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Combinatorial Identities with Several Kinds of Degenerate-Daehee Sequences

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Abstract.

In this paper, we mainly make use of the probabilistic method to calculate several different moment representations of the degenerate Daehee numbers of the third kind with degenerate log function. We also obtain the moment expressions of the degenerate Daehee numbers of higher-order and degenerate Daehee numbers of the second kind. When deriving the moment representations of degenerate Stirling numbers and the degenerate Bernoulli numbers, we arrive at the combinatorial identities of relationships of them and we prove them by the probabilistic method.

Keywords: Moment; Generating function; Degenerate Daehee numbers of the third kind; Uniform distribution; Gamma distribution.

Mathematics Subject Classification (2010)**:** 11B68, 60E07, 11B83, 62E15.

1 Introduction

It is common knowledge that the Daehee numbers of order r, denoted by $D_n^{(r)}$, are defined by the generating function[2][3][9]:

$$(\frac{\log(1+t)}{t})^r = \sum_{n\geq 0} D_n^{(r)} \frac{t^n}{n!}. \tag{1}$$

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When $r = 1$, $D_n^{(1)} = D_n$ are called the Daehee numbers.

The degenerate Daehee polynomials of higher-order $D_n^{(r)}(x|\lambda)$ are introduced as[4]:

$$\left(\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r (1+t)^x = \sum_{n\geq 0} D_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{2}$$

When $x = 0$, $D_n^{(r)}(0|\lambda) = D_n^{(r)}(\lambda)$ are called degenerate Daehee numbers of higher-order, which is defined by the following generating function:

$$\left(\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r = \sum_{n\geq 0} D_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{3}$$

We note that $\lim_{\lambda\to 0} D_n^{(r)}(\lambda) = D_n^{(r)}$.

Recently, D.S. Kim et al. presented the degenerate Daehee numbers of the second kind $D_{\lambda,2}(n)$ as follows[5]:

$$\frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}} - 1} = \sum_{n\geq 0} D_{\lambda,2}(n)\frac{t^n}{n!}. \tag{4}$$

We note that $\lim_{\lambda\to 0} D_{\lambda,2}(n) = D_n$.

Now we consider the degenerate log function, denoted by $log_\lambda(t)$, which is defined by the following generating function[7]:

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1). \tag{5}$$

In this case, S.S. Pyo et al. defined the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$ as follows[10]:

$$\frac{\log_\lambda(1+t)}{t} = \frac{\frac{1}{\lambda}((1+t)^\lambda - 1)}{t} = \sum_{n\geq 0} D_{\lambda,3}(n)\frac{t^n}{n!}. \tag{6}$$

We note that $\lim_{\lambda\to 0} D_{\lambda,3}(n) = D_n$.

In this paper, we make use of the special combinatorial sequences of the degenerate Bernoulli numbers $\beta_{n,\lambda}$, which are defined by the following generating function[1]:

$$\frac{t}{e_\lambda(t) - 1} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{n\geq 0} \beta_{n,\lambda}\frac{t^n}{n!} \tag{7}$$

The degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ are defined by the generating fuction[6]:

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \frac{(\frac{1}{\lambda}((1+t)^\lambda - 1))^k}{k!} = \sum_{n\geq k} S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{8}$$

The degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ are defined by the generating fuction[6]:

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^k}{k!} = \sum_{n\geq k} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{9}$$

If f and g are exponential generating functions, and

$$fg = (\sum_{r=0}^{\infty}\frac{a_r t^r}{r!})(\sum_{s=0}^{\infty}\frac{b_s t^s}{s!}),$$

then the coefficients of $\frac{t^n}{n!}$ in fg are given by

$$[\frac{t^n}{n!}](fg) = \sum_{r=0}^{n}\binom{n}{r}a_r b_{n-r}.$$

In this paper, we need the following notations: r.v denotes a random variable, $i.i.d$ shows that a sequence of random variables are independent identically distributed. The notation E denotes an expectation operator and definition is as follows: When $f(x)$ is a measurable function with continuous random variables X and $p(x)$ is a density function of X, we have

$$Ef(X) = \int_{-\infty}^{+\infty} f(x)p(x)dx.$$

Especially, setting $f(x)=x^n$,we obtain the moment of n-th order EX^n of random variables.

