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A Note On Almost Trans-1-Golden Submersions

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Abstract.

In this Note, two types of submersions whose total space is an almost trans-1-Golden manifold are studied. The study focuses on the transference of structures from the total space to the base one and the geometry of the fibers.

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MSC (2010): 53C15, 53C20, 53C25.

1 Introduction

The differential geometry of the Golden structures in Riemannian manifolds has interested several researchers, see for instance [1, 2, 3, 4, 5, 6] among many others. In [7], Hrectanu pointed out some submanifolds of almost Riemannian Golden manifold and in [8], their integrability has been studied.

Trans-1-Golden manifolds, which have been introduced by G. Beldjilali [3], are Riemannian manifolds furnished with a (Φ, η, ξ)-structure where:

1. Φ is a tensor field of type (1,1) such that $\Phi^2 = \Phi + I$, where I is the identity transformation,
2. ξ is a global vector field on M,
3. η is a differential 1-form, dual to ξ (i.e. $\eta(\xi) = 1$).

Given two almost Golden manifolds (M^n, g, Φ) and $(M'^{m'}, g', \Phi')$, B. Sahin and M.A. Akyol [10] introduced the study of Golden maps between them. They obtained many results concerning the consistency of certain maps. They also showed that this map is harmonic.

Considering two trans-1-Golden manifolds
$(M^{n+1}, g, \Phi, \xi, \eta)$ and $(M'^{m'+1}, g', \Phi', \xi', \eta')$,
one can define the concept of trans-1-Golden submersion. Since the manifolds (M^n, g, Φ) and $(M'^{m'}, g', \Phi')$, carry two Riemannian structures g and g', it can be referred to O'Neill [9] for the concept of Riemannian submersions. Moreover, to these Riemannian manifolds, one can add a tensor field Φ, of type $(1,1)$, following Watson [13] where he introduced the study of almost Hermitian submersions. In the same way, following again Watson [14], who studied two types of almost contact metric submersions, see also [11, 12], one can make an analogues study of two types of trans-1-Golden submersions. The way is to adapt the almost trans-1-Golden structure to the almost contact one and convert the language.

This paper is organized in the following way:

Section 2 is devoted to recall the defining relations of manifolds which will be used in this paper.

Section 3 concerns trans-1-Golden submersions where we develop fundamental properties of two types of this class of submersions.

The last section 4, deals with the relationships between the structure of the total space, the base space and the fibers following what has been obtained in [12].

2 Preliminaries

In this Section, we recall the defining relations of manifolds to be used in the sequel as treated by Beldjilali [2, 3].

An almost Golden Riemannian manifold is a triple (M, g, Φ) where:

1. (M, g) is a Riemannian manifold,
2. Φ is a tensor field of type (1,1) such that $\Phi^2 = \Phi + I$, where I is the identity transformation,
3. $g(\Phi X, Y) = g(X, \Phi Y)$ for all $X, Y \in \mathfrak{X}(M)$

The last condition means that the Riemannian metric g is compatible with the Golden structure Φ. In [10], it is shown that

$$g(\Phi X, \Phi Y) = g(\Phi X, Y) + g(X, Y).$$

Attentively looking the relation $\Phi^2 = \Phi + I$,we can deduce the quadratic equation $x^2 - x - 1 = 0$ whose solution is the number $\phi = \frac{1+\sqrt{5}}{2}$; Approximatively $\phi = 1,6180339...$ The second solution, denoted by ϕ^* is obtained by the relation $\phi^* = 1 - \phi$. In other words, $\phi^* = \frac{1-\sqrt{5}}{2}$. These two numbers are called golden ratio, they are eigenvalues of the automorphism Φ of the tangent bundle TM.

Now, let us turn our attention to the various classes of some interesting manifolds in this study.

