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On Four Special Cases of Generalized Tribonacci Sequence: Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas Sequences

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Abstract. In this paper, we investigate four new special cases, namely, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas sequences, of the generalized Tribonacci sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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1. Introduction

Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 1,$$

and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 1, K_2 = 3,$$

respectively. Tribonacci concept was introduced by 14 year old student M. Feinberg [6] in 1963. Basic properties of these sequences are given in [1,2,3,4,5,10,11,12,13,15,21,22,23].

It is the aim of this paper to explore some of the properties of generalized Tribonacci numbers and is to investigate, in details, four particular case, namely sequences of Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers. For completeness, we also present some of the well known propertises of Tribonacci and Tribonacci-Lucas numbers. Before, we recall the generalized 3-step Fibonacci sequence and its some properties.

The generalized 3-step Fibonacci sequence (also called the generalized Tribonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned}
\alpha &= \alpha(r, s, t) = \frac{r}{3} + A + B, \\
\beta &= \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B, \\
\gamma &= \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \\
B &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\
\Delta &= \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\
\omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma &= r, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\
\alpha\beta\gamma &= t.
\end{aligned}$$

If $\Delta(r, s, t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized 3-step Fibonacci numbers (generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{1.3}$$

where

$$\begin{aligned} b_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \\ b_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \\ b_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \end{aligned}$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [7]. This result of Howard and Saidak [7] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

LEMMA 1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized 3-step Fibonacci sequence (generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.4}$$

We next present Binet formula of generalized 3-step Fibonacci numbers (generalized Tribonacci numbers) $\{W_n\}$ by the use of generating function for W_n.

THEOREM 2. *(Binet formula of generalized 3-step Fibonacci numbers (generalized Tribonacci numbers))*

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.5}$$

where

$$\begin{aligned} d_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ d_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ d_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Proof. The proof follows from Lemma 1.

Note that from (1.3) and (1.5) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

In this paper we consider the case $r = 1$, $s = 1, t = 1$ and in this case we write $V_n = W_n$ and in this case we also call the sequence V_n as generalized Tribonacci sequence. So a generalized Tribonacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} \tag{1.6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.6) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized Tribonacci numbers. Binet formula of generalized Tribonacci numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.7}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \tag{1.8}$$
$$b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \tag{1.9}$$
$$b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.10}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x^2 - x - 1 = 0.$$

Moreover

$$\begin{aligned}
\alpha &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}, \\
\beta &= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\
\gamma &= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3},
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma &= 1, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized Tribonacci numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Tribonacci numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$V_2 - V_1 - V_0$
2	V_2	$-V_2 + 2V_1$
3	$V_2 + V_1 + V_0$	$-V_1 + 2V_0$
4	$2V_2 + 2V_1 + V_0$	$2V_2 - 2V_1 - 3V_0$
5	$4V_2 + 3V_1 + 2V_0$	$-3V_2 + 5V_1 + V_0$
6	$7V_2 + 6V_1 + 4V_0$	$V_2 - 4V_1 + 4V_0$
7	$13V_2 + 11V_1 + 7V_0$	$4V_2 - 3V_1 - 8V_0$
8	$24V_2 + 20V_1 + 13V_0$	$-8V_2 + 12V_1 + 5V_0$
9	$44V_2 + 37V_1 + 24V_0$	$5V_2 - 13V_1 + 7V_0$
10	$81V_2 + 68V_1 + 44V_0$	$7V_2 - 2V_1 - 20V_0$

Now we define four new special cases of the sequence $\{V_n\}$ besides the well known Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas (Tribonacci-Lucas-Lucas) sequence $\{K_n\}_{n\geq 0}$.

Tribonacci sequence$\{T_n\}_{n\geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$, Tribonacci-Perrin sequence $\{M_n\}_{n\geq 0}$, modified Tribonacci sequence $\{U_n\}_{n\geq 0}$, modified Tribonacci-Lucas sequence $\{G_n\}_{n\geq 0}$ and adjusted Tribonacci-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{1.11}$$

$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \tag{1.12}$$

$$M_{n+3} = M_{n+2} + M_{n+1} + M_n, \qquad M_0 = 3, M_1 = 0, M_2 = 2, \tag{1.13}$$

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \qquad U_0 = 1, U_1 = 1, U_2 = 1. \tag{1.14}$$

$$G_{n+3} = G_{n+2} + G_{n+1} + G_n, \qquad G_0 = 4, G_1 = 4, G_2 = 10, \tag{1.15}$$

$$H_{n+3} = H_{n+2} + H_{n+1} + H_n, \qquad H_0 = 4, H_1 = 2, H_2 = 0, \tag{1.16}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$, $\{M_n\}_{n\geq 0}$, $\{U_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$, and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \tag{1.17}$$

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \tag{1.18}$$

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} + M_{-(n-3)}, \tag{1.19}$$

$$U_{-n} = -U_{-(n-1)} - U_{-(n-2)} + U_{-(n-3)}, \tag{1.20}$$

$$G_{-n} = -G_{-(n-1)} - G_{-(n-2)} + G_{-(n-3)}, \tag{1.21}$$

$$H_{-n} = -H_{-(n-1)} - H_{-(n-2)} + H_{-(n-3)}, \tag{1.22}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.11)-(1.16) hold for all integer n.

T_n is the sequence A000073 in [14] and K_n is the sequence A001644 in [14] and U_n is the sequence A000213 in [14].

Next, we present the first few values of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927
T_{-n}		0	1	-1	0	2	-3	1	4	-8	5	7	-20	18
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757
K_{-n}		-1	-1	5	-5	-1	11	-15	3	23	-41	21	43	-105
M_n	3	0	2	5	7	14	26	47	87	160	294	541	995	1830
M_{-n}		-1	-2	6	-5	-3	14	-16	-1	31	-46	14	63	-123
U_n	1	1	1	3	5	9	17	31	57	105	193	355	653	1201
U_{-n}		-1	1	1	-3	3	1	-7	9	-1	-15	25	-11	-29
G_n	4	4	10	18	32	60	110	202	372	684	1258	2314	4256	7828
G_{-n}		2	-2	4	0	-6	10	-4	-12	26	-18	-20	64	-62
H_n	4	2	0	6	8	14	28	50	92	170	312	574	1056	1942
H_{-n}		-6	4	6	-16	14	8	-38	44	2	-84	126	-40	-170

For all integers n, Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers (using initial conditions in (1.8)-(1.10)) can be expressed using Binet's formulas as

$$\begin{aligned}
T_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
K_n &= \alpha^n+\beta^n+\gamma^n, \\
M_n &= \frac{(2\alpha+3)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(2\beta+3)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(2\gamma+3)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
U_n &= \frac{(\alpha^2+1)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(\beta^2+1)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(\gamma^2+1)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
G_n &= (\alpha+1)\alpha^n+(\beta+1)\beta^n+(\gamma+1)\gamma^n, \\
H_n &= (\alpha-1)^2\alpha^n+(\beta-1)^2\beta^n+(\gamma-1)^2\gamma^n,
\end{aligned}$$

respectively.

