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Certain Subclasses Of Meromorphic Functions With Fixed Second Coefficients Associated With Generalized Polylogarithm Functions

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Abstract. In this paper we introduce and study a subclass $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ of meromorphic univalent functions which is associated with generalized polylogarithm functions. We obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and meromorphically convexity for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ by fixing the second coefficient. Further, it is shown that the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ is closed under convex linear combination.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\Sigma_{\mathcal{S}}$, $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$, $(0 \le \alpha < 1)$ denote the subclasses of Σ that are meromorphically univalent functions , meromorphically starlike functions of order α and meromophically convex functions of order α respectively. Analytically, $f \in \Sigma^*(\alpha)$ if and only if, f is of the form (1.1) and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U},$$

similarly, $f \in \Sigma_K(\alpha)$, if and only if, f is of the form (1.1) and satisfies

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U},$$

and similar other classes of meromorphically univalent functions have been extensively studied by Altintas et al. [2], Aouf [3, 4], Ganigi and Uralegaddi [10], Kulkarni and Joshi [14], Mogra et al. [20], Uralegadi [28], Uralegaddi and Ganigi [29] and Uralegaddi and Somanatha [30] and others [1, 7, 8, 11, 13, 17, 18, 19, 21, 22, 24, 25, 26, 27, 32].

Let Σ_P be the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \;\; a_n \geq 0, \tag{1.2}$$

that are analytic and univalent in $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$ given by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \tag{1.3}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.4}$$

For $\varkappa \in \mathbb{N}$, the set of natural numbers with $\varkappa \geq 2$, an absolutely convergent series defined as

$$Li_\varkappa(z) = \sum_{n=1} \frac{z^n}{n^\varkappa}$$

is known as the polylogarithm. This class of functions was invented by Leibnitz [15]. For more works on polylogarithm and meromorphic function (see [1, 27, 32]).

We consider a linear operator

$$\Omega_\varkappa f(z) \;:\; \Sigma \to \Sigma$$

which is defined by the following Hadamard product (or convolution) :

$$\Omega_\varkappa f(z) = \phi_\varkappa(z) * f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{(n+2)^\varkappa} a_n z^n,$$

where

$$\phi_\varkappa(z) = z^{-2} Li_\varkappa(z).$$

Next, we define the linear operator

$$\sigma_\varkappa : \Sigma \to \Sigma$$

as follows:

$$\sigma_\varkappa f(z) = \left(\Omega_\varkappa f(z) - \frac{1}{2^\varkappa} a_0\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{(n+2)^\varkappa} a_n z^n.$$

For function f in the class Σ_P, we define a linear operator $\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z)$ as follows

$$\begin{aligned}
\mathscr{D}^{0}_{\mu,\varkappa} f(z) &= \sigma_{\varkappa} f(z)\\
\mathscr{D}^{1}_{\mu,\varkappa} f(z) &= (1-\mu)\sigma_{\varkappa} f(z) + \mu z(\sigma_{\varkappa} f(z))' = \mathscr{D}_{\mu,\varkappa} f(z)\\
\mathscr{D}^{2}_{\mu,\varkappa} f(z) &= \mathscr{D}_{\mu,\varkappa}\left(\mathscr{D}_{\mu,\varkappa} f(z)\right)\\
\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z) &= \mathscr{D}_{\mu,\varkappa}\left(\mathscr{D}^{\kappa-1}_{\mu,\varkappa} f(z)\right)\\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}} a_n z^n, \qquad \kappa \in \mathbb{N}.
\end{aligned}$$

Now, in the following definition, we define a subclass $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ for functions in the class Σ_P.

Definition 1.1. [27]) For $0 \le \alpha < 1$, $0 \le \mu,\ \lambda \le 1$, $\kappa, \varkappa \in \mathbb{N}$ and $\varkappa \ge 2$, let $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ denote a subclass of Σ consisting of functions of the form (1.1) satisfying the condition that

$$\Re\left(\frac{z(\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z) + \lambda z(\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z))'}\right) > \alpha, \quad z \in \mathbb{U}^*. \tag{1.5}$$

The main object of this paper is to obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and meromorphically convexity for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ by fixing the second coefficient. Further, it is shown that the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ is closed under convex linear combination. Our first theorem gives a necessary and sufficient condition for a function $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

