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A Geometric Construction of Multiwavelet Sets of $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$

Shiva Mittal

Department of Mathematics
S.P.M. Govt. Degree College, Prayagraj India-211 002
Email: shivamittal009@gmail.com.

Abstract.
In the present article we construct symmetric multiwavelet sets of finite order in $L^2(\mathbb{R})$ and multiwavelet sets in $H^2(\mathbb{R})$ by considering the geometric construction determining wavelet sets provided by N. Arcozzi, B. Behera and S. Madan for large classes of minimally supported frequency wavelets of $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$.

Keywords: MSF multiwavelets; multiwavelet sets; Hardy space.

MSC 2010: 42C15, 42C40.

1. Introduction and Preliminaries

The collection of all square integrable complex valued functions in $\mathbb{R}^n$, in which two functions are identified if they are equal almost everywhere (abbreviated, a.e.), is denoted by $L^2(\mathbb{R}^n)$. With the usual addition, the scalar multiplication and the inner product $\langle f, g\rangle$ of $f, g \in L^2(\mathbb{R}^n)$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx,$$

$L^2(\mathbb{R}^n)$ becomes a Hilbert space. For a function $f \in L^2(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(t)\, e^{-i<\xi,\, t>} dt,$$

and the inverse Fourier transform $\check{f}$ of f is defined by

$$\check{f}(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi)\, e^{i<\xi,\, t>} d\xi.$$

Let A denote an $n \times n$ expansive matrix, where $n \in \mathbb{Z}$ and A^* the transpose of A. By an expansive matrix, we mean a matrix for which the modulus of each eigen-value is greater than 1.

In this paper, we assume that a is an integer such that $|a| > 1$, and that L is a natural number for which $L/(|a| - 1)$ is an integer, say, d. The symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote, respectively, the set of natural numbers, the set of integers and the real line. By A, we denote an $n \times n$ expansive matrix such that $A\mathbb{Z}^n \subseteq \mathbb{Z}^n$, where $n \in \mathbb{N}$. The transpose of A is denoted by A^*. For a set E in the Euclidean space $\mathbb{R}^n$, the Lebesgue measure of E is denoted by $|E|$.

A finite set $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^n)$, is called an *orthonormal A-multiwavelet* of order L, if the system $\{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, ..., L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, where

$$\psi^l_{j,k}(x) = |detA|^{\frac{j}{2}} \psi^l\left(A^j x - k\right), \qquad x \in \mathbb{R}^n.$$

In case Ψ consists of a single element, say ψ, we say ψ to be an *n-dimensional orthonormal A-wavelet*, or simply an *A-wavelet*. The following result characterizes an orthonormal A-multiwavelet.

Theorem 1.1. [5,6,10,14] *A subset* $\Psi = \{\psi^1, ..., \psi^L\}$ *of* $L^2(\mathbb{R}^n)$ *is an orthonormal A-multiwavelet if and only if the following hold:*

(i) $\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(A^{*j}\xi)|^2 = 1, \quad a.e., \ \xi \in \mathbb{R}^n,$

(ii) $\sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(A^{*j}\xi)\overline{\hat{\psi}^l(A^{*j}(\xi + 2s\pi))} = 0, \quad a.e., \ \xi \in \mathbb{R}^n, \ s \in \mathbb{Z}^n \backslash A^*\mathbb{Z}^n$,

(iii) $||\psi^l|| = 1, \quad for \ l = 1, ..., L.$

A method to obtain A-multiwavelets in $L^2(\mathbb{R}^n)$ arises from the notion known as the A-Multiresolution analysis of multiplicity d [2,8,13,15,16], which is described below:

Definition 1.2. An *A-multiresolution analysis* (A-MRA) *of multiplicity* d associated to the lattice $\mathbb{Z}^n$ is a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ satisfying

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;

(b) $f(\cdot) \in V_j$, if and only if $f(A\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$;

(c) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(e) There exist functions $\varphi_1, \varphi_2, ..., \varphi_d \in L^2(\mathbb{R}^n)$ such that $\{\varphi_i(\cdot \ - k) : k \in \mathbb{Z}^n, \ i = 1, ..., d\}$ forms an orthonormal basis for V_0.