2. Generating Functions and Obtaining Binet Formula From Generating Function

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

LEMMA 3. *Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3}. \tag{2.1}$$

Proof. Take $r = 1, s = 1, t = 1$ in Lemma 1.

The previous lemma gives the following results as particular examples.

COROLLARY 4. *Generated functions of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n x^n &= \frac{x}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} K_n x^n &= \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} M_n x^n &= \frac{3 - 3x - x^2}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} U_n x^n &= \frac{1 - x^2}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} G_n x^n &= \frac{4 + 2x^2}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{4 - 2x - 6x^2}{1 - x - x^2 - x^3},
\end{aligned}$$

respectively.

We next present Binet formula of generalized Tribonacci numbers $\{V_n\}$ by the use of generating function for V_n.

THEOREM 5. *(Binet formula of generalized Tribonacci numbers)*

$$V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{d_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{d_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{2.2}$$

where

$$\begin{aligned}
d_1 &= V_0 \alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0), \\
d_2 &= V_0 \beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0), \\
d_3 &= V_0 \gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0).
\end{aligned}$$

Proof. Take $r = 1, s = 1, t = 1$ in Theorem 2.

Note that from (1.7) and (2.2) we have

$$\begin{aligned}
V_2-(\beta+\gamma)V_1+\beta\gamma V_0 &= V_0\alpha^2+(V_1-V_0)\alpha+(V_2-V_1-V_0),\\
V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0 &= V_0\beta^2+(V_1-V_0)\beta+(V_2-V_1-V_0),\\
V_2-(\alpha+\beta)V_1+\alpha\beta V_0 &= V_0\gamma^2+(V_1-V_0)\gamma+(V_2-V_1-V_0).
\end{aligned}$$

Next, using the last Theorem, we present the Binet formulas of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas sequences.

COROLLARY 6. *Binet formulas of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas sequences are*

$$\begin{aligned}
T_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\\
K_n &= \alpha^n+\beta^n+\gamma^n,\\
M_n &= \frac{(2\alpha+3)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(2\beta+3)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(2\gamma+3)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)},\\
U_n &= \frac{(\alpha^2+1)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(\beta^2+1)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(\gamma^2+1)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)},\\
G_n &= (\alpha+1)\alpha^n+(\beta+1)\beta^n+(\gamma+1)\gamma^n,\\
H_n &= (\alpha-1)^2\alpha^n+(\beta-1)^2\beta^n+(\gamma-1)^2\gamma^n,
\end{aligned}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [9]. Take $k=i=3$ in Corollary 3.1 in [9]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix}\alpha^2 & \alpha & 1\\ \beta^2 & \beta & 1\\ \gamma^2 & \gamma & 1\end{pmatrix},\ \Lambda_1=\begin{pmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{pmatrix},\\
\Lambda_2 &= \begin{pmatrix}\alpha^2 & \alpha^{n-1} & 1\\ \beta^2 & \beta^{n-1} & 1\\ \gamma^2 & \gamma^{n-1} & 1\end{pmatrix},\ \Lambda_3=\begin{pmatrix}\alpha^2 & \alpha & \alpha^{n-1}\\ \beta^2 & \beta & \beta^{n-1}\\ \gamma^2 & \gamma & \gamma^{n-1}\end{pmatrix}.
\end{aligned}$$

Then the Binet formula for Tribonacci numbers is

$$\begin{aligned}
T_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{3}T_{4-j}\det(\Lambda_j)=\frac{1}{\det(\Lambda)}(T_3\det(\Lambda_1)+T_2\det(\Lambda_2)+T_1\det(\Lambda_3))\\
&= \frac{1}{\det(\Lambda)}(2\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3))\\
&= \left(2\begin{vmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{vmatrix}+\begin{vmatrix}\alpha^2 & \alpha^{n-1} & 1\\ \beta^2 & \beta^{n-1} & 1\\ \gamma^2 & \gamma^{n-1} & 1\end{vmatrix}+\begin{vmatrix}\alpha^2 & \alpha & \alpha^{n-1}\\ \beta^2 & \beta & \beta^{n-1}\\ \gamma^2 & \gamma & \gamma^{n-1}\end{vmatrix}\right)\Big/\begin{vmatrix}\alpha^2 & \alpha & 1\\ \beta^2 & \beta & 1\\ \gamma^2 & \gamma & 1\end{vmatrix}.
\end{aligned}$$

Similarly, we obtain the Binet formula for Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas as

$$\begin{aligned}
K_n &= \frac{1}{\det(\Lambda)}(7\det(\Lambda_1)+3\det(\Lambda_2)+\det(\Lambda_3)) \\
M_n &= \frac{1}{\det(\Lambda)}(5\det(\Lambda_1)+2M_2\det(\Lambda_2)), \\
U_n &= \frac{1}{\det(\Lambda)}(3\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3)), \\
G_n &= \frac{1}{\det(\Lambda)}(18\det(\Lambda_1)+10\det(\Lambda_2)+4\det(\Lambda_3)), \\
H_n &= \frac{1}{\det(\Lambda)}(6\det(\Lambda_1)+2\det(\Lambda_3)),
\end{aligned}$$

respectively.

3. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$.

THEOREM 7 (Simson Formula of Generalized Tribonacci Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{3.1}$$

Proof. (3.1) is given in Soykan [17].

The previous theorem gives the following results as particular examples.

COROLLARY 8. *For all integers n, Simson formula of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers are given as*

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = -1,$$

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = -44,$$

$$\begin{vmatrix} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{vmatrix} = -41,$$

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -4,$$

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -88,$$

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = -176,$$

respectively.

4. Some Identities

In this section, we obtain some identities of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers. First, we can give a few basic relations between $\{T_n\}$ and $\{K_n\}$.

LEMMA 9. *The following equalities are true:*

$$\begin{aligned} 22T_n &= K_{n+4} - 5K_{n+3} + 8K_{n+2}, \\ 22T_n &= -4K_{n+3} + 9K_{n+2} + K_{n+1}, \\ 22T_n &= 5K_{n+2} - 3K_{n+1} - 4K_n, \\ 22T_n &= 2K_{n+1} + K_n + 5K_{n-1}, \\ 22T_n &= 3K_n + 7K_{n-1} + 2K_{n-2}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} K_n &= 5T_{n+4} - 6T_{n+3} - 5T_{n+2}, \\ K_n &= -T_{n+3} + 5T_{n+1}, \\ K_n &= -T_{n+2} + 4T_{n+1} - T_n, \\ K_n &= 3T_{n+1} - 2T_n - T_{n-1}, \\ K_n &= T_n + 2T_{n-1} + 3T_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$K_n = a \times T_{n+4} + b \times T_{n+3} + c \times T_{n+2}$$

and solving the system of equations

$$\begin{aligned} K_0 &= a \times T_4 + b \times T_3 + c \times T_2 \\ K_1 &= a \times T_5 + b \times T_4 + c \times T_3 \\ K_2 &= a \times T_6 + b \times T_5 + c \times T_4 \end{aligned}$$

we find that $a = \frac{1}{22}, b = -\frac{5}{22}, c = \frac{4}{11}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{T_n\}$ and $\{M_n\}$.

