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Numerical stability of coupled differential equation with piecewise constant arguments

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Abstract.

This paper deals with the stability of numerical solutions for a coupled differential equation with piecewise constant arguments. A sufficient condition such that the system is asymptotically stable is derived. Furthermore, when the linear θ-method is applied to this system, it is shown that the linear θ-method is asymptotically stable if and only if $\frac{1}{2} < \theta \leq 1$. Finally, some numerical experiments are given.

Keywords: Coupled differential equation; Piecewise constant arguments; Linear θ-method; Stability.

1 Introduction

In this article, we consider the following coupled differential equation with piecewise constant arguments (EPCA)

$$\begin{aligned} x'(t) &= ax(t) + by([t]), \\ y'(t) &= cy(t) + dx([t]), \\ x(0) &= x_0, \quad y(0) = y_0, \end{aligned} \tag{1}$$

where $a, b, c, d \in \mathbf{R}$, $x_0, y_0 \in \mathbf{R}$ are given initial values and $[\cdot]$ denotes the greatest integer function. System (1) can be written in matrix form

$$\begin{aligned} &\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{X}([t]), \\ &\mathbf{X}(0) = \mathbf{X}_0, \end{aligned} \tag{2}$$

where $\mathbf{X}(t) = (x(t), y(t))^T$, $\mathbf{X}(0) = (x_0, y_0)^T$ and

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix}.$$

The general form of EPCA is

$$\begin{aligned} &X'(t) = f(t, X(t), X(\alpha(t))), \\ &X(0) = X_0, \end{aligned} \tag{3}$$

where the argument $\alpha(t)$ has intervals of constancy. In recent years, much research has been focused on the solutions of EPCA. Many properties such as stability [1,2,3], oscillation [4,5], periodicity [6,7], bifurcation [8,9] and asymptotic behavior [10,11] are included. These systems can be found in a wide variety of scientific and engineering applications such as biology, ecology, spread of some infectious diseases in humans and so on. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [12]. On the other hand, it is observed that the research on the numerical solutions of EPCA has been conducted for several years. Liu et al. [13] investigated the stability of the Runge-Kutta methods for EPCA with one delay $u'(t) = au(t) + a_0u([t])$. In [14, 15], oscillations of numerical solutions in θ-methods and Runge-Kutta methods for a linear EPCA $x'(t) + ax(t) + a_1x([t-1]) = 0$ were considered, respectively. Wen et al. [16] studied the dissipativity of analytic solution and numerical solution of a class of nonlinear EPCA. For more information of numerical treatment for EPCA, the interest reader can see [17-20]. To the best of our knowledge, until now very few results dealing with the numerical solution of multi-dimensional EPCA have been reported except for [21]. Different from [21], in our paper, we will consider the numerical stability of (1) in a more direct and easy to understand way.

In the next section, we will give the expression of analytic solution of (1) and a sufficient condition under which the analytic solution of (1) is asymptotically stable will be shown.

2 Analytical Stability

In this section, we shall address a sufficient condition under which the analytic solution of (1) is asymptotically stable.

Definition 2.1 ([12]) A solution of (2) on $[0,\infty)$ is a function $\mathbf{X}(t)$ satisfies the conditions: $\mathbf{X}(t)$ is continuous on $[0,\infty)$; the derivative $\mathbf{X}'(t)$ exists at each point $t\in[0,\infty)$, with the possible exception of the points $[t]\in[0,\infty)$ where one-sided derivatives exist; System (2) is satisfied on each interval $[k,k+1)\subset[0,\infty)$ with integral end-points.