An almost trans-1-Golden manifold is a quintuple (M, g, Φ, η, ξ) where:

- (M, g, Φ) is an almost Golden manifold,
- ξ is a global vector field on M,
- η is a differential 1-form, dual to ξ (i.e. $\eta(\xi) = 1$).

The triple (Φ, ξ, η) is called almost trans-1-Golden structure.

Note that, in [3], this manifold is defined by

$$(\nabla_X \Phi)Y = \sqrt{5} g(X, Y)\xi + \eta(Y)X.$$

If an almost trans-1-Golden manifold $(M^{n+1}, \Phi, g, \eta, \xi)$ is such that

$$\nabla \Phi = 0,$$

it is called a $\mathcal{C}-$ Golden manifold.

An almost trans-1-Golden $(M^{n+1}, \Phi, g, \eta, \xi)$ verifying the condition

$$(\nabla_X \Phi)Y = \sqrt{5}(g(X, Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi)$$

is called a $\mathcal{G}-$Golden manifold.

An almost trans-1-Golden manifold, defined by

$$(\nabla_X \Phi)Y = \sigma\sqrt{5}(g(X, Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi)$$

where σ is a function on M, is called a generalized $\mathcal{G}-$Golden manifold.

As an example of trans-1-Golden manifolds, let us symplify the one found in Beldjilali [3; $ex.3.8$]. Let $\mathbb{R}^{n+1}$ be an Euclidean space with Cartesian coordinates $\{x_1, ...x_n, z\}$. We can put
$\xi = \frac{\delta}{\delta z}$,
$\eta = dz - r dx_1$,
$\Phi = \phi^* I + \sqrt{5}\eta \otimes \xi$,
$g = \eta \otimes \eta + \mu \sum_i^n dx_i^2$,
where r and μ are two functions on $\mathbb{R}^{n+1}$.

It is not hard to prove that $\Phi^2 = \Phi X + X$ and $g(\Phi X, Y) = g(X, \Phi Y)$ so that (Φ, g, ξ, η) is an almost metric trans-1-Golden structure.

Recently, pursuing the study of Golden geometry, Beldjilali [1], introduced a new class as follows.

Let M^{2n+1} be a differentiable manifold of odd dimension $2n+1$. By an *almost Golden Riemannian almost contact metric manifold*, one understands a quintuple

$$(M^{2n+1}, g, \xi, \eta, \Phi)$$

where the tensor field Φ satifies the relation

$$\Phi = \phi I - \sqrt{5}\eta \otimes \xi.$$

He shows that if this manifold is integrable, which means that the Nijenhuis tensor N_Φ vanishes with

$$N_\Phi(D, E) = \Phi^2[D, E] + [\Phi D, \Phi E] - \Phi[\Phi D, E] - \Phi[D, \Phi E],$$

then it leads to a $\mathcal{G}-$Golden manifold and $d\eta = 0$.

3 Almost Trans-1-Golden Submersions

We begin by recalling, from O'Neill [7] the concept of Riemannian submersion before attacking that of almost trans-1-Golden one.

Let (M, g) and (M', g') be two smooth, connected and complete Riemannian manifolds. By a Riemannian submersion, one understands a smooth surjective mapping

$$\pi : M \longrightarrow M'$$

such that:

(i) π has maximal rank, and

(ii) $\pi_{*|(Ker\pi_*)^\perp}$ is a linear isometry.

Here, π_* denotes the differential of π whose kernel is denoted by $\ker\pi_*$ and $(\ker\pi_*)^\perp$ is orthogonal to the kernel, $\ker\pi_*$, of π_*.

Vectors in $\ker\pi_*$ are vertical while those in $(\ker\pi_*)^\perp$ are horizontal. For each $x' \in M'$, $\pi^{-1}(x')$ is a closed , embedded submanifold of M, called the fiber of π over x'. Noting by $F_{x'} = \pi^{-1}(x')$ it is known that $\dim F_{x'} = \dim M - \dim M'$. The tangent bundle TM of the total space M has an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ is the vertical distribution while $H(M)$ designates the horizontal one.