LEMMA 10. *The following equalities are true:*

$$\begin{aligned} 41T_n &= 4M_{n+4} - 10M_{n+3} + 11M_{n+2}, \\ 41T_n &= -6M_{n+3} + 15M_{n+2} + 4M_{n+1}, \\ 41T_n &= 9M_{n+2} - 2M_{n+1} - 6M_n, \\ 41T_n &= 7M_{n+1} + 3M_n + 9M_{n-1}, \\ 41T_n &= 10M_n + 16M_{n-1} + 7M_{n-2}, \end{aligned}$$

and

$$\begin{aligned} M_n &= 6T_{n+4} - 8T_{n+3} - 5T_{n+2}, \\ M_n &= -2T_{n+3} + T_{n+2} + 6T_{n+1}, \\ M_n &= -T_{n+2} + 4T_{n+1} - 2T_n, \\ M_n &= 3T_{n+1} - 3T_n - T_{n-1}, \\ M_n &= 2T_{n-1} + 3T_{n-2}, \end{aligned}$$

Now, we give a few basic relations between $\{T_n\}$ and $\{U_n\}$.

LEMMA 11. *The following equalities are true:*

$$\begin{aligned} 2T_n &= -U_{n+4} + U_{n+3} + 2U_{n+2}, \\ 2T_n &= U_{n+2} - U_{n+1}, \\ 2T_n &= U_n + U_{n-1}, \end{aligned}$$

and

$$\begin{aligned} U_n &= T_{n+4} - 3T_{n+2}, \\ U_n &= T_{n+3} - 2T_{n+2} + T_{n+1}, \\ U_n &= -T_{n+2} + 2T_{n+1} + T_n, \\ U_n &= T_{n+1} - T_{n-1}, \\ U_n &= T_n + T_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{T_n\}$ and $\{G_n\}$.

LEMMA 12. *The following equalities are true:*

$$\begin{aligned} 22T_n &= 7G_{n+4} - 13G_{n+3} + G_{n+2}, \\ 22T_n &= -6G_{n+3} + 8G_{n+2} + 7G_{n+1}, \\ 22T_n &= 2G_{n+2} + G_{n+1} - 6G_n, \\ 22T_n &= 3G_{n+1} - 4G_n + 2G_{n-1}, \\ 22T_n &= -G_n + 5G_{n-1} + 3G_{n-2}, \end{aligned}$$

and

$$\begin{aligned} G_n &= 4T_{n+4} - 6T_{n+3}, \\ G_n &= -2T_{n+3} + 4T_{n+2} + 4T_{n+1}, \\ G_n &= 2T_{n+2} + 2T_{n+1} - 2T_n, \\ G_n &= 4T_{n+1} + 2T_{n-1}, \\ G_n &= 4T_n + 6T_{n-1} + 4T_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{T_n\}$ and $\{H_n\}$.

LEMMA 13. *The following equalities are true:*

$$\begin{aligned} 44T_n &= -3H_{n+4} + 4H_{n+3} + 9H_{n+2}, \\ 44T_n &= H_{n+3} + 6H_{n+2} - 3H_{n+1}, \\ 44T_n &= 7H_{n+2} - 2H_{n+1} + H_n, \\ 44T_n &= 5H_{n+1} + 8H_n + 7H_{n-1}, \\ 44T_n &= 13H_n + 12H_{n-1} + 5H_{n-2}, \end{aligned}$$

and

$$
\begin{aligned}
H_n &= 6T_{n+4} - 2T_{n+3} - 16T_{n+2}, \\
H_n &= 4T_{n+3} - 10T_{n+2} + 6T_{n+1}, \\
H_n &= -6T_{n+2} + 10T_{n+1} + 4T_n, \\
H_n &= 4T_{n+1} - 2T_n - 6T_{n-1}, \\
H_n &= 2T_n - 2T_{n-1} + 4T_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{K_n\}$ and $\{M_n\}$.

LEMMA 14. *The following equalities are true:*

$$
\begin{aligned}
41K_n &= -37M_{n+4} + 72M_{n+3} + 11M_{n+2}, \\
41K_n &= 35M_{n+3} - 26M_{n+2} - 37M_{n+1}, \\
41K_n &= 9M_{n+2} - 2M_{n+1} + 35M_n, \\
41K_n &= 7M_{n+1} + 44M_n + 9M_{n-1}, \\
41K_n &= 51M_n + 16M_{n-1} + 7M_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
22M_n &= -23K_{n+4} + 49K_{n+3} - 8K_{n+2}, \\
22M_n &= 26K_{n+3} - 31K_{n+2} - 23K_{n+1}, \\
22M_n &= -5K_{n+2} + 3K_{n+1} + 26K_n, \\
22M_n &= -2K_{n+1} + 21K_n - 5K_{n-1}, \\
22M_n &= 19K_n - 7K_{n-1} - 2K_{n-2}.
\end{aligned}
$$

Now, we give a few basic relations between $\{U_n\}$ and $\{K_n\}$.

LEMMA 15. *The following equalities are true:*

$$
\begin{aligned}
K_n &= 2U_{n+3} - 3U_{n+2}, \\
K_n &= -U_{n+2} + 2U_{n+1} + 2U_n, \\
K_n &= U_{n+1} + U_n - U_{n-1}, \\
K_n &= 2U_n + U_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
11U_n &= -6K_{n+4} + 8K_{n+3} + 7K_{n+2}, \\
11U_n &= 2K_{n+3} + K_{n+2} - 6K_{n+1}, \\
11U_n &= 3K_{n+2} - 4K_{n+1} + 2K_n, \\
11U_n &= -K_{n+1} + 5K_n + 3K_{n-1}, \\
11U_n &= 4K_n + 2K_{n-1} - K_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{K_n\}$ and $\{G_n\}$.

LEMMA 16. *The following equalities are true:*

$$
\begin{aligned}
2K_n &= -3G_{n+4} + 4G_{n+3} + 3G_{n+2}, \\
2K_n &= G_{n+3} - 3G_{n+1}, \\
2K_n &= G_{n+2} - 2G_{n+1} + G_n, \\
2K_n &= -G_{n+1} + 2G_n + G_{n-1}, \\
2K_n &= G_n - G_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
G_n &= K_{n+3} - K_{n+2}, \\
G_n &= K_{n+1} + K_n, \\
G_n &= 2K_n + K_{n-1} + K_{n-2}.
\end{aligned}
$$

Now, we give a few basic relations between $\{K_n\}$ and $\{H_n\}$.