Theorem 2.1 ([22]) System (2) has on $[0,\infty)$ a unique solution

$$\mathbf{X}(t)=\mathbf{Q}_0(\{t\})\mathbf{B}_0^{[t]}\mathbf{X}_0,$$

where

$$\mathbf{Q}_0(t)=e^{\mathbf{A}t}+(e^{\mathbf{A}t}-\mathbf{I})\mathbf{A}^{-1}\mathbf{B},\quad \mathbf{B}_0=e^{\mathbf{A}}+(e^{\mathbf{A}}-\mathbf{I})\mathbf{A}^{-1}\mathbf{B},$$

where $\{t\}$ is the fractional part of t

Definition 2.2 If any solution $\mathbf{X}(t)$ of (2) satisfies

$$\lim_{t\to\infty}\mathbf{X}(t)=0,$$

then the zero solution of (2) is called asymptotically stable.

Lemma 2.1 ([12]) The zero solution of (2) is asymptotically stable if and only if $\rho(\mathbf{B}_0)<1$, where $\rho(\mathbf{B}_0)=\max_i(|\lambda_i|)$, λ_i, $i=1,2$ are the eigenvalues of $\mathbf{B}_0$.

Theorem 2.2 ([21]) The zero solution of (2) is asymptotically stable if

$$\begin{cases}\mu[\mathbf{A}]<0,\\ \|\mathbf{B}\|<-\mu[\mathbf{A}],\end{cases}$$

where $\|\cdot\|$ denotes the matrix norm induced by a vector norm on C^2 and $\mu[\cdot]$ denotes the logarithmic norm of the matrix, defined by

$$\mu[\mathbf{L}]=\lim_{\Delta\to 0^+}\frac{\|\mathbf{I}_2+\Delta\mathbf{L}\|-1}{\Delta},$$

here $\mathbf{I}_2$ is the 2×2 identity matrix.

Corollary 2.1 System (1) is asymptotically stable, if

$$\max\{a, c\} < 0, \quad \max\{|b|, |d|\} < -\max\{a, c\}. \qquad (4)$$

Proof: System (1) is asymptotically stable if and only if System (2) is asymptotically stable. So we only need to consider the stability of System (2). By the following formulas

$$\mu[\mathbf{L}] = \lambda_{\max}\left(\frac{\mathbf{L} + \mathbf{L}^*}{2}\right), \quad \|\mathbf{L}\| = \sqrt{\rho(\mathbf{L}^*\mathbf{L})},$$

we can obtain

$$\mu[\mathbf{A}] = \max\{a, c\}, \quad \|\mathbf{B}\| = \max\{|b|, |d|\}.$$

In view of Theorem 2.2, Condition (4) is got.

Remark 2.1 From Corollary 2.1, the following inequality can be easily obtained

$$\left|\frac{bd}{ac}\right| < 1, \qquad (5)$$

in fact, there are four cases in (4). (i) If $|b| > |d|$, $0 > a > c$ then $|b| < -a$; (ii) If $|b| < |d|$, $0 > a > c$ then $|d| < -a$; (iii) If $|b| > |d|$, $a < c < 0$ then $|b| < -c$; (iv) If $|b| < |d|$, $a < c < 0$ then $|d| < -c$. Therefore, regardless of which case, the coefficients a, b, c and d are all satisfy (5).

3 Stability of the Linear θ -Method

Let $h = \frac{1}{m}$ be a given stepsize with integer $m \geq 1$ and the gridpoints $t_n = nh \quad n = 1, 2, \cdots$, we consider the linear θ -method to (1),

$$\begin{aligned} x_{n+1} &= x_n + h\{\theta(ax_{n+1} + by^h([(n+1)h])) + (1-\theta)(ax_n + by^h([nh]))\}, \\ y_{n+1} &= y_n + h\{\theta(cy_{n+1} + dx^h([(n+1)h])) + (1-\theta)(cy_n + dx^h([nh]))\}, \end{aligned} \qquad (6)$$

where θ is a parameter with $0 \leq \theta \leq 1$. $x^h([nh])$ and $x^h([(n+1)h])$ are approximations to $x([t])$ of (1) at t_n and t_{n+1}, respectively. Similarly, $y^h([nh])$ and $y^h([(n+1)h])$ are approximations to $y([t])$ of (1) at t_n and t_{n+1}, respectively. Let $n = km + l$, $l = 0, 1, \cdots, m-1$. Then $x^h([t_n + \delta h])$ and $y^h([t_n + \delta h])$, $0 \le \delta \le 1$ can be defined as x_{km} and y_{km} according to Definition 2.1, respectively. So (6) reads

