infty}D_{i,\lambda}(x)\frac{t^i}{i!}\sum_{j=0}^{\infty}\tilde{d}_j(\lambda)\frac{t^j}{j!}\sum_{k=0}^{\infty}C_k\frac{t^k}{k!}\\&=\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x\frac{(1+t)^{\lambda}-1}{\lambda t}\frac{t}{ln(1+t)}\\&=(1+t)^x=\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we can complete the proof.

Theorem 3.14 For $n\geq 0$, we get

$$\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}(n)\tilde{d}_j(\lambda)C_k=n!. \tag{42}$$

Proof When (41) $x=n$, we can complete the proof.

Theorem 3.15 For $n\geq 0$, we have

$$\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}(x)\tilde{d}_j(\lambda)C_k\lambda^k=d_n(x|\lambda). \tag{43}$$

Proof By the method of generating function, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}(x)\tilde{d}_j(\lambda)C_k\lambda^k\frac{t^n}{n!}\\&=\sum_{i=0}^{\infty}D_{i,\lambda}(x)\frac{t^i}{i!}\sum_{j=0}^{\infty}\tilde{d}_j(\lambda)\frac{t^j}{j!}\sum_{k=0}^{\infty}C_k\lambda^k\frac{t^k}{k!}\\&=\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}(1+t)^x\frac{(1+t)^{\lambda}-1}{\lambda t}\frac{\lambda t}{ln(1+\lambda t)}\\&=\frac{ln(1+t)}{ln(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x=\sum_{n=0}^{\infty}d_n(x|\lambda)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.16 For $n\geq 0$, we have

$$\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}C_j(\lambda)L(k,\lambda)=(-1)^{\lambda}\binom{n}{\lambda}b_{n-\lambda,\lambda}. \tag{44}$$

Proof By the method of generating function, we have

$$\sum_{n=\lambda}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}D_{i,\lambda}C_j(\lambda)L(k,\lambda)\frac{t^n}{n!}$$
$$=\sum_{i=0}^{\infty}D_{i,\lambda}\frac{t^i}{i!}\sum_{j=0}^{\infty}C_j(\lambda)\frac{t^j}{j!}\sum_{k=\lambda}^{\infty}L(k,\lambda)\frac{t^k}{k!}$$
$$=\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}\frac{t(1+t)^{\lambda}}{ln(1+t)}\frac{(-1)^{\lambda}t^{\lambda}}{(1+t)^{\lambda}\lambda!}$$
$$=\frac{\lambda t}{(1+t)^{\lambda}-1}\frac{(-1)^{\lambda}t^{\lambda}}{\lambda!}=\sum_{n=0}^{\infty}b_{n,\lambda}\frac{t^n}{n!}\frac{(-1)^{\lambda}t^{\lambda}}{\lambda!}$$
$$=\sum_{n=0}^{\infty}(-1)^{\lambda}b_{n,\lambda}\frac{t^{n+\lambda}}{n!\lambda!}=\sum_{n=\lambda}^{\infty}(-1)^{\lambda}\binom{n}{\lambda}b_{n-\lambda,\lambda}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.17 For $n \geq 0$, we have

$$\sum_{i+j+k=n}\binom{n}{i,j,k}b_{i,\lambda}B_{j,\lambda}d_k(\lambda)=\sum_{m=0}^{n}\binom{n}{m}B_mD_{n-m,\lambda}. \tag{45}$$

Proof By the method of generating function, we have

$$\sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}b_{i,\lambda}B_{j,\lambda}d_k(\lambda)\frac{t^n}{n!}$$
$$=\sum_{i=0}^{\infty}b_{i,\lambda}\frac{t^i}{i!}\sum_{j=0}^{\infty}B_{j,\lambda}\frac{t^j}{j!}\sum_{k=0}^{\infty}d_k(\lambda)\frac{t^k}{k!}$$
$$=\frac{\lambda t}{(1+t)^{\lambda}-1}\frac{ln(1+\lambda t)^{\frac{1}{\lambda}}}{e^t-1}\frac{ln(1+t)}{ln(1+\lambda t)^{\frac{1}{\lambda}}}$$
$$=\frac{t}{e^t-1}\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\sum_{n=0}^{\infty}D_{n,\lambda}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B_mD_{n-m,\lambda}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.18 For $n \geq 0$, $k \geq 1$, we have

$$\sum_{m=0}^{n}\binom{n}{m}D^{(k)}_{m,\lambda}(k)\hat{C}^{(k)}_{n-m}=b^{(k)}_{n,\lambda}(x). \tag{46}$$

Proof By the method of generating function, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}^{(k)}(k)\hat{C}_{n-m}^{(k)}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(k)\frac{t^n}{n!}\sum_{n=0}^{\infty}\hat{C}_{n}^{(k)}\frac{t^n}{n!}\\&=\left(\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}\right)^{k}(1+t)^{k}\left(\frac{t}{(1+t)ln(1+t)}\right)^{k}\\&=\left(\frac{\lambda t}{(1+t)^{\lambda}-1}\right)^{k}(1+t)^{x}=\sum_{n=0}^{\infty}b_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

Theorem 3.19 For $n \geq 0$, $k \geq 1$, we have

$$\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}^{(k)}(x)(\lambda k)_{n-m}=\hat{D}_{n,\lambda}^{(k)}(x). \tag{47}$$

Proof By the method of generating function, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}^{(k)}(x)(\lambda k)_{n-m}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\lambda k)_n\frac{t^n}{n!}\\&=\left(\frac{\lambda ln(1+t)}{(1+t)^{\lambda}-1}\right)^{k}(1+t)^{x}(1+t)^{\lambda k}\\&=\sum_{n=0}^{\infty}\hat{D}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of equation, we complete the proof.

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