LEMMA 17. *The following equalities are true:*

$$
\begin{aligned}
2K_n &= H_{n+3} - H_{n+2}, \\
2K_n &= H_{n+1} + H_n, \\
2K_n &= 2H_n + H_{n-1} + H_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
H_n &= -3K_{n+4} + 4K_{n+3} + 3K_{n+2}, \\
H_n &= K_{n+3} - 3K_{n+1}, \\
H_n &= K_{n+2} - 2K_{n+1} + K_n, \\
H_n &= -K_{n+1} + 2K_n + K_{n-1}, \\
H_n &= K_n - K_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{M_n\}$ and $\{U_n\}$.

LEMMA 18. *The following equalities are true:*

$$
\begin{aligned}
2M_n &= U_{n+4} + 3U_{n+3} - 8U_{n+2}, \\
2M_n &= 4U_{n+3} - 7U_{n+2} + U_{n+1}, \\
2M_n &= -3U_{n+2} + 5U_{n+1} + 4U_n, \\
2M_n &= 2U_{n+1} + U_n - 3U_{n-1}, \\
2M_n &= 3U_n - U_{n-1} + 2U_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
41U_n &= -17M_{n+4} + 22M_{n+3} + 25M_{n+2}, \\
41U_n &= 5M_{n+3} + 8M_{n+2} - 17M_{n+1}, \\
41U_n &= 13M_{n+2} - 12M_{n+1} + 5M_n, \\
41U_n &= M_{n+1} + 18M_n + 13M_{n-1}, \\
41U_n &= 19M_n + 14M_{n-1} + M_{n-2}.
\end{aligned}
$$

Now, we give a few basic relations between $\{M_n\}$ and $\{G_n\}$.

LEMMA 19. *The following equalities are true:*

$$
\begin{aligned}
22M_n &= -40G_{n+4} + 57G_{n+3} + 32G_{n+2}, \\
22M_n &= 17G_{n+3} - 8G_{n+2} - 40G_{n+1}, \\
22M_n &= 9G_{n+2} - 23G_{n+1} + 17G_n, \\
22M_n &= -14G_{n+1} + 26G_n + 9G_{n-1}, \\
22M_n &= 12G_n - 5G_{n-1} - 14G_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
41G_n &= -2M_{n+4} + 46M_{n+3} - 26M_{n+2}, \\
41G_n &= 44M_{n+3} - 28M_{n+2} - 2M_{n+1}, \\
41G_n &= 16M_{n+2} + 42M_{n+1} + 44M_n, \\
41G_n &= 58M_{n+1} + 60M_n + 16M_{n-1}, \\
41G_n &= 118M_n + 74M_{n-1} + 58M_{n-2},
\end{aligned}
$$

Next, we present a few basic relations between $\{M_n\}$ and $\{H_n\}$.

LEMMA 20. *The following equalities are true:*

$$\begin{aligned}
44M_n &= 3H_{n+4}+18H_{n+3}-31H_{n+2},\\
44M_n &= 21H_{n+3}-28H_{n+2}+3H_{n+1},\\
44M_n &= -7H_{n+2}+24H_{n+1}+21H_n,\\
44M_n &= 17H_{n+1}+14H_n-7H_{n-1},\\
44M_n &= 31H_n+10H_{n-1}+17H_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
41H_n &= -98M_{n+4}+122M_{n+3}+120M_{n+2},\\
41H_n &= 24M_{n+3}+22M_{n+2}-98M_{n+1},\\
41H_n &= 46M_{n+2}-74M_{n+1}+24M_n,\\
41H_n &= -28M_{n+1}+70M_n+46M_{n-1},\\
41H_n &= 42M_n+18M_{n-1}-28M_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{U_n\}$ and $\{G_n\}$.

LEMMA 21. *The following equalities are true:*

$$\begin{aligned}
22U_n &= -7G_{n+4}+2G_{n+3}+21G_{n+2},\\
22U_n &= -5G_{n+3}+14G_{n+2}-7G_{n+1},\\
22U_n &= 9G_{n+2}-12G_{n+1}-5G_n,\\
22U_n &= -3G_{n+1}+4G_n+9G_{n-1},\\
22U_n &= G_n+6G_{n-1}-3G_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
G_n &= 2U_{n+4}-U_{n+3}-3U_{n+2},\\
G_n &= U_{n+3}-U_{n+2}+2U_{n+1},\\
G_n &= 3U_{n+1}+U_n,\\
G_n &= 4U_n+3U_{n-1}+3U_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{U_n\}$ and $\{H_n\}$.

LEMMA 22. *The following equalities are true:*

$$\begin{aligned}22U_n &= -4H_{n+4}+9H_{n+3}+H_{n+2},\\ 22U_n &= 5H_{n+3}-3H_{n+2}-4H_{n+1},\\ 22U_n &= 2H_{n+2}+H_{n+1}+5H_n,\\ 22U_n &= 3H_{n+1}+7H_n+2H_{n-1},\\ 22U_n &= 10H_n+5H_{n-1}+3H_{n-2},\end{aligned}$$

and

$$\begin{aligned}H_n &= -5U_{n+4}+10U_{n+3}-U_{n+2},\\ H_n &= 5U_{n+3}-6U_{n+2}-5U_{n+1},\\ H_n &= -U_{n+2}+5U_n,\\ H_n &= -U_{n+1}+4U_n-U_{n-1},\\ H_n &= 3U_n-2U_{n-1}-U_{n-2}.\end{aligned}$$

Next, we present a few basic relations between $\{G_n\}$ and $\{H_n\}$.

LEMMA 23. *The following equalities are true:*

$$\begin{aligned}2G_n &= H_{n+4}-H_{n+2},\\ 2G_n &= H_{n+3}+H_{n+1},\\ 2G_n &= H_{n+2}+2H_{n+1}+H_n,\\ 2G_n &= 3H_{n+1}+2H_n+H_{n-1},\\ 2G_n &= 5H_n+4H_{n-1}+3H_{n-2},\end{aligned}$$

and

$$\begin{aligned}H_n &= -2G_{n+4}+G_{n+3}+5G_{n+2},\\ H_n &= -G_{n+3}+3G_{n+2}-2G_{n+1},\\ H_n &= 2G_{n+2}-3G_{n+1}-G_n,\\ H_n &= -G_{n+1}+G_n+2G_{n-1},\\ H_n &= G_{n-1}-G_{n-2}.\end{aligned}$$

5. Sum Formulas

5.1. Sum of Terms. The following proposition presents some formulas of generalized Tribonacci numbers with positive subscripts.

Proposition 24. *If $r=1, s=1, t=1$ then for $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+3} - V_{n+1} - V_2 + V_0)$.

(b): $\sum_{k=0}^{n} V_{2k} = \frac{1}{2}(V_{2n+1} + V_{2n} - V_1 + V_0)$.

(c): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{2}(V_{2n+2} + V_{2n+1} - V_2 + V_1)$.