$$\begin{aligned} x_{n+1} &= x_n + h\{\theta(ax_{n+1} + by_{km}) + (1-\theta)(ax_n + by_{km})\}, \\ y_{n+1} &= y_n + h\{\theta(cy_{n+1} + dx_{km}) + (1-\theta)(cy_n + dx_{km})\}, \end{aligned} \tag{7}$$

that is

$$\begin{aligned} x_{n+1} &= \frac{1 + h(1-\theta)a}{1 - h\theta a} x_n + \frac{hb}{1 - h\theta a} y_{km}, \\ y_{n+1} &= \frac{1 + h(1-\theta)c}{1 - h\theta c} y_n + \frac{hd}{1 - h\theta c} x_{km}, \end{aligned} \tag{8}$$

Denote

$$\mathbf{M}_1 = \begin{pmatrix} \dfrac{1 + h(1-\theta)a}{1 - h\theta a} & 0 \\ 0 & \dfrac{1 + h(1-\theta)c}{1 - h\theta c} \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} 0 & \dfrac{hb}{1 - h\theta a} \\ \dfrac{hd}{1 - h\theta c} & 0 \end{pmatrix},$$

thus (8) gives

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{M}_1 \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \mathbf{M}_2 \begin{pmatrix} x_{km} \\ y_{km} \end{pmatrix}. \tag{9}$$

Let

$$\mathbf{Z}_n = (x_n, y_n, x_{n-1}, y_{n-1}, \cdots, x_{km+1}, y_{km+1}, x_{km}, y_{km})^T,$$

then (9) can be written as

$$\mathbf{Z}_{n+1} = \mathbf{G}\mathbf{Z}_n, \tag{10}$$

where

$$\mathbf{G} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{M}_2 \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ & \cdots & & \cdots & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix},$$

with

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The stability of the linear θ -method can be given by the following theorem.

Theorem 3.1 The linear θ -method applied to System (1) with Condition (4) is asymptotically stable if and only if $\frac{1}{2} < \theta \le 1$.

Proof: Assume that $\frac{1}{2} < \theta \le 1$, after some calculations, we get the characteristic polynomial of $\mathbf{G}$ as follows

$$P(z) = \det[\mathbf{I}z^{l+1} - \mathbf{M}_1 z^l - \mathbf{M}_2] \tag{11}$$

$$= \det\left[\begin{pmatrix} z^{l+1} & 0 \\ 0 & z^{l+1} \end{pmatrix} - \begin{pmatrix} \frac{1+h(1-\theta)a}{1-h\theta a} z^l & 0 \\ 0 & \frac{1+h(1-\theta)c}{1-h\theta c} z^l \end{pmatrix} - \begin{pmatrix} 0 & \frac{hb}{1-h\theta a} \\ \frac{hd}{1-h\theta c} & 0 \end{pmatrix}\right]$$

$$= z^{2(l+1)} - \frac{1+h(1-\theta)a}{1-h\theta a} z^{2l+1} - \frac{1+h(1-\theta)c}{1-h\theta c} z^{2l+1} + \frac{1+h(1-\theta)a}{1-h\theta a} \frac{1+h(1-\theta)c}{1-h\theta c} z^{2l}$$

$$- \frac{h^2 bd}{(1-h\theta a)(1-h\theta c)}.$$

For brevity, we set

$$R(x_1) = \frac{1+x_1(1-\theta)}{1-x_1\theta}, \quad R(x_2) = \frac{1+x_2(1-\theta)}{1-x_2\theta},$$

where

$$x_1 = ha, \quad x_2 = hc,$$

then (11) gives

$$P(z) = z^{2(l+1)} - R(x_1)z^{2l+1} - R(x_2)z^{2l+1} + R(x_1)R(x_2)z^{2l} - \frac{bd}{ac}(R(x_1)-1)(R(x_2)-1) \quad (12)$$

$$= z^{2l}(z - R(x_1))(z - R(x_2)) - \frac{bd}{ac}(R(x_1)-1)(R(x_2)-1)$$

$$= (z - R(x_1))(z - R(x_2))\left[z^{2l} - \frac{bd}{ac}\frac{(R(x_1)-1)(R(x_2)-1)}{(z - R(x_1))(z - R(x_2))} \right].$$