A vector field X of the horizontal distribution is called a *basic* vector field if it is π-related to a vector field X_* which means $\pi_* X = X_*$.

In this paper, we will denote horizontal vector fields by X, Y and Z, while those of the vertical distribution will be denoted by U, V and W. On the base space, tensors and other operators will be specified by a prime ($'$), while those of the fibers will be denoted by a caret $\hat{}$. For instance, ∇, ∇' and $\hat{\nabla}$ will designate the Levi-Civita connection of the total space, the base space and the fibers, respectively.

Proposition 3.1. [7] *Let $(M,g) \xrightarrow{\pi} (M',g')$ be a Riemannian submersion, X and Y two basic vector fields on M, then:*

(1) $g(X,Y) = g'(\pi_* X, \pi_* Y)$;

(2) $\mathcal{H}[X,Y]$ is the basic vector field associated to $[X_*, Y_*]$;

(3) $\mathcal{H}(\nabla_X Y)$is the basic vector field associated to $(\nabla'_{X_*} Y_*)$.

Proof. See O'Neill [7]. □

Let $(M^{n+1}, g, \Phi, \xi, \eta)$ and $(M'^{n'+1}, g', \Phi', \xi', \eta')$ be two almost trans-1-Golden manifolds. By an almost trans-1-Golden submersion of type I, one understands a Riemannian submersion:

$$(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{n'+1}, g', \Phi', \xi', \eta')$$

satisfying

(i) $\pi_*\Phi = \Phi'\pi_*$,

(ii) $\pi_*\xi = \xi'$.

Next, we overview some of the fundamental properties of this type of submersions.

Proposition 3.2. *Let $(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M^{m'+1}, g', \Phi', \xi', \eta')$ be an almost trans-1-Golden submersion of type I. Then,*

(1) $\pi^*\eta' = \eta$,

(2) $U \in V(M)$ implies that $\Phi U \in V(M)$;

(3) $\xi \in H(M)$;

(4) $\eta(U) = 0$ for all $U \in V(M)$;

Proof. (1) Consider a basic vector X and let us calculate.

$$\pi^*\eta'(X_*) = \eta'(\pi_*X) = g'(\xi', \pi_*X).$$

But $\xi' = \pi_*\xi$ so that $g'(\xi', \pi_*X) = g'(\pi_*\xi, \pi_*X) = g(\xi, X) = \eta(X)$ which shows that $\pi^*\eta' = \eta$ as claimed.

(2) It is known that if U is vertical, then it is in the kernel of π_*; Since $\Phi'\pi_*U = \pi_*\Phi U$ and $\pi_*U = 0$, then $\pi_*\Phi U = 0$ which shows that ΦU is in the kernel of π_* so that it is vertical.

(3) To show that ξ is horizontal, Watson [14] decomposes ξ into a vertical part ξ_1 and a horizontal one, say, ξ_2 such that $\xi = \xi_1 + \xi_2$. As $\eta(\xi) = 1$ then $\eta(\xi_1) + \eta(\xi_2) = 1$. But $\eta(\xi_2) = g(\xi_2, \xi_2) = g'(\pi_*\xi_2, \pi_*\xi_2) = g'(\xi', \xi') = 1$. Thus $\eta(\xi_1) = 0$ from which $\eta(\xi) = \eta(\xi_2)$ which shows that ξ is horizontal.

(4) It is known that $\eta(U) = g(\xi, U)$. Since ξ is horizontal, then $g(\xi, U) = 0 = \eta(U)$.

□

As consequence of assertion (1) in the above proposition, we have $\pi^*d\eta' = d\eta$. Statement (2) means that the vertical distribution is invariant by Φ.