Next we shall introduce several moment representations of some combinatorial sequences used in this paper.

Lemma 1.1 [11] Suppose that r.v $L_1, L_2, \cdots,$ $i.i.d \sim L[0,1]$ and r.v $L_e = \sum_{k\geq 1}\frac{L_k}{2k\pi}$, then Bernoulli numbers B_n satisfy the following moment representation,

$$B_n = E(iL_e - \frac{1}{2})^n, \quad n \geqslant 0. \tag{10}$$

Lemma 1.2 [12] Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\varGamma_1, \varGamma_2, \cdots, i.i.d \sim \varGamma(1,1)$, u_i and $\varGamma_j$ are independent for all i,j, when $n,k \geqslant 1$, then Stirling numbers of the first kind have the following moment representation,

$$S_1(n,k) = (-1)^{n-k}\binom{n}{k}E(u_1\varGamma_1 + u_2\varGamma_2 + \cdots + u_k\varGamma_k)^{n-k}. \tag{11}$$

It is demanded that $S_1(n,0) = S_1(0,k) = 0$, $S_1(0,0) = 1$.

Lemma 1.3 [8] Suppose that r.v $u \sim U[0,1]$, $\varGamma \sim \varGamma(1,1)$, u and $\varGamma$ are independent, then Daehee numbers have the following moment representation,

$$D_n = (-1)^n E(u\varGamma)^n, \quad n \geqslant 0. \tag{12}$$

Lemma 1.4 [8] Suppose that r.v $u_1, u_2, \cdots,\ i.i.d \sim U[0,1]$ for all i, when $n, k \geqslant 1$, then Stirling numbers of the second kind have the following moment representation,

$$S_2(n,k) = \binom{n}{k} E(u_1 + u_2 + \cdots + u_k)^{n-k}. \tag{13}$$

It is demanded that $S_2(n,0) = S_2(0,k) = 0$, $S_2(0,0) = 1$.

Lemma 1.5 [8] Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, u_i and Γ_j are independent for all i, j, when $n, k \geqslant 1$, then Stirling numbers of the first kind have the following moment representation,

$$D_n^{(k)} = (-1)^n E(u_1\Gamma_1 + \cdots + u_k\Gamma_k)^n. \tag{14}$$

2 Moment Representations of the Degenerate Daehee Sequences

In this section, we use probabilistic method to derive several moment representations about the degenerate Daehee numbers of higher-order and the degenerate Daehee numbers of the third kind. On the foundation of the moment representations of the degenerate Daehee numbers of the third kind, we conclude moment representations of the degenerate Stirling numbers of the first and the second kind, the degenerate Daehee numbers of the second kind and degenerate Bernoulli numbers.

Theorem 2.1 Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, u_i and Γ_j are independent for $i, j = 1, 2, ...$, r.v $X_1 \sim \Gamma(u, 1)$, $u \sim U[0,1]$ and r.v $\Gamma \sim \Gamma(r-n+1, 1)$ and $r-n+1 > 0$, for $r \in \mathbb{N}_+$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\lambda \neq 0$, the degenerate Daehee numbers of higher-order $D_n^{(r)}(\lambda)$ satisfy the following moment representation:

$$D_n^{(r)}(\lambda) = (-1)^n E(u_1\Gamma_1 + \cdots + u_r\Gamma_r + \lambda r X_1 - \lambda\Gamma)^n. \tag{15}$$

Proof From the generating function of the degenerate Daehee numbers of higher-order, we replace t by $-it$ in Eq.(3),

$$\begin{aligned}\sum_{n\geq 0} D_n^{(r)}(\lambda)\frac{(-it)^n}{n!} &= \left(\frac{\log(1-it)}{\log(1-\lambda it)^{\frac{1}{\lambda}}}\right)^r \\ &= \left(\frac{\log(1-it)}{-it}\right)^r \left(\frac{-\lambda it}{\log(1-\lambda it)}\frac{1}{1-\lambda it}\right)^r (1-\lambda it)^r.\end{aligned} \tag{16}$$

Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1], \Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, u_i and Γ_j are independent for for $i, j = 1, 2, ...$, for the right side of the Eq.(16),

$$\left(\frac{\log(1-it)}{-it}\right)^r = \left(\sum_{m\geq 0}\frac{(it)^m}{m+1}\right)^r = \left(\sum_{m\geq 0} Eu^m (it)^m\right)^r$$

$$
\begin{aligned}
&= \sum_{n\geq 0} \sum_{m_1+\cdots+m_r=n} Eu_1^{m_1} \cdots Eu_r^{m_r} (it)^n \\
&= \sum_{n\geq 0} \sum_{m_1+\cdots+m_r=n} \binom{n}{m_1,\cdots,m_r} Eu_1^{m_1} \cdots Eu_r^{m_r} m_1! \cdots m_r! \frac{(it)^n}{n!} \\
&= \sum_{n\geq 0} \sum_{m_1+\cdots+m_r=n} \binom{n}{m_1,\cdots,m_r} Eu_1^{m_1} \cdots Eu_r^{m_r} E\Gamma_1^{m_1} \cdots E\Gamma_r^{m_r} \frac{(it)^n}{n!} \\
&= \sum_{n\geq 0} E(\sum_{m_1+\cdots+m_r=n} \binom{n}{m_1,\cdots,m_r} (u_1\Gamma_1)^{m_1} \cdots (u_r\Gamma_r)^{m_r}) \frac{(it)^n}{n!} \\
&= \sum_{n\geq 0} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r)^n \frac{(it)^n}{n!}.
\end{aligned}
$$

Suppose that r.v $X_1 \sim \Gamma(u,1)$, r.v $u \sim U[0,1]$, the characteristic function of r.v X_1 denoted by

$$
\begin{aligned}
&Ee^{itX} = E\{E[e^{itX}|u]\} = E(\frac{1}{1-it})^x \\
&= \int_0^1 (\frac{1}{1-it})^x dx = \frac{-it}{\log(1-it)} \frac{1}{1-it} = \sum_{n\geq 0} EX_1^n \frac{(it)^n}{n!}
\end{aligned}
$$

so we have

$$
\begin{aligned}
&(\frac{-\lambda it}{\log(1-\lambda it)} \frac{1}{1-\lambda it})^r = (\sum_{m\geq 0} EX_1^m \frac{(\lambda it)^m}{m!})^r \\
&= \sum_{n\geq 0} \sum_{m_1+\cdots+m_r=n} \binom{n}{m_1,\cdots,m_r} EX_1^{m_1} \cdots EX_1^{m_r} \frac{(\lambda it)^n}{n!} \\
&= \sum_{n\geq 0} E(rX_1)^n \frac{(\lambda it)^n}{n!} = \sum_{n\geq 0} E(\lambda r X_1)^n \frac{(it)^n}{n!}.
\end{aligned}
$$

Suppose that r.v $\Gamma \sim \Gamma(r-n+1,1)$ and $r-n+1>0$, Eq.(16) can be expressed by

$$
\begin{aligned}
&\sum_{n\geq 0} D_n^{(r)}(\lambda) \frac{(-it)^n}{n!} \\
&= \sum_{n\geq 0} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r)^n \frac{(it)^n}{n!} \sum_{n\geq 0} E(\lambda r X_1)^n \frac{(it)^n}{n!} \sum_{n\geq 0} (r)_n \frac{(-\lambda it)^n}{n!} \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r)^k E(\lambda r X_1)^{n-k} \frac{(it)^n}{n!} \\
&\quad \times \sum_{n\geq 0} (-\lambda)^n < r-n+1 >_n \frac{(it)^n}{n!}
\end{aligned}
$$

$$= \sum_{n\geq 0} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r + \lambda r X_1)^n \frac{(it)^n}{n!} \sum_{n\geq 0} (-\lambda)^n E\Gamma^n \frac{(it)^n}{n!}$$
$$= \sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r + \lambda r X_1)^k E(-\lambda\Gamma)^{n-k} \frac{(it)^n}{n!}$$
$$= \sum_{n\geq 0} E(u_1\Gamma_1 + \cdots + u_r\Gamma_r + \lambda r X_1 - \lambda\Gamma)^n \frac{(it)^n}{n!}. \quad (17)$$

By comparing the coefficients $\frac{(it)^n}{n!}$ in both sides, we arrive at the moment expression of degenerate Daehee number of higher-order $D_n^{(r)}(\lambda)$.