2. Coefficient Inequality by Fixing the Second Coefficient

Furthermore, we say that a function $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, whenever $f(z)$ is of the form (1.2). For the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, we derive the following characterization property:

Theorem 2.1. *Let $f \in \Sigma_P$ be given by (1.2). Then $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ if and only if*

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n \le (1-\alpha). \tag{2.1}$$

Proof. If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then

$$\Re\left(\frac{z(\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z) + \lambda z(\mathscr{D}^{\kappa}_{\mu,\varkappa} f(z))'}\right) = \Re\left(\frac{-1+\sum\limits_{n=0}^{\infty} n\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}{-1+\sum\limits_{n=0}^{\infty}(\lambda-1+n\lambda)\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}\right).$$

By letting $z \to 1^-$, we have

$$\left(\frac{-1+\sum\limits_{n=1}^{\infty} n\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n}{-1+\sum\limits_{n=1}^{\infty}(\lambda-1+n\lambda)\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n}\right) > \alpha.$$

This shows that (2.1) holds.

Conversely assume that (2.1) holds. It is sufficient to show that

$$\left|\frac{\omega-1}{\omega+1-2\alpha}\right|<1,$$

where

$$\omega=\frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}.$$

Using (2.1) that

$$\left|\frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'-\left[(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'\right]}{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'+(1-2\alpha)\left[(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'\right]}\right|$$

$$=\left|\frac{\sum\limits_{n=1}^{\infty}(1-\lambda)\,(n+1)\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}{-2(1-\alpha)+\sum\limits_{n=1}^{\infty}\left[\left[(1+(1-2\alpha)\lambda)\right]n+(1-2\alpha)(\lambda-1)\right]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}\right|$$

$$\le\frac{\sum\limits_{n=1}^{\infty}(1-\lambda)\,(n+1)\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n}{2(1-\alpha)-\sum\limits_{n=1}^{\infty}\left[\left[(1+(1-2\alpha)\lambda)\right]n+(1-2\alpha)(\lambda-1)\right]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n}$$

$$\le 1.$$

Thus we have $f\in\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. □

For a function defined by (1.2) and in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, Theorem 2.1, immediately yields

$$a_1\le\frac{(1-\alpha)}{(1+\alpha(1-2\lambda))\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}.\tag{2.2}$$

Hence we may take

$$a_1=\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}},\quad c\,(0<c<1).\tag{2.3}$$

Motivated by the works of Aouf and Darwish [5], Aouf and Joshi [6], Ghanim and Darus [11], Magesh et al. [17], Sivasubramanian et al. [24] and Uralegaddi [28], we now introduce the following class of functions and use the similar techniques to prove our results.

Let $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ be the subclass of $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ consisting of functions of the form

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}z+\sum_{n=2}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^n,\tag{2.4}$$

where $0<c<1$.

In our next theorem, we now find out the coefficient inequality for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$.

Theorem 2.2. *Let the function $f(z)$ defined by (2.4). Then $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ if and only if,*

$$\sum_{n=2}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n \leq (1-\alpha)(1-c). \tag{2.5}$$

The result is sharp.

Proof. By putting

$$a_1 = \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}, \qquad 0<c<1, \tag{2.6}$$

in (2.1), the result is easily derived. The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}z+\frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}z^n, \quad n\geq 2. \tag{2.7}$$

□

Corollary 2.3. *If the function f defined by (2.4) is in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$, then*

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \quad n\geq 2. \tag{2.8}$$

The result is sharp for the function $f(z)$ given by (2.7).

Next we obtain growth and distortion properties for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$.

Theorem 2.4. *If the function $f(z)$ defined by (2.4) is in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ for $0<|z|=r<1$, then we have*

$$\frac{1}{r}-\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r-\frac{(1-\alpha)(1-c)}{(2+\alpha(1-3\lambda))\ \frac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r^2 \leq |f(z)|$$
$$\leq \frac{1}{r}+\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r+\frac{(1-\alpha)(1-c)}{(2+\alpha(1-3\lambda))\ \frac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r^2.$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}z+\frac{(1-\alpha)(1-c)}{(2+\alpha(1-3\lambda))\ \frac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}z^2.$$

Proof. Since $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$, Theorem 2.2 yields,