Proof. Take $r=1, s=1, t=1$ in Theorem 2.1 in [16] or (or take $x=1, r=1, s=1, t=1$ in Theorem 2.1 in [18]).

As special cases of above proposition, we have the following six corollaries.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 25. *For $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} - T_{n+1} - 1)$.

(b): $\sum_{k=0}^{n} T_{2k} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1)$.

(c): $\sum_{k=0}^{n} T_{2k+1} = \frac{1}{2}(T_{2n+2} + T_{2n+1})$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 26. *For $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+3} - K_{n+1})$.

(b): $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(K_{2n+1} + K_{2n} + 2)$.

(c): $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 27. *For $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} M_k = \frac{1}{2}(M_{n+3} - M_{n+1} + 1)$.

(b): $\sum_{k=0}^{n} M_{2k} = \frac{1}{2}(M_{2n+1} + M_{2n} + 3)$.

(c): $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{2}(M_{2n+2} + M_{2n+1} - 2)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 28. *For $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} U_k = \frac{1}{2}(U_{n+3} - U_{n+1})$.

(b): $\sum_{k=0}^{n} U_{2k} = \frac{1}{2}(U_{2n+1} + U_{2n})$.

(c): $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{2}(U_{2n+2} + U_{2n+1})$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

COROLLARY 29. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} G_k = \frac{1}{2}(G_{n+3} - G_{n+1} - 6)$.

(b): $\sum_{k=0}^{n} G_{2k} = \frac{1}{2}(G_{2n+1} + G_{2n})$.

(c): $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{2}(G_{2n+2} + G_{2n+1} - 6)$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 30. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} H_k = \frac{1}{2}(H_{n+3} - H_{n+1} + 4)$.

(b): $\sum_{k=0}^{n} H_{2k} = \frac{1}{2}(H_{2n+1} + H_{2n} + 2)$.

(c): $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{2}(H_{2n+2} + H_{2n+1} + 2)$.

The following proposition presents some formulas of generalized Tribonacci numbers with negative subscripts.

PROPOSITION 31. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} V_{-k} = \frac{1}{2}\left(-3V_{-n-1} - 2V_{-n-2} - V_{-n-3} + V_2 - V_0\right)$.

(b): $\sum_{k=1}^{n} V_{-2k} = \frac{1}{2}\left(-V_{-2n+1} + V_{-2n} + V_1 - V_0\right)$.

(c): $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{2}\left(-V_{-2n} - V_{-2n-1} + V_2 - V_1\right)$.

Proof. Take $r = 1, s = 1, t = 1$ in Theorem 3.1 in [16] or (or take $x = 1, r = 1, s = 1, t = 1$ in Theorem 3.1 in [18]).

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 32. *For $n \geq 1$, Tribonacci numbers have the following properties.*

(a): $\sum_{k=1}^{n} T_{-k} = \frac{1}{2}(-3T_{-n-1} - 2T_{-n-2} - T_{-n-3} + 1)$.

(b): $\sum_{k=1}^{n} T_{-2k} = \frac{1}{2}(-T_{-2n+1} + T_{-2n} + 1)$.

(c): $\sum_{k=1}^{n} T_{-2k+1} = \frac{1}{2}(-T_{-2n} - T_{-2n-1})$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 33. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n} K_{-k} = \frac{1}{2}(-3K_{-n-1} - 2K_{-n-2} - K_{-n-3})$.

(b): $\sum_{k=1}^{n} K_{-2k} = \frac{1}{2}(-K_{-2n+1} + K_{-2n} - 2)$.

(c): $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{2}(-K_{-2n} - K_{-2n-1} + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

COROLLARY 34. *For $n \geq 1$, Tribonacci-Perrin numbers have the following properties.*

(a): $\sum_{k=1}^{n} M_{-k} = \frac{1}{2}\left(-3M_{-n-1} - 2M_{-n-2} - M_{-n-3} - 1\right)$.

(b): $\sum_{k=1}^{n} M_{-2k} = \frac{1}{2}\left(-M_{-2n+1} + M_{-2n} - 3\right)$.

(c): $\sum_{k=1}^{n} M_{-2k+1} = \frac{1}{2}\left(-M_{-2n} - M_{-2n-1} + 2\right)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

COROLLARY 35. *For $n \geq 1$, modified Tribonacci numbers have the following properties.*

(a): $\sum_{k=1}^{n} U_{-k} = \frac{1}{2}\left(-3U_{-n-1} - 2U_{-n-2} - U_{-n-3}\right)$.

(b): $\sum_{k=1}^{n} U_{-2k} = \frac{1}{2}\left(-U_{-2n+1} + U_{-2n}\right)$.

(c): $\sum_{k=1}^{n} U_{-2k+1} = \frac{1}{2}\left(-U_{-2n} - U_{-2n-1}\right)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

COROLLARY 36. *For $n \geq 1$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} G_{-k} = \frac{1}{2}\left(-3G_{-n-1} - 2G_{-n-2} - G_{-n-3} + 6\right)$.

(b): $\sum_{k=1}^{n} G_{-2k} = \frac{1}{2}\left(-G_{-2n+1} + G_{-2n}\right)$.

(c): $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{2}\left(-G_{-2n} - G_{-2n-1} + 6\right)$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 37. *For $n \geq 1$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} H_{-k} = \frac{1}{2}\left(-3H_{-n-1} - 2H_{-n-2} - H_{-n-3} - 4\right)$.

(b): $\sum_{k=1}^{n} H_{-2k} = \frac{1}{2}\left(-H_{-2n+1} + H_{-2n} - 2\right)$.

(c): $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{2}\left(-H_{-2n} - H_{-2n-1} - 2\right)$.

The following proposition presents some formulas of generalized Tribonacci numbers with positive subscripts.

PROPOSITION 38. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} (-1)^k V_k = \frac{1}{2}((-1)^n \left(V_{n+3} - 2V_{n+2} + V_{n+1}\right) + V_2 - 2V_1 + V_0)$.

(b): $\sum_{k=0}^{n} (-1)^k V_{2k} = \frac{1}{2}((-1)^n V_{2n+2} - (-1)^n V_{2n+1} - V_2 + V_1 + 2V_0)$.

(c): $\sum_{k=0}^{n} (-1)^k V_{2k+1} = \frac{1}{2}((-1)^n V_{2n+1} + (-1)^n V_{2n} + V_1 - V_0)$.

Proof. Take $x = -1, r = 1, s = 1, t = 1$ in Theorem 2.1 in [18].

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 39. *For $n \geq 0$, Tribonacci numbers have the following properties.*

(a): $\sum_{k=0}^{n}(-1)^k T_k = \frac{1}{2}((-1)^n (T_{n+3} - 2T_{n+2} + T_{n+1}) - 1)$.

(b): $\sum_{k=0}^{n}(-1)^k T_{2k} = \frac{1}{2}(-1)^n (T_{2n+2} - T_{2n+1})$.