Proposition 3.3. *The fibers of an almost trans-1-Golden submersion of type I are almost Riemannian Golden manifolds while the base space is an almost trans-1-Golden manifold.*

Proof. Noting by $(n+1)$ the dimension of the total space and by $(n'+1)$ that of the base space, it is known that the fibers have $(n+1)-(n'+1) = (n-n')$ as dimension. Note by $\hat{\Phi}$, the restriction of Φ to the fibers, putting $m = n - n'$, we have the following diagram

$$(F^m, \hat{g}, \hat{\Phi}) \hookrightarrow (M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{n'+1}, g', \Phi', \xi', \eta')$$

We have an almost Riemannian Golden manifold as the fiber. The problem will be to describe this structure.

Note by $\hat{\Phi}$ the restriction of Φ to the fibers, from [10], we have

$$\hat{g}(\hat{\Phi}U, \hat{\Phi}U) = \hat{g}(U, \hat{\Phi}U) + \hat{g}(U, U)$$

which is clear and shows the compatibility of $\hat{g}$ with $\hat{\Phi}$. We then conclude that $(\hat{g}, \hat{\Phi})$ is an almost Riemannian Golden structure. □

Now, let us deal with another type of almost trans-1-Golden submersions.

Let $(M^{n+1}, g, \Phi, \xi, \eta)$ be an almost trans-1-Golden manifold and $(M'^{m'}, g', \Phi')$ an almost Riemannian Golden one.

A Riemannian submersion

$$(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{m'}, g', \Phi')$$

is called an almost trans-1-Golden submersion of type II if it satisfies

$$\pi_* \Phi = \Phi' \pi_*.$$

Proposition 3.4. *Let $(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{m'}, g', \Phi')$ be an almost trans-1-Golden submersion of type II. Then, $U \in V(M)$ implies that $\Phi U \in V(M)$;*

Proof. Let U be vertical vector field. According to the definition of this type of submersion $\pi_*\Phi U = \Phi' \pi_* U$. But U beeing vertical, then $\pi_* U = 0$. Thus $\pi_* \Phi U = 0$ implyies that ΦU is in the kernel of π_* and then ΦU is vertical. □

Proposition 3.5. *The fibers of an almost trans-1-Golden submersion of type II are almost trans-1-Golden manifolds while the base space is an almost Riemannian Golden manifold.*

Proof. It is clear that the dimension of the fibers is $(n-n')+1$. Let $(\hat{g}, \hat{\Phi}, \hat{\xi}, \hat{\eta})$ be the restriction of the almost trans-1-Golden structure (g, φ, ξ, η) of the total space to the fibers . We have to show that $(\hat{g}, \hat{\Phi}, \hat{\xi}, \hat{\eta})$ is an almost trans-1-Golden structure. As in the case of submersions of type I, we have the following diagram

$$(F^m, \hat{g}, \hat{\Phi}, \hat{\eta}, \hat{\xi}) \hookrightarrow (M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{n'}, g', \Phi')$$

This is a problem to be examined by establishing the relations between $\hat{\Phi}$ and Φ; $\hat{\eta}$ with η; and $\hat{\xi}$ with ξ. □

4 Transference of Structures

Proposition 4.1. *Let $M^{n+1} \xrightarrow{\pi} M'^{n'+1}$ be an almost trans-1-Golden submersion of type I. If the total space is a $\mathcal{C}-$Golden manifold, then the fibers are almost Riemannian Golden manifolds while the base space is a $\mathcal{C}-$Golden manifold.*

Proof. See Proposition 3.3.

□

Proposition 4.2. *Let $M^{n+1} \xrightarrow{\pi} M'^{n'+1}$ be an almost trans-1-Golden submersion of type I. If the total space is a $\mathcal{G}-$Golden manifold, then the fibers are almost Riemannian Golden manifolds while the base space is a $\mathcal{G}-$Golden manifold.*

Proof. See again Proposition 3.3. □

Proposition 4.3. *Let $M^{n+1} \xrightarrow{\pi} M'^{n'+1}$ be an almost trans-1-Golden submersion of type I. If the total space is a generalized $\mathcal{G}-$Golden manifold, then the fibers are almost Riemannian Golden manifolds while the base space is a generalized $\mathcal{G}-$Golden manifold.*

Proof. Referring to Proposition 3.3 the proof follows. □

Now, turning to the almost trans-1-Golden submersions of type II, analogue properties are established as follows.