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \quad n\geq 2. \tag{2.9}$$

Thus, for $0 < |z| = r < 1$

$$
\begin{aligned}
|f(z)| &\leq \frac{1}{|z|} + \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}|z| + \sum_{n=2}^{\infty} a_n |z|^n \\
&\leq \frac{1}{r} + \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r + r^2 \sum_{n=2}^{\infty} a_n \\
&\leq \frac{1}{r} + \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r + \frac{(1-\alpha)(1-c)}{(2+\alpha(1-3\lambda))\,\dfrac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|f(z)| &\geq \frac{1}{|z|} - \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}|z| - \sum_{n=2}^{\infty} a_n |z|^n \\
&\geq \frac{1}{r} - \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r - r^2 \sum_{n=2}^{\infty} a_n \\
&\geq \frac{1}{r} - \frac{(1-\alpha)c}{(1+\alpha(1-2\lambda))\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r - \frac{(1-\alpha)(1-c)}{(2+\alpha(1-3\lambda))\,\dfrac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r^2.
\end{aligned}
$$

Thus the proof of the theorem is complete. □

Theorem 2.5. *If the function $f(z)$ defined by (2.4) is in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ for $0 < |z| = r < 1$, then we have*

$$
\begin{aligned}
&\frac{1}{r^2} - \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} - \frac{(1-\alpha)(1-c)}{[2+\alpha(1-3\lambda)]\,\dfrac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r \leq |f'(z)| \\
&\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} + \frac{(1-\alpha)(1-c)}{[2+\alpha(1-3\lambda)]\,\dfrac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}r.
\end{aligned}
$$

The result is sharp for the function $f(z)$ given by

$$
f(z) = \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}z + \frac{(1-\alpha)(1-c)}{[2+\alpha(1-3\lambda)]\,\dfrac{[1+3\mu]^{\kappa}}{4^{\varkappa}}}z^2.
$$

Proof. In view of Theorem 2.2, it follows that

$$
na_n \leq \frac{2n(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\,\dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \quad n \geq 2. \tag{2.10}
$$

Thus, for $0 < |z| = r < 1$ and making use of (2.10), we obtain

$$\begin{aligned} |f'(z)| &\leq \left|\frac{-1}{z^2}\right| + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} + \sum_{n=2}^{\infty} n a_n |z|^{n-1}, \quad |z| = r \\ &\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} + r\sum_{n=2}^{\infty} n a_n \\ &\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} + \frac{(1-\alpha)(1-c)}{[2+\alpha(1-3\lambda)]\,\frac{[1+3\mu]^{\kappa}}{4^{\varkappa}}} r \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq \left|\frac{-1}{z^2}\right| - \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} - \sum_{n=2}^{\infty} n a_n |z|^{n-1}, \quad |z| = r \\ &\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} - r\sum_{n=2}^{\infty} n a_n \\ &\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} - \frac{(1-\alpha)(1-c)}{[2+\alpha(1-3\lambda)]\,\frac{[1+3\mu]^{\kappa}}{4^{\varkappa}}} r. \end{aligned}$$

Hence the result follows. □

Next, we shall show that the class $\mathcal{M}_P(\alpha, \lambda, c)$ is closed under convex linear combination.

Theorem 2.6. *If*

$$f_1(z) = \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z \tag{2.11}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} z^n, \quad n \geq 2. \tag{2.12}$$

Then $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ *if and only if it can expressed in the form*

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z) \tag{2.13}$$

where $\mu_n \geq 0$ *and* $\sum_{n=2}^{\infty} \mu_n \leq 1$.

Proof. From (2.11)(2.12)(2.13), we have

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)\mu_n}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} z^n. \end{aligned}$$

Since

$$\sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)\mu_n}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} \frac{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}{(1-\alpha)(1-c)}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leq 1$$

it follows from Theorem 2.1 that the function $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$. Conversely, suppose that $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$. Since

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \quad n \geq 2.$$

Setting

$$\mu_n = \frac{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}{(1-\alpha)(1-c)} a_n$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

It follows that

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z).$$

Hence the proof complete. □

Theorem 2.7. *The class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$ is closed under linear combination.*

Proof. Suppose that the function f be given by (2.4), and let the function g be given by

$$g(z) = \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} |b_n| z^n, \quad n \geq 2.$$