It is obvious that $P(z)$ has two zeros $R(x_1)$ and $R(x_2)$ of order 1, respectively. By virtue of [23] and the property of stability function of the linear θ-method we have $|R(x_1)| < 1$ and $|R(x_2)| < 1$ for $\frac{1}{2} < \theta \leq 1$.

Let

$$f(z) = z^{2l}, \quad g(z) = -\frac{bd}{ac}\frac{(R(x_1)-1)(R(x_2)-1)}{(z - R(x_1))(z - R(x_2))},$$

for any m and $|z| = 1$, by (5) we have

$$|g(z)| = \left| -\frac{bd}{ac}\frac{(R(x_1)-1)(R(x_2)-1)}{(z - R(x_1))(z - R(x_2))} \right| \leq \left| \frac{bd}{ac} \right| < 1 = \left| z^{2l} \right| = |f(z)|.$$

By Rouches theorem, we know that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the unit circle. It is observed that $f(z)$ has $2l$ zeros, so $f(z) + g(z)$ also has $2l$ zeros inside the unit circle. Hence all roots of characteristic polynomial $P(z)$ have modulus less than 1, which means that $\rho(\mathbf{G}) < 1$, where $\rho(\mathbf{G})$ denotes the spectral radius of matrix $\mathbf{G}$.

According to Lemma 5.6.10 in [24], there exists a norm $\|\cdot\|$ such that $\|\mathbf{G}\| < 1$. So from (10) we have

$$\|Z_{n+1}\| = \|\mathbf{G}\| \|Z_n\| < \|Z_n\|,$$

which implies that the linear θ -method is asymptotically stable.

Conversely, for (6), the linear θ -method is asymptotically stable implies that

$$\lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} y_n = 0.$$

We focus on the special case that $b = d = 0$ and $a < 0$, $c < 0$. Obviously, (4) is satisfied in this case. Then (9) can be written as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{M}_1 \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{13}$$

Similar to the analysis in [25], we can easily get this proof.

4 Numerical Experiments

In this section, four numerical examples be addressed to test the correctness of the results in the paper.

Firstly, we consider the following coupled EPCA

$$\begin{aligned} &x'(t) = -5x(t) + 2y([t]), \\ &y'(t) = -4y(t) + 3.5x([t]), \\ &x(0) = y(0) = 1. \end{aligned} \tag{14}$$

By computation we have that $\max\{a, c\} = -4 < 0$ and $\max\{|b|,|d|\} = 3.5 < 4 =$

$-\max\{a, c\}$. So Corollary 2.1 is satisfied, then the analytic solution of (14) is asymptotically stable. In Fig. 1, we plot the 2-norm of numerical solution of (14) with $\theta = 0.8$ and $m = 50$. From this figure we can see that the numerical solution of (14) is asymptotically stable, which is agree with Theorem 3.1.