The functions $\varphi_1, \varphi_2, ..., \varphi_d$ are called *scaling functions* of the A-MRA, and the vector $\Phi_{vec} = (\varphi_1, ..., \varphi_d)^*$ is called a *scaling vector* for the A-MRA.

An A-multiresolution analysis of multiplicity d gives rise to an A-multiwavelet Ψ of order L, where $L = (|detA| - 1)d$ as described in [8].

It is well known that $|supp\ \hat{\psi}|$, where ψ is an n-dimensional orthonormal A-wavelet, is at least $(2\pi)^n$. An A-wavelet ψ for which $|supp\ \hat{\psi}| = (2\pi)^n$, is said to be a *minimally supported frequency* (MSF) *A-wavelet* [10–12]. It is also known that for an MSF A-wavelet ψ, there exists a measurable set W of measure $(2\pi)^n$ such that $|\hat{\psi}| = \chi_W$. We call the set W to be an *A-wavelet set.*

Based on the notion of multiwavelets [5, 6, 9, 10, 14], wavelet sets have been generalized into multiwavelet sets by Bownik, Rzeszotnik and Speegle in [7]. The study related to wavelet sets and also to multiwavelet sets has attracted attention of several workers [1, 3, 7, 12, 17–20].

The concept of an MSF *A-wavelet* has been generalized to that of an MSF *A-multiwavelet of order L* [4, 7] as follows:

Definition 1.3. An MSF *A-multiwavelet* of order L is an orthonormal A-multiwavelet $\Psi = \{\psi^1, ..., \psi^L\}$ such that $|\hat{\psi}^l| = \chi_{W_l}$, for some measurable sets $W_l \subset \mathbb{R}^n$, $l = 1, ..., L$.

Stated below is a characterization of MSF A-multiwavelets:

Theorem 1.4. *A set* $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^n)$ *such that* $|\hat{\psi}^l| = \chi_{W_l}$, *for* $l = 1, ..., L$, *is an orthonormal A-multiwavelet if and only if*

(i) $\sum_{k\in\mathbb{Z}^n} \chi_{W_l}(\xi + 2\pi k) \cdot \chi_{W_m}(\xi + 2\pi k) = \delta_{l,m}, \quad a.e., \ \xi \in \mathbb{R}^n, \ l, m = 1, ..., L,$

(ii) $\sum_{j\in\mathbb{Z}}\sum_{l=1}^{L} \chi_{W_l}(A^{*j}\xi) = 1, \quad a.e., \ \xi \in \mathbb{R}^n.$

Definition 1.5. A set $W \subset \mathbb{R}^n$ is an *A-multiwavelet set* of order L, if $W = \dot{\bigcup}_{l=1}^{L} W_l$, for some measurable sets $W_1, ..., W_L \subset \mathbb{R}^n$ satisfying

(i) $\sum_{k\in\mathbb{Z}^n} \chi_{W_l}(\xi + 2k\pi) \cdot \chi_{W_m}(\xi + 2k\pi) = \delta_{l,m}$, a.e., $\xi \in \mathbb{R}^n$, $l, m = 1, ..., L$,

(ii) $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \chi_{W_l}(A^{*j}\xi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$.

A characterization of A-multiwavelet sets of order L established in [7], is as follows:
Theorem 1.6. *A measurable set $W \subset \mathbb{R}^n$ is an A-multiwavelet set of order L if and only if*

(i) $\sum_{k\in\mathbb{Z}^n} \chi_W(\xi + 2k\pi) = L$, *a.e.*, $\xi \in \mathbb{R}^n$, *and*

(ii) $\sum_{j\in\mathbb{Z}} \chi_W(A^{*j}\xi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$.

The notions of an orthonormal a-multiwavelet of order L, minimally supported frequency multiwavelet, a-multiwavelet sets of order L, a-multiresolution analysis of finite muliplicity can be defined for $L^2(\mathbb{R})$ from the results mentioned earlier.