Assuming that f and g are in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$, it is enough to prove that the function H defined by

$$h(z) = \zeta f(z) + (1-\zeta) g(z), \quad 0 \leq \zeta \leq 1$$

is also in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$. Since

$$h(z) = \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} |a_n\zeta + (1-\zeta)b_n| z^n,$$

we observe that

$$\sum_{n=2}^{\infty} [n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ |a_n\zeta + (1-\zeta)b_n| \le (1-\alpha)(1-c),$$

with the aid of Theorem 2.2. Thus $h \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$. □

Next we determine the radii of meromophically starlikeness of order δ and meromophically convexity of order δ for functions in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$.

Theorem 2.8. *Let the function $f(z)$ defined by (2.4) be in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$, then we have*

(i) *f is meromophically starlike of order $\delta (0 \le \delta < 1)$ in the disk $|z| < r_1(\alpha, \lambda, c, \delta)$ where $r_1(\alpha, \lambda, c, \delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r^2 + \frac{(n+2-\delta)(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} r^{n+1} \le (1-\delta),\ \ n \ge 2.$$

(ii) *f is meromophically convex of order $\delta (0 \le \delta < 1)$ in the disk $|z| < r_2(\alpha, \lambda, c, \delta)$ where $r_2(\alpha, \lambda, c, \delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r^2 + \frac{n(n+2-\delta)(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} r^{n+1} \le (1-\delta),\ \ n \ge 2.$$

Each of these results is sharp for the function $f_n(z)$ given by (2.7).

Proof. It is enough to highlight that

$$\left|\frac{zf'(z)}{f(z)} + 1\right| \le 1-\delta, \quad |z| < r_1.$$

Thus, we have

$$\left|\frac{zf'(z)}{f(z)} + 1\right| = \left| \frac{\frac{-1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} n a_n z^n + \frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} a_n z^n}{\frac{1}{z} + \frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z + \sum_{n=2}^{\infty} a_n z^n} \right|. \tag{2.14}$$

Hence (2.14) holds true if

$$\frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r^2+\sum_{n=2}^{\infty}(n+1)a_n r^{n+1} \leq (1-\delta)\left[1-\frac{(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r^2-\sum_{n=2}^{\infty}a_n r^{n+1}\right], \tag{2.15}$$

or,

$$\frac{(3-\delta)(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r^2+\sum_{n=2}^{\infty}(n+2-\delta)a_n r^{n+1} \leq (1-\delta) \tag{2.16}$$

and it follows that from (2.5), we may take

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}\mu_n, \quad n\geq 2, \tag{2.17}$$

where $\mu_n \geq 0$ and $\sum\limits_{n=2}^{\infty}\mu_n \leq 1$.

For each fixed r, we choose the positive integer $n_0 = n_0(r_0)$ for which

$$\frac{(n+2-\delta)}{[n+\alpha-\alpha\lambda(1+n)]\,\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}r^{n+1},$$

is maximal. Then it follows that

$$\sum_{n=2}^{\infty}(n+2-\delta)a_n r^{n+1} \leq \frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]\,\frac{[1+\mu(n_0+1)]^{\kappa}}{(n_0+2)^{\varkappa}}}r^{n_0+1}. \tag{2.18}$$

Then f is starlike of order δ in $0<|z|<r_1(\alpha,\lambda,c,\delta)$ provided that

$$\frac{(3-\delta)(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r^2+\frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]\,\frac{[1+\mu(n_0+1)]^{\kappa}}{(n_0+2)^{\varkappa}}}r^{n_0+1} \leq (1-\delta). \tag{2.19}$$

We find the value $r_0 = r_0(k,c,\delta,n)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\delta)(1-\alpha)c}{[1+\alpha(1-2\lambda)]\,\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}r_0^2+\frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]\,\frac{[1+\mu(n_0+1)]^{\kappa}}{(n_0+2)^{\varkappa}}}r_0^{n_0+1} = (1-\delta). \tag{2.20}$$

It is the value for which the function $f(z)$ is starlike in $0<|z|<r_0$.

(ii) In a similar manner, we can prove our result providing the radius of meromorphically convexity of order δ $(0\leq\delta<1)$ for functions in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa,\ c)$, so we skip the proof of (ii). □

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