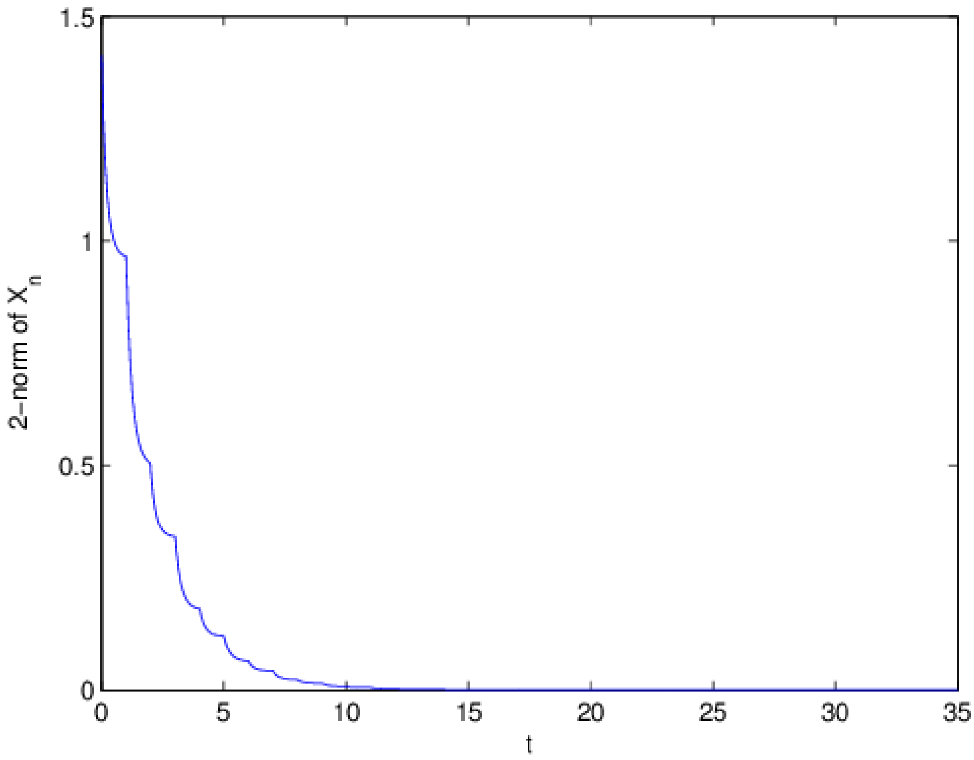

Fig. 1 The numerical solution of (14) with $\theta = 0.8$ and $m = 50$.

Secondly, we consider another coupled EPCA

$$\begin{aligned} &x'(t) = -4.5x(t) + 1.5y([t]), \\ &y'(t) = -5y(t) + 3x([t]), \qquad (15) \\ &x(0) = y(0) = 1. \end{aligned}$$

Through some simple computations we get that $\max\{a, c\} = -4.5 < 0$ and $\max\{|b|, |d|\} = 3 < 4.5 = -\max\{a, c\}$. Thus, Corollary 2.1 holds, then the analytic solution of (15) is asymptotically stable. Set $\theta = 0.6$ and $m = 50$, in Fig. 2, we draw the 2-norm of numerical solution of (15). We observe from this figure that the numerical solution of (15) is asymptotically stable, which is in accordance with Theorem 3.1.