The classical *Hardy space* $H^2(\mathbb{R})$ defined by

$$H^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0, \text{ for a.e., } \xi \leq 0 \right\},$$

is a closed subspace of $L^2(\mathbb{R})$. A function $\psi \in H^2(\mathbb{R})$ is an orthonormal wavelet for $H^2(\mathbb{R})$ if the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $H^2(\mathbb{R})$. For simplicity, we call such a ψ an *H^2-wavelet*.

A finite set $\Psi = \{\psi^1, ..., \psi^L\} \subset H^2(\mathbb{R})$, is called an *orthonormal a-multiwavelet* of order L for $H^2(\mathbb{R})$ if the system $\{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}, l = 1, ..., L\}$ is an orthonormal basis for $H^2(\mathbb{R})$, where

$$\psi^l_{j,k}(x) = |a|^{\frac{j}{2}} \psi^l\left(a^j x - k\right), \qquad x \in \mathbb{R}.$$

In case Ψ consists of a single element, say ψ, we say ψ to be an *orthonormal a-wavelet*, or simply an *a-wavelet* for $H^2(\mathbb{R})$. The following result characterizes an orthonormal a-multiwavelet for $H^2(\mathbb{R})$ analogous to that given in [5, 6, 9, 14].

Theorem 1.7. *A subset $\Psi = \{\psi^1, ..., \psi^L\}$ of $H^2(\mathbb{R})$ is an orthonormal a-multiwavelet if and only if the following hold*:

(i) $\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(a^j\xi)|^2 = \chi_{\mathbb{R}^+}(\xi)$, *a.e.*, $\xi \in \mathbb{R}$,

(ii) $\sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\psi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j(\xi + 2s\pi))} = 0$, *a.e.*, $\xi \in \mathbb{R}$, $s \in \mathbb{Z}\backslash a\mathbb{Z}$,

(iii) $||\psi^l|| = 1$, *for* $l = 1, ..., L$.

Definition 1.8. An *a-multiresolution analysis* (a-MRA) *of multiplicity d* for $H^2(\mathbb{R})$ is a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $H^2(\mathbb{R})$ satisfying

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(b) $f(\cdot) \in V_j$, if and only if $f(a\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$,

(c) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,

(d) $\overline{\cup_{j\in\mathbb{Z}} V_j} = H^2(\mathbb{R})$,

(e) There exist functions $\varphi_1, \varphi_2, ..., \varphi_d \in H^2(\mathbb{R})$ such that $\{\varphi_i(\cdot - k) : k \in \mathbb{Z},\ i = 1, ..., d\}$ forms an orthonormal basis for V_0.

Analogous to definition of an MSF a-multiwavelet of order L for $L^2(\mathbb{R})$, we define an MSF a-multiwavelet of order L for $H^2(\mathbb{R})$.

Definition 1.9. An MSF *a-multiwavelet* of order L is an orthonormal a-multiwavelet $\Psi = \{\psi^1, ..., \psi^L\} \subset H^2(\mathbb{R})$ such that $|\hat{\psi}^l| = \chi_{W_l}$, for some measurable sets $W_l \subset \mathbb{R}^+$, $l = 1, ..., L$, and each W_l has minimal Lebesgue measure.

Stated below is a characterization of MSF a-multiwavelets for $H^2(\mathbb{R})$.

Theorem 1.10. *A set $\Psi = \{\psi^1, ..., \psi^L\} \subset H^2(\mathbb{R})$ such that $|\hat{\psi}^l| = \chi_{W_l}$, for $l = 1, ..., L$, is an orthonormal a-multiwavelet for $H^2(\mathbb{R})$ if and only if*

(i) $\sum_{k\in\mathbb{Z}} \chi_{W_l}(\xi + 2\pi k)\ \chi_{W_m}(\xi + 2\pi k) = \delta_{l,m}$, *a.e.*, $\xi \in \mathbb{R}$, $l, m = 1, ..., L$,

(ii) $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \chi_{W_l}(a^j\xi) = \chi_{\mathbb{R}^+}(\xi)$, *a.e.*, $\xi \in \mathbb{R}$.