Proposition 4.4. *If the total space of a type II, almost trans-1-Golden submersion*

$(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{m'}, g', \Phi')$ *is a* $\mathcal{C}-$*Golden manifold, then the fibers are* $\mathcal{C}-$*Golden manifolds while the base space is an almost Riemannian Golden manifold.*

Proof. Referring to Proposition 3.5 the proof follows. □

Proposition 4.5. *Assume that the total space of a type* II, *almost trans-1-Golden submersion*
$(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{m'}, g', \Phi')$ *is a* $\mathcal{G}-$*Golden manifold, then the fibers are* $\mathcal{G}-$*Golden manifolds while the base space is an almost Riemannian Golden manifold.*

Proof. The proof follows by referring to Proposition 3.5. □

Proposition 4.6. *Suppose that the total space of a type* II, *almost trans-1-Golden submersion* $(M^{n+1}, g, \Phi, \xi, \eta) \xrightarrow{\pi} (M'^{m'}, g', \Phi')$
is a generalized $\mathcal{G}-$*Golden manifold, then the fibers are generalized* $\mathcal{G}-$*Golden manifold while the base space is an almost Riemannian Golden manifold.*

Proof. The proof follows by referring again to Proposition 3.5. □

References

[1] BELDJILALI, G., A New class of Golden Riemannian Manifolds, *Int. Elect. J. Geom.*,**13**(1)(2020),1-8.

[2] BELDJILALI, G., 3-dimensional Golden almost contact metric manifolds, *Palestine J. Math*,**9**(1)(2020),594-603.

[3] BELDJILALI, G., s-Golden Manifolds, *Meditrr. J. Math.* (2019), 16-56.

[4] BELDJILALI, G., Induced structures in Golden Riemannian Manifolds, *Beitr. Algebra Geom.* **59**(2018),761-777.

[5] CRASMAREANU, M.,HRETCANU, C.E., Golden differential geometry, *Chaos, Solitons & Fractals* **38**(5)(2008),1124-1146.

[6] ETAYO, F., SANTAMARIA, R.,UPADHYAY, A.:On the geometry of almost Golden Riemannian manifolds,*Meditrr. J. Math.***14**(2017),187

[7] HRETCANU, C. E., Submanifolds in Riemannian manifold with Golden structure. *Workshop on Finsler Geometry and its Applications,* Hungary (2017).

[8] GEZER, A., CENGIZ, On integrability of Golden Riemannian structures, *Turkish J. Math.* **37**(2013),693-703.

[9] O'NEILL, B., The fundamental equations of a submersion,*Michigan J. Math.***13**(1966),459-469.

[10] SAHIN, B.,AKYOL, M.A.;: Golden maps between Golden Riemannian manifolds and constancy of certain maps, *Math. Commun.* **19**, (2014)333-342.

[11] TSHIKUNA-MATAMBA,T., Superminimal fibres in an almost contact metric submersion, *Iranian J. Math. Sci. and Info.,* **3**(2)(2008),77-88.

[12] TSHIKUNA-MATAMBA, T.,On the structure of the base space and the fibres of an almost contact metric submersion, *Houston J. Math.*,**23**(1997),291-305.

[13] WATSON, B.,Almost Hermitian submersion, *J.Diff. Geom.***11**(1)(1976),147-165.

[14] WATSON, B., The differential geometry of two types of almost contact metric submersions, *The Math.Heritage of C. F. Gauss, Ed. G. M. Rassias,* World Sci. Publ. Co. Singapore, 1991, pp. 827-861.