When r.v $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, we obtain that $\Gamma_1+\Gamma_2+\cdots+\Gamma_{r-n+1} \sim \Gamma(r-n+1,1)$, $(r-n+1>0)$ and $E(\Gamma_1+\Gamma_2+\cdots+\Gamma_{r-n+1})^n =< r-n+1 >_n$, so we get the following theorom.

Theorem 2.2 Under the condition of Theorem 2.1, for all $i, j \geq 1$, we get the moment representation of the degenerate Daehee numbers of higher-order $D_n^{(r)}(\lambda)$:

$$D_n^{(r)}(\lambda) = (-1)^n E\left(\sum_{i=1}^{r}(u_i\Gamma_i + \lambda X_1) - \lambda \sum_{j=1}^{r-n+1} \Gamma_j\right)^n. \quad (18)$$

Suppose r.v $X_2 \sim \Gamma(-x,1)$ and $x<0$, the characteristic function of r.v X_2 denoted by

$$Ee^{itX} = (1-it)^{-(-x)} = \sum_{n\geq 0} EX_2^n \frac{(it)^n}{n!} \quad (19)$$

So we can easily obtain the following corollary.

Corollary 2.3 Under the condition of Theorem 2.1 and Theorem 2.2, suppose r.v $X_2 \sim \Gamma(-x,1)$ and $x<0$, we can easily obtain the moment representation of the degenerate Daehee polynomials of higher-order $D_n^{(r)}(x|\lambda)$:

$$D_n^{(r)}(x|\lambda) = (-1)^n E(u_1\Gamma_1 + \cdots + u_r\Gamma_r + \lambda r X_1 - \lambda\Gamma + X_2)^n. \quad (20)$$

$$D_n^{(r)}(x|\lambda) = (-1)^n E\left(\sum_{i=1}^{r}(u_i\Gamma_i + \lambda X_1) - \lambda \sum_{j=1}^{r-n+1} \Gamma_j + X_2\right)^n. \quad (21)$$

Theorem 2.4 Suppose that r.v $u \sim U[0,1]$, $\Gamma' \sim \Gamma(\lambda-n,1)$, $(\lambda-n>0)$, u and Γ' are independent, then for $n \in \mathbb{N}$ we get the first moment representation of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$:

$$D_{\lambda,3}(n) = E(u\Gamma')^n. \quad (22)$$

Proof In light of the Eq.(6) we have

$$\sum_{n\geq 0} D_{\lambda,3}(n)\frac{t^n}{n!} = \frac{\frac{1}{\lambda}((1+t)^{\lambda}-1)}{t} = \frac{1}{\lambda t}\sum_{n\geq 1}(\lambda)_n\frac{t^n}{n!}$$
$$= \sum_{n\geq 1}(\lambda-1)_{n-1}\frac{t^{n-1}}{n!} = \sum_{n\geq 0}(\lambda-1)_n\frac{t^n}{(n+1)!} = \sum_{n\geq 0} <\lambda-n>_n \frac{t^n}{(n+1)!}$$
$$= \sum_{n\geq 0}\frac{<\lambda-n>_n}{n+1}\frac{t^n}{n!} = \sum_{n\geq 0} Eu^n E\Gamma'^n\frac{t^n}{n!} = \sum_{n\geq 0} E(u\Gamma')^n\frac{t^n}{n!}. \tag{23}$$

By comparing the coefficients $\frac{t^n}{n!}$, we conclude the proof.

When r.v $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, we obtain that $\Gamma_1+\Gamma_2+\cdots+\Gamma_{\lambda-n} \sim \Gamma(\lambda-n,1)$, $(\lambda-n>0)$, we get the following theorem.

Theorem 2.5 Under the condition of Theorem 2.4, we get the second moment representation of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$:

$$D_{\lambda,3}(n) = E[u(\Gamma_1+\cdots+\Gamma_{\lambda-n})]^n. \tag{24}$$

Corollary 2.6 For r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, we get

$$D_{\lambda,3}(n) = (-1)^n D_n^{(\lambda-n)}. \tag{25}$$

Corollary 2.7 For r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, we get

$$D_{\lambda,3}(n) = E(u_1\Gamma_1+\cdots+u_{\lambda-n}\Gamma_{\lambda-n})^n \tag{26}$$

$$D_{\lambda,3}(n) = \sum_{i_1+\cdots+i_{\lambda-n}=n}(-1)^n\binom{n}{i_1,\cdots,i_{\lambda-n}}D_{i_1}\cdots D_{i_{\lambda-n}}. \tag{27}$$