Definition 1.11. A set $W \subset \mathbb{R}^+$ is an *a-multiwavelet set* of order L for $H^2(\mathbb{R})$, if $W = \dot{\bigcup}_{l=1}^{L} W_l$, for some measurable sets $W_1, ..., W_L \subset \mathbb{R}^+$ satisfying

(i) $\sum_{k\in\mathbb{Z}} \chi_{W_l}(\xi + 2k\pi)\ \chi_{W_m}(\xi + 2k\pi) = \delta_{l,m}$, a.e., $\xi \in \mathbb{R}$, $l, m = 1, ..., L$, *and*

(ii) $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \chi_{W_l}(a^j\xi) = \chi_{\mathbb{R}^+}(\xi)$, *a.e.*, $\xi \in \mathbb{R}$.

The following is an analogous characterization of a-multiwavelet sets of order L for $H^2(\mathbb{R})$ that is established in [7] in case of $L^2(\mathbb{R})$.

Theorem 1.12. *A measurable set $W \subset \mathbb{R}^+$ is an a-multiwavelet set of order L for $H^2(\mathbb{R})$ if and only if*

(i) $\sum_{k\in\mathbb{Z}} \chi_W(\xi + 2k\pi) = L$, a.e., $\xi \in \mathbb{R}$,

(ii) $\sum_{j\in\mathbb{Z}} \chi_W(a^j\xi) = \chi_{\mathbb{R}^+}(\xi)$, a.e., $\xi \in \mathbb{R}$.

A symmetric multiwavelet sets W is of the form $W = W^- \cup W^+$, where W^+ is a subset of $\mathbb{R}^+$, and $W^+ = -W^-$. In Section 2, we present a method to construct large families of symmetric a-multiwavelet sets of order L, where

$$W^+ = I_1 \cup I_2 \cup .....I_1 \cup I_n,$$

for $n \geq 1$ and the subsequent subsections 2.1 and 2.2 provide a family of symmetric six-interval a-multiwavelet sets of order L and a family of symmetric four-interval a-multiwavelet sets of order L with examples in $L^2(\mathbb{R})$. In Section 3 we obtain a-multiwavelet sets of order L in $H^2(\mathbb{R})$.

2. A Geometric Construction of Symmetric Multiwavelet Sets in $L^2(\mathbb{R})$

Let a be a real number with $|a| > 1$ and L be a positive integer. Consider the set D, in the first quadrant of the Cartesian plane, of the points P such that

$$P \equiv P[\lambda, m] = (a^{-\lambda}, a^{-\lambda}m),$$

where $m \in \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{Z}$.

Let $n \in \mathbb{N}$ and $P_j = P[\lambda_j, m_j] = (a^{-\lambda_j}, a^{-\lambda_j} m_j)$, for $j = 1, 2..., n$. Define the points c_j's $j = 0, 1, 2..., n-1$, as follows

$$c_0 = 0, \quad \text{and}$$
$$c_j = -\frac{[a^{-\lambda_j} m_j - a^{-\lambda_{j+1}} m_{j+1}]}{(a^{-\lambda_j} - a^{-\lambda_{j+1}})}, \quad j = 1, 2..., n-1.$$

Clearly, c_j is the negative of the slope of the straight line joining P_j and P_{j+1} where $j = 0, 1, 2..., n-1$.

The order sequence of points $\Omega = (P_1, ..., P_n)$ is said to be an *MSF polygon for multiwavelet* if the points c_j, $j = 0, 1, 2..., n$ satisfy

$$0 = c_0 < c_1 \cdots < c_n = \frac{L}{2} \tag{2.1}$$

and

$$\lambda_1 = 0 \quad \text{and} \quad 2am_1 = a^{-\lambda_n}[2m_n + L]. \tag{2.2}$$

Theorem 2.1. *Let $\Omega = (P_1, ..., P_n)$ be an MSF polygon for multiwavelet as described above. Let*

$$I_j = \left[2\pi(c_{j-1} + m_j), 2\pi(c_j + m_j)\right], \text{ for } j = 1, 2..., n.$$

If $W^+ = I_1 \cup I_2 \cup ... \cup I_n$, then $W = W^+ \cup W^-$ is a symmetric a-multiwavelet set of order L for $L^2(\mathbb{R})$.