(c): $\sum_{k=0}^{n}(-1)^k T_{2k+1} = \frac{1}{2}((-1)^n T_{2n+1} + (-1)^n T_{2n} + 1)$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 40. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties.*

(a): $\sum_{k=0}^{n}(-1)^k K_k = \frac{1}{2}((-1)^n (K_{n+3} - 2K_{n+2} + K_{n+1}) + 4)$.

(b): $\sum_{k=0}^{n}(-1)^k K_{2k} = \frac{1}{2}((-1)^n K_{2n+2} - (-1)^n K_{2n+1} + 4)$.

(c): $\sum_{k=0}^{n}(-1)^k K_{2k+1} = \frac{1}{2}((-1)^n K_{2n+1} + (-1)^n K_{2n} - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

COROLLARY 41. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties.*

(a): $\sum_{k=0}^{n}(-1)^k M_k = \frac{1}{2}((-1)^n (M_{n+3} - 2M_{n+2} + M_{n+1}) + 5)$.

(b): $\sum_{k=0}^{n}(-1)^k M_{2k} = \frac{1}{2}((-1)^n M_{2n+2} - (-1)^n M_{2n+1} + 4)$.

(c): $\sum_{k=0}^{n}(-1)^k M_{2k+1} = \frac{1}{2}((-1)^n M_{2n+1} + (-1)^n M_{2n} - 3)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

COROLLARY 42. *For $n \geq 0$, modified Tribonacci numbers have the following properties.*

(a): $\sum_{k=0}^{n}(-1)^k U_k = \frac{1}{2}(-1)^n (U_{n+3} - 2U_{n+2} + U_{n+1})$.

(b): $\sum_{k=0}^{n}(-1)^k U_{2k} = \frac{1}{2}((-1)^n U_{2n+2} - (-1)^n U_{2n+1} + 2)$.

(c): $\sum_{k=0}^{n}(-1)^k U_{2k+1} = \frac{1}{2}((-1)^n U_{2n+1} + (-1)^n U_{2n})$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

COROLLARY 43. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k G_k = \frac{1}{2}((-1)^n (G_{n+3} - 2G_{n+2} + G_{n+1}) + 6)$.

(b): $\sum_{k=0}^{n}(-1)^k G_{2k} = \frac{1}{2}((-1)^n G_{2n+2} - (-1)^n G_{2n+1} + 2)$.

(c): $\sum_{k=0}^{n}(-1)^k G_{2k+1} = \frac{1}{2}((-1)^n G_{2n+1} + (-1)^n G_{2n})$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 44. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k H_k = \frac{1}{2}(-1)^n(H_{n+3} - 2H_{n+2} + H_{n+1})$.

(b): $\sum_{k=0}^{n}(-1)^k H_{2k} = \frac{1}{2}((-1)^n H_{2n+2} - (-1)^n H_{2n+1} + 10)$.

(c): $\sum_{k=0}^{n}(-1)^k H_{2k+1} = \frac{1}{2}((-1)^n H_{2n+1} + (-1)^n H_{2n} - 2)$.

The following proposition presents some formulas of generalized Tribonacci numbers with negative subscripts.

PROPOSITION 45. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(-1)^k V_{-k} = \frac{1}{2}((-1)^n V_{-n-1} + (-1)^n V_{-n-3} - V_2 + 2V_1 - V_0)$.

(b): $\sum_{k=1}^{n}(-1)^k V_{-2k} = \frac{1}{2}((-1)^n V_{-2n} - (-1)^n V_{-2n-1} + V_2 - V_1 - 2V_0)$.

(c): $\sum_{k=1}^{n}(-1)^k V_{-2k+1} = \frac{1}{2}((-1)^n V_{-2n+1} - (-1)^n V_{-2n} - V_1 + V_0)$.

Proof. Take $x = -1, r = 1, s = 1, t = 1$ in Theorem 3.1 in [18].

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 46. *For $n \geq 1$, Tribonacci numbers have the following properties.*

(a): $\sum_{k=1}^{n}(-1)^k T_{-k} = \frac{1}{2}((-1)^n T_{-n-1} + (-1)^n T_{-n-3} + 1)$.

(b): $\sum_{k=1}^{n}(-1)^k T_{-2k} = \frac{1}{2}(-1)^n(T_{-2n} - T_{-2n-1})$.

(c): $\sum_{k=1}^{n}(-1)^k T_{-2k+1} = \frac{1}{2}((-1)^n T_{-2n+1} - (-1)^n T_{-2n} - 1)$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Lucas numbers.

COROLLARY 47. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n}(-1)^k K_{-k} = \frac{1}{2}((-1)^n K_{-n-1} + (-1)^n K_{-n-3} - 4)$.

(b): $\sum_{k=1}^{n}(-1)^k K_{-2k} = \frac{1}{2}((-1)^n K_{-2n} - (-1)^n K_{-2n-1} - 4)$.

(c): $\sum_{k=1}^{n}(-1)^k K_{-2k+1} = \frac{1}{2}((-1)^n K_{-2n+1} - (-1)^n K_{-2n} + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

COROLLARY 48. *For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(-1)^k M_{-k} = \frac{1}{2}((-1)^n M_{-n-1} + (-1)^n M_{-n-3} - 5)$.

(b): $\sum_{k=1}^{n}(-1)^k M_{-2k} = \frac{1}{2}((-1)^n M_{-2n} - (-1)^n M_{-2n-1} - 4)$.

(c): $\sum_{k=1}^{n}(-1)^k M_{-2k+1} = \frac{1}{2}((-1)^n M_{-2n+1} - (-1)^n M_{-2n} + 3)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

COROLLARY 49. *For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} (-1)^k U_{-k} = \frac{1}{2}((-1)^n U_{-n-1} + (-1)^n U_{-n-3})$.

(b): $\sum_{k=1}^{n} (-1)^k U_{-2k} = \frac{1}{2}((-1)^n U_{-2n} - (-1)^n U_{-2n-1} - 2)$.

(c): $\sum_{k=1}^{n} (-1)^k U_{-2k+1} = \frac{1}{2}((-1)^n U_{-2n+1} - (-1)^n U_{-2n})$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

COROLLARY 50. *For $n \geq 1$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} (-1)^k G_{-k} = \frac{1}{2}((-1)^n G_{-n-1} + (-1)^n G_{-n-3} - 6)$.

(b): $\sum_{k=1}^{n} (-1)^k G_{-2k} = \frac{1}{2}((-1)^n G_{-2n} - (-1)^n G_{-2n-1} - 2)$.

(c): $\sum_{k=1}^{n} (-1)^k G_{-2k+1} = \frac{1}{2}((-1)^n G_{-2n+1} - (-1)^n G_{-2n})$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 51. *For $n \geq 1$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} (-1)^k H_{-k} = \frac{1}{2}((-1)^n H_{-n-1} + (-1)^n H_{-n-3})$.

(b): $\sum_{k=1}^{n} (-1)^k H_{-2k} = \frac{1}{2}((-1)^n H_{-2n} - (-1)^n H_{-2n-1} - 10)$.