Theorem 2.8 Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma', \Gamma'_1, \Gamma'_2, \cdots, i.i.d \sim \Gamma(\lambda-n,1)$, u_i and Γ'_j are independent for $i,j = 1,2,...$, then for $n \geq k$ we obtain the moment representation of the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$:

$$S_{1,\lambda}(n,k) = \binom{n}{k}E(u_1\Gamma'_1+\cdots+u_k\Gamma'_k)^{n-k}. \tag{28}$$

Proof According to the Eq.(8) we have

$$\frac{k!}{n!}S_{1,\lambda}(n,k) = [t^n](\log_\lambda(1+t))^k$$
$$= [t^{n-k}](\frac{\log_\lambda(1+t)}{t})^k = [t^{n-k}](\sum_{m\geq 0} E(u\Gamma')^m \frac{t^m}{m!})^k$$
$$= [t^{n-k}]\sum_{n\geq 0} E(\sum_{m_1+\cdots+m_k=n} \binom{n}{m_1,\cdots,m_k}(u_1\Gamma_1')^{m_1}\cdots(u_k\Gamma_k')^{m_k})\frac{t^n}{n!}$$
$$= [t^{n-k}]\sum_{n\geq 0} E(u_1\Gamma_1'+\cdots+u_k\Gamma_k')^n \frac{t^n}{n!}.$$

This concludes the proof.

Corollary 2.9 Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1', \Gamma_2', \cdots, i.i.d \sim \Gamma(\lambda-n,1)$, u_i and Γ_j' are independent for $i,j = 1,2,...$, then for $n \geq k$ we get

$$S_{1,\lambda}(n,k) = \binom{n}{k}\sum_{i_1+\cdots+i_k=n-k}\binom{n-k}{i_1,\cdots,i_k}D_{\lambda,3}(i_1)\cdots D_{\lambda,3}(i_k). \tag{29}$$

Next we discover the relationship of generating functions of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$ and Stirling numbers of the first find $S_1(n,k)$, and derive the third moment representation of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$ in the following theorem.

Theorem 2.10 Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, then for $n \in \mathbb{N}$ we have

$$D_{\lambda,3}(n) = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}E(u_1\Gamma_1+\cdots+u_m\Gamma_m)^{n-m}(\lambda-1)^m. \tag{30}$$

Proof The level generating function of the Stirling numbers of the first kind $S_1(n,k)$ is denoted by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k. \tag{31}$$

Thus it follows from the proof of theorem 2.4 and Eq.(32),

$$\sum_{n\geq 0} D_{\lambda,3}(n)\frac{t^n}{n!} = \frac{\frac{1}{\lambda}((1+t)^\lambda - 1)}{t}$$
$$= \sum_{n\geq 0}(\lambda-1)_n\frac{t^n}{(n+1)!} = \sum_{n\geq 0}\sum_{m=0}^{n} S_1(n,m)(\lambda-1)^m\frac{t^n}{(n+1)!}$$

$$= \sum_{n\geq 0}\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}E(u_1\Gamma_1+\cdots+u_m\Gamma_m)^{n-m}(\lambda-1)^m\frac{t^n}{(n+1)!}.$$

By comparing the coefficients $\frac{t^n}{n!}$, therefore we derive the Eq.(30).

Now we obtain the forth moment representation of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$ according to the relationship between the Daehee numbers D_n and the Stirling numbers of the second kind $S_2(n,k)$ in the following theorem.

Theorem 2.11 Suppose that r.v $u, u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma', \Gamma'_1, \Gamma'_2, \cdots,$ $i.i.d \sim \Gamma(\lambda-n,1)$, $(\lambda-n>0)$, $\Gamma_1 \sim \Gamma(1,1)$, then for $n\in\mathbb{N}$ we have:

$$D_{\lambda,3}(n) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^l E(u\Gamma)^l E(u_1+\cdots+u_l)^{k-l} \times E(u_1\Gamma'_1+\cdots+u_m\Gamma'_m)^{n-k}. \tag{32}$$