Proof. It is parallel to that in [1].

Denote $K(\Omega)$ by a multiwavelet set associated to Ω. If Ω_1 and Ω_2 are different MSF polygons, then $K(\Omega_1) \neq K(\Omega_2)$.

Remarks 2.2.

(i) Geometrically, (2.1) says that the straight lines joining P_j and P_{j+1}, for $j = 1, 2, ..., n-1$ must have negative decreasing slopes in $\left(-\frac{L}{2}, 0\right)$.

(ii) (2.2) can be expressed in the following way. If we decompose m_1 as follows:

$$2m_1 = a^s(2t + L), \quad s, \ t \in N \cup \{0\},$$

then by (2.2), we have $a^s(2t + L) = 2m_1 = a^{-\lambda_n - 1}[L + 2m_n]$. It further implies that $\lambda_n = -s - 1$, and $m_n = t$. This shows that, there is a bijection between the values of m_1 and pairs (P_1, P_n).

2.1. Symmetric Six-interval a-Multiwavelet Sets of Order L

The following example provides a family of symmetric six-interval a-multiwavelet sets of order L.

Example 2.1.1. Let $n = 3$. Consider

$$\lambda_1 = 0, \ m_1 = \frac{a^s}{2}(2t + L), \ \lambda_2 = -v, \ m_2 = 0, \ \lambda_3 = -s - 1, \ m_3 = t,$$

where s, t and v are non-negative integers such that $s \geq 1, t \geq 1$. Then we have

$$P_1 = P\left[0, \frac{a^s}{2}(2t+L)\right] = \left(1, \frac{a^s}{2}(2t+L)\right), \quad P_2 = P[-v, 0] = (a^v, 0),$$

and $P_3 = P[-s-1, t] = (a^{s+1}, a^{s+1}t)$.

From these, we obtain

$$c_0 = 0, \quad c_3 = \frac{L}{2}, \quad c_1 = \frac{a^s(2t+L)}{2(a^v-1)}, \quad \text{and} \quad c_2 = \frac{ta^{s+1}}{(a^v - a^{s+1})}.$$

Clearly, $c_1 > 0$. That $c_1 < c_2$ provides

$$L < 2t(a-1). \tag{2.3}$$

Further, that $c_2 < c_3$ implies

$$a^{s+1}(2t+L) < La^v. \tag{2.4}$$

With (2.3) and (2.4), $\Omega = (P_1, P_2, P_3)$ forms an MSF polygon for multwavelet. Now,

$$\begin{aligned} I_1 =& [2\pi c_0, 2\pi c_1] + 2\pi m_1 \\ =& \left[\pi a^s(2t+L), \frac{\pi a^{s+v}(2t+L)}{(a^v-1)}\right], \\ I_2 =& [2\pi c_1, 2\pi c_2] + 2\pi m_2 \\ =& \left[\frac{\pi a^s(2t+L)}{(a^v-1)}, \frac{2\pi t a^{s+1}}{(a^v - a^{s+1})}\right], \end{aligned}$$

and

$$\begin{aligned} I_3 =& [2\pi c_3, 2\pi c_3] + 2\pi m_3 \\ =& \left[\frac{2\pi t a^v}{(a^v - a^{s+1})}, \pi(2t+L)\right]. \end{aligned}$$

Then $W = W^+ \cup W^-$, where $W^+ = I_1 \cup I_2 \cup I_3$ is a symmetric a-multiwavelet set of order L consisting of six disjoint intervals, where s, t and v are non-negative integers such that $s \geq 1, t \geq 1$ satisfying (2.3) and (2.4).