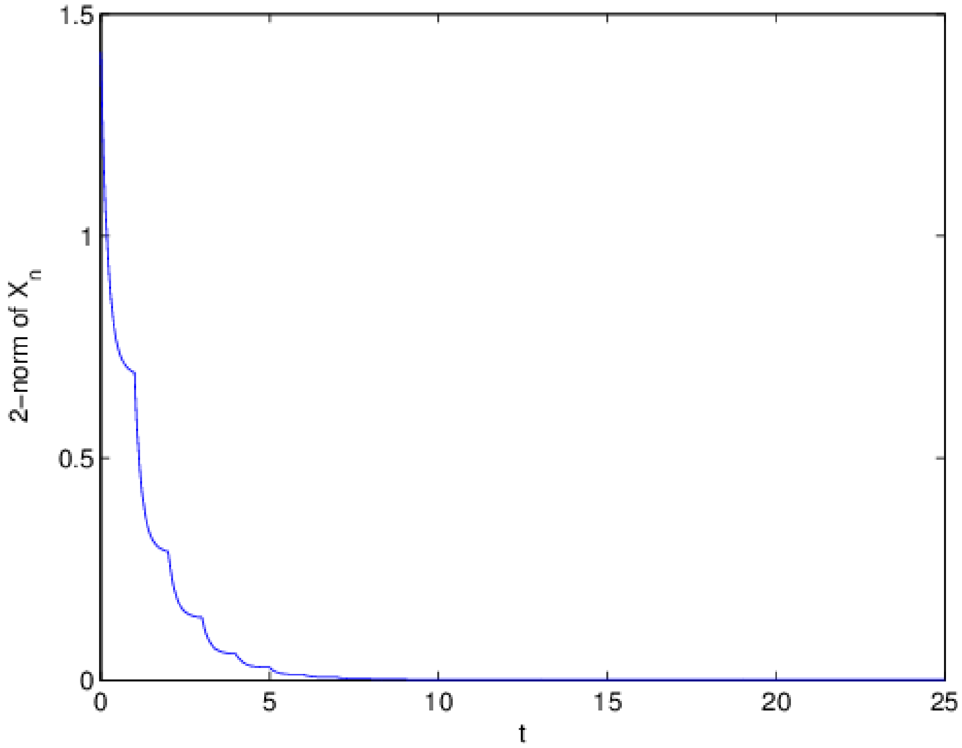

Fig. 2 The numerical solution of (15) with $\theta = 0.6$ and $m = 50$.

Thirdly, for the coupled EPCA

$$\begin{aligned} x'(t) &= -6x(t) + 3y([t]), \\ y'(t) &= -7y(t) + 2x([t]), \\ x(0) &= y(0) = 1. \end{aligned} \qquad (16)$$

we can easily see that the coefficients $a = -6$, $b = 3$, $c = -7$ and $d = 2$ meet Corollary 2.1, so the analytic solution of (16) is asymptotically stable. In Fig. 3, we draw the 2-norm of numerical solution of (16) with $\theta = 0.7$ and $m = 50$. From this figure we can see that the numerical solution of (16) is asymptotically stable, which shows no difference with Theorem 3.1.

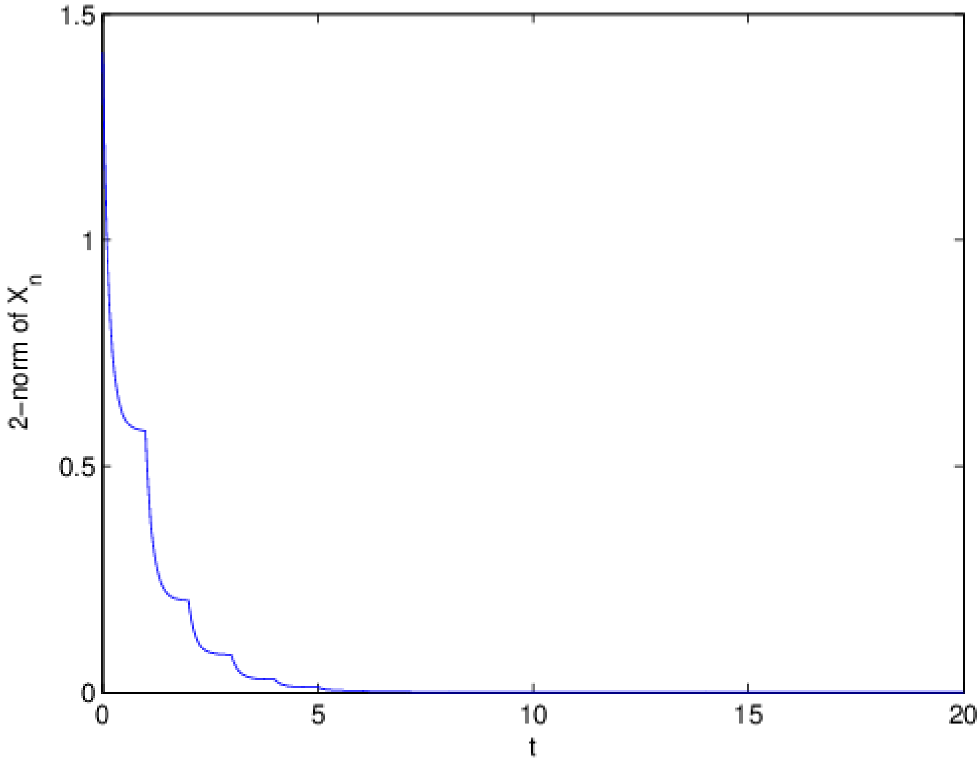

Fig. 3 The numerical solution of (16) with $\theta = 0.7$ and $m = 50$.