(c): $\sum_{k=1}^{n} (-1)^k H_{-2k+1} = \frac{1}{2}((-1)^n H_{-2n+1} - (-1)^n H_{-2n} + 2)$.

5.2. Sum of Squares of Terms. The following proposition presents some formulas of generalized Tribonacci numbers with positive subscripts.

PROPOSITION 52. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} V_k^2 = \frac{1}{4}(-V_{n+3}^2 - 4V_{n+2}^2 - 5V_{n+1}^2 + 4V_{n+3}V_{n+2} + 2V_{n+3}V_{n+1} + V_2^2 + 4V_1^2 + 5V_0^2 - 4V_2V_1 - 2V_2V_0)$.

(b): $\sum_{k=0}^{n} V_{k+1}V_k = \frac{1}{4}(V_{n+3}^2 + 2V_{n+2}^2 + V_{n+1}^2 - 2V_{n+3}V_{n+2} - 2V_{n+2}V_{n+1} - V_2^2 - 2V_1^2 - V_0^2 + 2V_2V_1 + 2V_1V_0)$.

(c): $\sum_{k=0}^{n} V_{k+2}V_k = \frac{1}{4}(V_{n+3}^2 + V_{n+1}^2 - 2V_{n+3}V_{n+1} - V_2^2 - V_0^2 + 2V_2V_0)$.

Proof. Take $x = 1, r = 1, s = 1, t = 1$ in Theorem 4.1 in [20], see also [19].

From the last proposition, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 53. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+3}T_{n+2} + 2T_{n+3}T_{n+1} + 1)$.

(b): $\sum_{k=0}^{n} T_{k+1}T_k = \frac{1}{4}(T_{n+3}^2 + 2T_{n+2}^2 + T_{n+1}^2 - 2T_{n+3}T_{n+2} - 2T_{n+2}T_{n+1} - 1)$.

(c): $\sum_{k=0}^{n} T_{k+2}T_k = \frac{1}{4}(T_{n+3}^2 + T_{n+1}^2 - 2T_{n+3}T_{n+1} - 1)$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last Proposition, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 54. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} K_k^2 = \frac{1}{4}(-K_{n+3}^2 - 4K_{n+2}^2 - 5K_{n+1}^2 + 4K_{n+3}K_{n+2} + 2K_{n+3}K_{n+1} + 28)$.

(b): $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{4}(K_{n+3}^2 + 2K_{n+2}^2 + K_{n+1}^2 - 2K_{n+3}K_{n+2} - 2K_{n+2}K_{n+1} - 8)$.

(c): $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{4}(K_{n+3}^2 + K_{n+1}^2 - 2K_{n+3}K_{n+1})$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 55. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} M_k^2 = \frac{1}{4}(-M_{n+3}^2 - 4M_{n+2}^2 - 5M_{n+1}^2 + 4M_{n+3}M_{n+2} + 2M_{n+3}M_{n+1} + 37)$.

(b): $\sum_{k=0}^{n} M_{k+1}M_k = \frac{1}{4}(M_{n+3}^2 + 2M_{n+2}^2 + M_{n+1}^2 - 2M_{n+3}M_{n+2} - 2M_{n+2}M_{n+1} - 13)$.

(c): $\sum_{k=0}^{n} M_{k+2}M_k = \frac{1}{4}(M_{n+3}^2 + M_{n+1}^2 - 2M_{n+3}M_{n+1} - 1)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the above proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 56. *For $n \geq 0$, modified Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n} U_k^2 = \frac{1}{4}(-U_{n+3}^2 - 4U_{n+2}^2 - 5U_{n+1}^2 + 4U_{n+3}U_{n+2} + 2U_{n+3}U_{n+1} + 4)$.

(b): $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{4}(U_{n+3}^2 + 2U_{n+2}^2 + U_{n+1}^2 - 2U_{n+3}U_{n+2} - 2U_{n+2}U_{n+1})$.

(c): $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{4}(U_{n+3}^2 + U_{n+1}^2 - 2U_{n+3}U_{n+1})$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 57. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} G_k^2 = \frac{1}{4}(-G_{n+3}^2 - 4G_{n+2}^2 - 5G_{n+1}^2 + 4G_{n+3}G_{n+2} + 2G_{n+3}G_{n+1} + 4)$.

(b): $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{4}(G_{n+3}^2 + 2G_{n+2}^2 + G_{n+1}^2 - 2G_{n+3}G_{n+2} - 2G_{n+2}G_{n+1} - 36)$.

(c): $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{4}(G_{n+3}^2 + G_{n+1}^2 - 2G_{n+3}G_{n+1} - 36)$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 58. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} H_k^2 = \frac{1}{4}(-H_{n+3}^2 - 4H_{n+2}^2 - 5H_{n+1}^2 + 4H_{n+3}H_{n+2} + 2H_{n+3}H_{n+1} + 96)$.

(b): $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{4}(H_{n+3}^2 + 2H_{n+2}^2 + H_{n+1}^2 - 2H_{n+3}H_{n+2} - 2H_{n+2}H_{n+1} - 8)$.

(c): $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{4}(H_{n+3}^2 + H_{n+1}^2 - 2H_{n+3}H_{n+1} - 16)$.

The following proposition presents some formulas of generalized Tribonacci numbers with positive subscripts.

PROPOSITION 59. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n}(-1)^k V_k^2 = \frac{1}{4}((-1)^n (V_{n+3}^2 - 2V_{n+2}^2 + 3V_{n+1}^2 - 2V_{n+1}V_{n+3}) + V_2^2 - 2V_1^2 + 3V_0^2 - 2V_0V_2)$.

(b): $\sum_{k=0}^{n}(-1)^k V_{k+1}V_k = \frac{1}{4}((-1)^n (V_{n+3}^2 - V_{n+1}^2 - 2V_{n+3}V_{n+2} + 2V_{n+2}V_{n+1}) + V_2^2 - V_0^2 - 2V_1V_2 + 2V_1V_0)$.

(c): $\sum_{k=0}^{n}(-1)^k V_{k+2}V_k = \frac{1}{4}((-1)^n (V_{n+3}^2 - 2V_{n+2}^2 - V_{n+1}^2 + 2V_{n+3}V_{n+1} - 4V_{n+2}V_{n+1}) + V_2^2 - 2V_1^2 - V_0^2 + 2V_2V_0 - 4V_1V_0)$.

Proof. Take $x = -1, r = 1, s = 1, t = 1$ in Theorem 4.29 in [20]

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 60. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k T_k^2 = \frac{1}{4}((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 + 3T_{n+1}^2 - 2T_{n+1}T_{n+3}) - 1)$.

(b): $\sum_{k=0}^{n}(-1)^k T_{k+1}T_k = \frac{1}{4}((-1)^n (T_{n+3}^2 - T_{n+1}^2 - 2T_{n+3}T_{n+2} + 2T_{n+2}T_{n+1}) - 1)$.