Example 2.1.2. To get a symmetric six-interval 3-multiwavelet set of order 2, we select non-negative integers s, t and v as $t = 1$, $s = 1$ and $v = 3$ in Example 2.1.1. Clearly, s, t and v satisfy (2.3) and (2.4). We get

$$I_1 = \left[12\pi, \frac{162\pi}{13}\right], \quad I_2 = \left[\frac{6\pi}{13}, \pi\right], \quad I_3 = [3\pi, 4\pi],$$

with $|I_1| = \frac{6\pi}{13}$, $|I_2| = \frac{7\pi}{13}$ and $|I_3| = \pi$. It follows that $|W^+| = 2\pi$.

Hence, $W = W^+ \cup W^-$, where

$$W^+ = \left[\frac{6\pi}{13}, \pi\right] \cup [3\pi, 4\pi] \cup \left[12\pi, \frac{162\pi}{13}\right]$$

is a symmetric 3-multiwavelet set of order 2 consisting of six disjoint intervals.

The following example provides a family of symmetric six-interval a-multiwavelet sets of order L, which is different from that we obtained in Example 2.1.1.

Example 2.1.3. Let $n = 3$. Consider

$$\lambda_1 = 0,\ m_1 = \frac{1}{2}a^{s+1}L,\ \lambda_2 = u,\ m_2 = v, \lambda_3 = -s-2,\ m_3 = 0,$$

where s, u and v are non-negative integers such that $s \geq 0, u \geq 1$. Then, we have

$$P_1 = P\left[0, \frac{1}{2}a^{s+1}L\right] = \left(1, \frac{1}{2}a^{s+1}L\right),\ \ P_2 = P[u,v] = (a^{-u}, va^{-u}),$$

$$\text{and } \ P_3 = P[-s-2, 0] = (a^{s+2}, 0).$$

This gives

$$c_0 = 0,\ \ c_3 = \frac{L}{2},\ \ c_1 = \frac{2v - a^{s+u+1}L}{2(a^u-1)},\ \text{ and }\ c_2 = \frac{v}{(a^{s+u+2}-1)}.$$

Because $c_1 < c_2$,

$$v < \frac{(a^{s+u+2}-1)a^{s+1}L}{2(a^{s+2}-1)}. \tag{2.5}$$

Further, since $c_2 < c_3$,

$$2v < (a^{s+u+2}-1)L, \tag{2.6}$$

and $0 < c_1$ implies that

$$2v > a^{s+u+1}L. \tag{2.7}$$

Clearly, (2.5) implies (2.6). Combining (2.5) and (2.7), we get

$$a^{s+u+1}L < 2v < \frac{(a^{s+u+2}-1)a^{s+1}L}{(a^{s+2}-1)}. \tag{2.8}$$

With (2.8), $\Omega = (P_1, P_2, P_3)$ forms an MSF polygon for multiwavelet. Now,

$$\begin{aligned} I_1 =& [2\pi c_0, 2\pi c_1] + 2\pi m_1 \\ =& \left[\pi a^{s+1}L, \frac{\pi(2v - a^{s+1}L)}{(a^u-1)}\right], \\ I_2 =& [2\pi c_1, 2\pi c_2] + 2\pi m_2 \\ =& \left[\frac{\pi(2v - a^{s+1}L)a^u}{(a^u-1)}, \frac{2\pi v a^{s+u+2}}{(a^{s+u+2}-1)}\right], \end{aligned}$$

and

$$\begin{aligned} I_3 =& [2\pi c_3, 2\pi c_3] + 2\pi m_3 \\ =& \left[\frac{2\pi v}{(a^{s+u+2}-1)}, \pi L\right]. \end{aligned}$$

Then $W = W^+ \cup W^-$, where $W^+ = I_1 \cup I_2 \cup I_3$ is a symmetric a-multiwavelet set of order L consisting of six disjoint intervals, where s, u and v are non-negative integers such that $s \geq 0,\ u \geq 1$ satisfying (2.8).