Finally, we consider a coupled EPCA

$$\begin{aligned} x'(t) &= -4.6x(t) + 2.7y([t]), \\ y'(t) &= -3y(t) - 2x([t]), \\ x(0) &= y(0) = 1. \end{aligned} \tag{17}$$

It is not difficult to see that the coefficients $a = -4.6$, $b = 2.7$, $c = -3$ and $d = -2$ satisfy Corollary 2.1, hence the analytic solution of (17) is asymptotically stable. Set $\theta = 0.9$ and $m = 50$, in Fig. 4, we plot the 2-norm of numerical solution of (17). We observe from this figure that the numerical solution of (17) is asymptotically stable, which is identical with Theorem 3.1.

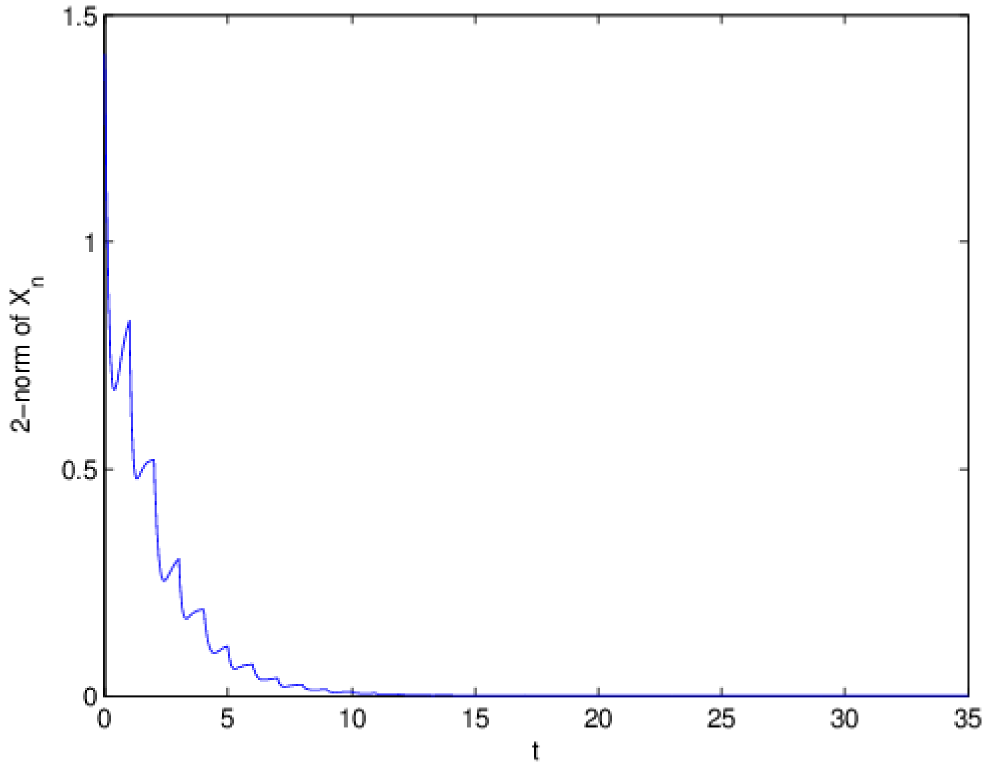

Fig. 4 The numerical solution of (17) with $\theta = 0.9$ and $m = 50$.

Conclusions

In this paper, the stability of numerical solutions for a coupled differential equation with piecewise constant arguments are investigated. It is proven that the linear θ-method is asymptotically stable if and only if $\frac{1}{2} < \theta \leq 1$. The process is more convenient than that in [21]. We will consider the more general case in our future work.

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