(c): $\sum_{k=0}^{n}(-1)^k T_{k+2}T_k = \frac{1}{4}((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 - T_{n+1}^2 + 2T_{n+3}T_{n+1} - 4T_{n+2}T_{n+1}) - 1)$.

Taking $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 61. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k K_k^2 = \frac{1}{4}((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 + 3K_{n+1}^2 - 2K_{n+1}K_{n+3}) + 16)$.

(b): $\sum_{k=0}^{n}(-1)^k K_{k+1}K_k = \frac{1}{4}(-1)^n (K_{n+3}^2 - K_{n+1}^2 - 2K_{n+3}K_{n+2} + 2K_{n+2}K_{n+1})$.

(c): $\sum_{k=0}^{n}(-1)^k K_{k+2}K_k = \frac{1}{4}((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 - K_{n+1}^2 + 2K_{n+3}K_{n+1} - 4K_{n+2}K_{n+1}) + 4)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

COROLLARY 62. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k M_k^2 = \frac{1}{4}((-1)^n (M_{n+3}^2 - 2M_{n+2}^2 + 3M_{n+1}^2 - 2M_{n+1}M_{n+3}) + 19)$.

(b): $\sum_{k=0}^{n}(-1)^k M_{k+1}M_k = \frac{1}{4}((-1)^n (M_{n+3}^2 - M_{n+1}^2 - 2M_{n+3}M_{n+2} + 2M_{n+2}M_{n+1}) - 5)$.

(c): $\sum_{k=0}^{n}(-1)^k M_{k+2}M_k = \frac{1}{4}((-1)^n (M_{n+3}^2 - 2M_{n+2}^2 - M_{n+1}^2 + 2M_{n+3}M_{n+1} - 4M_{n+2}M_{n+1}) + 7)$.

Taking $V_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the above proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

COROLLARY 63. *For $n \geq 0$, modified Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k U_k^2 = \frac{1}{4}(-1)^n (U_{n+3}^2 - 2U_{n+2}^2 + 3U_{n+1}^2 - 2U_{n+1}U_{n+3})$.

(b): $\sum_{k=0}^{n}(-1)^k U_{k+1}U_k = \frac{1}{4}(-1)^n (U_{n+3}^2 - U_{n+1}^2 - 2U_{n+3}U_{n+2} + 2U_{n+2}U_{n+1})$.

(c): $\sum_{k=0}^{n}(-1)^k U_{k+2}U_k = \frac{1}{4}((-1)^n (U_{n+3}^2 - 2U_{n+2}^2 - U_{n+1}^2 + 2U_{n+3}U_{n+1} - 4U_{n+2}U_{n+1}) - 4)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $V_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

COROLLARY 64. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k G_k^2 = \frac{1}{4}((-1)^n (G_{n+3}^2 - 2G_{n+2}^2 + 3G_{n+1}^2 - 2G_{n+1}G_{n+3}) + 36)$.

(b): $\sum_{k=0}^{n}(-1)^k G_{k+1}G_k = \frac{1}{4}((-1)^n (G_{n+3}^2 - G_{n+1}^2 - 2G_{n+3}G_{n+2} + 2G_{n+2}G_{n+1}) + 36)$.

(c): $\sum_{k=0}^{n}(-1)^k G_{k+2}G_k = \frac{1}{4}((-1)^n (G_{n+3}^2 - 2G_{n+2}^2 - G_{n+1}^2 + 2G_{n+3}G_{n+1} - 4G_{n+2}G_{n+1}) + 68)$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 65. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k H_k^2 = \frac{1}{4}((-1)^n (H_{n+3}^2 - 2H_{n+2}^2 + 3H_{n+1}^2 - 2H_{n+1}H_{n+3}) + 40)$.

(b): $\sum_{k=0}^{n}(-1)^k H_{k+1}H_k = \frac{1}{4}(-1)^n (H_{n+3}^2 - H_{n+1}^2 - 2H_{n+3}H_{n+2} + 2H_{n+2}H_{n+1})$.

(c): $\sum_{k=0}^{n}(-1)^k H_{k+2}H_k = \frac{1}{4}((-1)^n (H_{n+3}^2 - 2H_{n+2}^2 - H_{n+1}^2 + 2H_{n+3}H_{n+1} - 4H_{n+2}H_{n+1}) - 56)$.

6. Matrices related with Generalized Tribonacci numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{6.1}$$

For matrix formulation (6.1), see [8]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. From (1.6) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{6.2}$$

and from (6.1) (or using (6.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = T$ in (6.2) we have

$$\begin{pmatrix} T_{n+2} \\ T_{n+1} \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} T_{n+1} \\ T_n \\ T_{n-1} \end{pmatrix}. \tag{6.3}$$

We also define

$$B_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} & V_{n-2} \end{pmatrix}$$

THEOREM 66. *For all integer $m, n \geq 0$, we have*

(a): $B_n = A^n$

(b): $C_1 A^n = A^n C_1$

(c): $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a): By expanding the vectors on the both sides of (6.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1}+V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3}+V_{n-4} & V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} V_n+V_{n-1}+V_{n-2} & V_{n-1}+2V_{n-2}+2V_{n-3}+V_{n-4} & V_{n-1}+V_{n-2}+V_{n-3} \\ V_n & V_{n-1}+V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3} & V_{n-2} \end{pmatrix} \\ &= C_n, \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix A^n can be given as

$$A^n = A^{n-1} + A^{n-2} + 2A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

for all integer m and n.

THEOREM 67. *For $m, n \geq 0$ we have*

$$V_{n+m} = V_nT_{m+1} + (V_{n-1}+V_{n-2})T_m + V_{n-1}T_{m-1} \tag{6.4}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

REMARK 68. *By induction, it can be proved that for all integers $m, n \leq 0$, (6.4) holds. So for all integers m, n, (6.4) is true.*

COROLLARY 69. *For all integers m, n, we have*

$$T_{n+m} = T_n T_{m+1} + (T_{n-1} + T_{n-2}) T_m + T_{n-1} T_{m-1}, \tag{6.5}$$

$$K_{n+m} = K_n T_{m+1} + (K_{n-1} + K_{n-2}) T_m + K_{n-1} T_{m-1}, \tag{6.6}$$

$$M_{n+m} = M_n T_{m+1} + (M_{n-1} + M_{n-2}) T_m + M_{n-1} T_{m-1}, \tag{6.7}$$

$$U_{n+m} = U_n T_{m+1} + (U_{n-1} + U_{n-2}) T_m + U_{n-1} T_{m-1}, \tag{6.8}$$

$$G_{n+m} = G_n T_{m+1} + (G_{n-1} + G_{n-2}) T_m + G_{n-1} T_{m-1}, \tag{6.9}$$

$$H_{n+m} = H_n T_{m+1} + (H_{n-1} + H_{n-2}) T_m + H_{n-1} T_{m-1}. \tag{6.10}$$

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