Example 2.1.4. In order to get a symmetric six-interval 2-multiwavelet set of order 3, we select non-negative integers s, u and v as $s = 0$, $u = 2$, and $v = 13$ in Example 2.1.3. Clearly, s, u and v satisfy (2.8). We get

$$I_1 = \left[6\pi, \frac{20\pi}{3}\right], \quad I_2 = \left[\frac{80\pi}{3}, \frac{416\pi}{15}\right], \quad I_3 = \left[\frac{26\pi}{15}, 3\pi\right],$$

with $|I_1| = \frac{2\pi}{3}$, $|I_2| = \frac{16\pi}{15}$, and $|I_3| = \frac{19\pi}{15}$. It follows that $|W^+| = 3\pi$.

Hence, $W = W^+ \cup W^-$, where

$$W^+ = \left[\frac{26\pi}{15}, 3\pi\right] \cup \left[6\pi, \frac{20\pi}{3}\right] \cup \left[\frac{80\pi}{3}, \frac{416\pi}{15}\right],$$

is a symmetric 2-multiwavelet set of order 3 consisting of six disjoint intervals.

2.2. Symmetric Four-interval a-Multiwavelet Sets of Order L

The following example provides a family of symmetric four-interval a-multiwavelet sets of order L.

Example 2.2.1. If we select $s = 0$ in Example 2.1.1, we get symmetric four-interval a-multiwavelet set of order L.

Now, we get

$$I_1 = \left[\pi(2t + L), \frac{\pi a^v(2t + L)}{(a^v - 1)}\right], \quad I_2 = \left[\frac{\pi(2t + L)}{(a^v - 1)}, \frac{2\pi t a}{(a^v - a)}\right],$$

and

$$I_3 = \left[\frac{2\pi t a^v}{(a^v - a)}, \pi(2t + L)\right].$$

Since $c_1 + m_1 = c_3 + m_3 = \pi(2t + L)$, we have

$$I \equiv I_3 \cup I_1 = \left[\frac{2\pi t a^v}{(a^v - a)}, \frac{\pi a^v(2t + L)}{(a^v - 1)}\right].$$

Then $W = W^+ \cup W^-$, where $W^+ = I \cup I_2$ is a symmetric a-multiwavelet set of order L consisting of four disjoint intervals, where t and v are non-negative integers such that $t \geq 1$ satisfying

$$L < 2t(a - 1), \text{ and } a(2t + L) < La^v. \tag{2.9}$$

Example 2.2.2. To obtain a symmetric four-interval 3-multiwavelet set of order 2, we select non-negative integers t and v as $t = 1$, $s = 1$ and $v = 2$ in Example 2.2.1. Clearly, t and v satisfy (2.9). We get

$$I = \left[3\pi, \frac{9\pi}{2}\right], \quad I_2 = \left[\frac{\pi}{2}, \pi\right],$$

with $|I| = \frac{3\pi}{2}$, and $|I_2| = \frac{\pi}{2}$. Therefore, $|W^+| = 2\pi$.

Hence, $W = W^+ \cup W^-$, where

$$W^+ = \left[\frac{\pi}{2}, \pi\right] \cup \left[3\pi, \frac{9\pi}{2}\right]$$

is a symmetric 3-multiwavelet set of order 2 consisting of four disjoint intervals.

3. A Geometric Construction of Multiwavelet Sets for $H^2(\mathbb{R})$

In this section, we extend the geometric construction determining symmetric multiwavelet sets for $L^2(\mathbb{R})$ given in Section 2 to obtain some a-multiwavelet sets of order L for $H^2(\mathbb{R})$.

Consider the set D, in the first quadrant of the Cartesian plane, of points P such that

$$P \equiv P[\lambda, m] = (a^{-\lambda}, a^{-\lambda}m),$$

where $m \in \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{Z}$. Let $n \in \mathbb{N}$ and $P_j = P[\lambda_j, m_j] = (a^{-\lambda_j}, a^{-\lambda_j}m_j)$, $j = 1, 2..., n$ with $m_j \neq m_{j+1}, m_0 \neq m_n + L$, and $Q_n = P[\lambda_n + 1, m_n + L]$. Without loss of generality, we can take $\lambda_1 = 0$, and $m_1 = 0$. Define the points c_j's, for $j = 0, 1, 2..., n$ as follows:

$$\begin{aligned} c_0 =& \frac{m_n + L}{(a^{\lambda_n+1} - 1)}, \\ c_j =& -\frac{[a^{-\lambda_j}m_j - a^{-\lambda_{j+1}}m_{j+1}]}{(a^{-\lambda_j} - a^{-\lambda_{j+1}})}, \quad j = 1, 2..., n-1, \\ \text{and } \ c_n =& c_0 + L. \end{aligned}$$

Clearly, c_0 is the negative of the slope of the straight line joining P_1 and Q_n, and for $j = 0, 1, 2..., n-1$, c_j is the negative of the slope of the straight line joining P_j and P_{j+1}. The order sequence of points $\Omega = (P_1, ..., P_n)$ is said to be an H^2-MSF polygon if the points c_j, $j = 0, 1, 2..., n$ satisfy

$$0 < c_0 < c_1 \cdots < c_n = c_0 + L.$$

Theorem 3.1. *Let $\Omega = (P_1, ..., P_n)$ be an H^2-MSF polygon as described above. Let*

$$I_j = [2\pi c_{j-1}, 2\pi c_j] + 2\pi m_j, \quad j = 1, 2..., n.$$

Then $W = I_1 \cup I_2 \cup ... \cup I_n$ is an a-multiwavelet set of order L for $H^2(\mathbb{R})$.

Denote the multiwavelet set associated with Ω by $K(\Omega)$. If Ω_1 and Ω_2 are different H^2-MSF polygons, then $K(\Omega_1) \neq K(\Omega_2)$.

Example 3.2. Let $n = 2$. Consider $\lambda_1 = 0, m_1 = 0, \lambda_2 = r, m_2 = k$, where r is an integer and $k \in \mathbb{N} \cup \{0\}$. Then we have

$$P_1 = (1, 0), \ \ P_2 = (a^{-r}, a^{-r}k) \ \text{ and } \ Q_2 = (a^{-(r+1)}, a^{-(r+1)}(k + L)).$$

This gives

$$c_0 = \frac{k + L}{(a^{r+1} - 1)}, \ \ c_1 = \frac{k}{(a^r - 1)} \ \text{ and } \ c_2 = \frac{k + a^{r+1}L}{(a^{r+1} - 1)}.$$

Further, the condition $0 < c_0 < c_1 < c_2$ is equivalent to

$$0 < \frac{(a^r - 1)L}{a^r(a - 1)} < k < \frac{a(a^r - 1)L}{(a - 1)}. \tag{3.1}$$

If $r = 0$, then inequality (3.1) gives k is negative number, which is not possible. Hence $r \geq 1$. Now,

$$\begin{aligned} I_1 =& [2\pi c_0, 2\pi c_1] + 2\pi m_1 \\ =& [2\pi c_0, 2\pi c_1] \\ =& \left[\frac{2\pi(k + L)}{(a^{r+1} - 1)}, \frac{2\pi k}{(a^r - 1)}\right] \end{aligned}$$

and

$$\begin{aligned} I_2 =& [2\pi c_1, 2\pi c_2] + 2\pi m_2 \\ =& \left[\frac{2\pi a^r k}{(a^r - 1)}, \frac{2\pi a^{r+1}(k+L)}{(a^{r+1}-1)} \right]. \end{aligned}$$

Then

$$W = I_1 \cup I_2 = \left[\frac{2\pi(k+L)}{(a^{r+1}-1)}, \frac{2\pi k}{(a^r - 1)} \right] \cup \left[\frac{2\pi a^r k}{(a^r - 1)}, \frac{2\pi a^{r+1}(k+L)}{(a^{r+1}-1)} \right],$$

is an a-multiwavelet set of order L for $H^2(\mathbb{R})$ consisting of two disjoint intervals, where k and r are natural numbers which satisfy (3.1).

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