Proof The generating function of the Stirling numbers of the second kind $S_2(n,k)$ is denoted by

$$\frac{(e^t-1)^k}{k!} = \sum_{k=0}^{n} S_2(n,k)\frac{t^k}{n!}. \tag{33}$$

Thus from the proof of theorem 2.4, we have

$$\sum_{n\geq 0} D_{\lambda,3}(n)\frac{t^n}{n!} = \frac{\frac{1}{\lambda}((1+t)^\lambda-1)}{t}$$
$$= \frac{\log(e^{\frac{1}{\lambda}((1+t)^\lambda-1)}-1+1)}{t}$$
$$= \sum_{l\geq 0} D_l\frac{(e^{\frac{1}{\lambda}((1+t)^\lambda-1)}-1)^l}{l!}$$
$$= \sum_{l\geq 0} D_l\sum_{m\geq l} S_2(m,l)\frac{1}{m!}(\frac{1}{\lambda}((1+t)^\lambda-1))^m$$
$$= \sum_{l\geq 0} D_l\sum_{m\geq l} S_2(m,l)\frac{1}{m!}(t\sum_{k\geq 0}E(u\Gamma')^k\frac{t^k}{k!})^m$$
$$= \sum_{m\geq 0}\sum_{l=0}^{m} D_l S_2(m,l)\frac{t^m}{m!}$$
$$\times\sum_{n\geq 0}E\sum_{k_1+\cdots+k_m=n}\binom{n}{k_1,\cdots,k_m}(u_1\Gamma'_1)^{k_1}\cdots(u_m\Gamma'_m)^{k_m}\frac{t^n}{n!}$$

$$= \sum_{m\geq 0}\sum_{l=0}^{m}(-1)^l E(u\Gamma)^l \binom{m}{l} E(u_1+\cdots+u_l)^{m-l}\frac{t^m}{m!}$$
$$\times \sum_{n\geq 0} E(u_1\Gamma_1'+\cdots+u_m\Gamma_m')^n \frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^l E(u\Gamma)^l E(u_1+\cdots+u_l)^{k-l}$$
$$\times E(u_1\Gamma_1'+\cdots+u_m\Gamma_m')^{n-k}\frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$, therefore we derive the Eq.(32).

Theorem 2.12 Suppose that r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $X_1, X_2, \cdots, i.i.d \sim \Gamma(\frac{1}{\lambda}-n, 1)$, $(\frac{1}{\lambda}-n>0)$, u_i and X_j are independent for $i,j=1,2,...$, then for $n \geq k$ we obtain the moment representation of the degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$:

$$S_{2,\lambda}(n,k) = \lambda^{n-k}\binom{n}{k}E(u_1X_1+\cdots+u_kX_k)^{n-k}. \tag{34}$$

Proof By setting t to λt and λ to $\frac{1}{\lambda}$ in Eq.(8), we have

$$\sum_{n\geq k} S_{1,\frac{1}{\lambda}}(n,k)\frac{(\lambda t)^n}{n!} = \frac{(\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1))^k}{k!} = \lambda^k \sum_{n\geq k} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{35}$$

By comparing the coefficients $\frac{t^n}{n!}$ in Eq.(35), we have

$$S_{2,\lambda}(n,k) = \lambda^{n-k}S_{1,\frac{1}{\lambda}}(n,k).$$

According the moment representation of $S_{1,\lambda}(n,k)$, we have

$$S_{2,\lambda}(n,k) = \lambda^{n-k}\binom{n}{k}E(u_1X_1+\cdots+u_kX_k)^{n-k},$$

here r.v $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, and r.v $X_1, X_2, \cdots, i.i.d \sim \Gamma(\frac{1}{\lambda}-n, 1)$, and $\frac{1}{\lambda}-n>0$.

Theorem 2.13 Suppose that r.v $u, u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, $\Gamma' \sim \Gamma(\lambda-n, 1)$, $X_1, X_2, \cdots, i.i.d \sim \Gamma(\frac{1}{\lambda}-n, 1)$, $(\frac{1}{\lambda}-n>0)$, u_i and Γ_j, X_j are independent for $i,j=1,2,...$, then for $n \in \mathbb{N}$ we obtain the moment representation of the degenerate Bernoulli numbers $\beta_{n,\lambda}$:

$$\beta_{n,\lambda} = \sum_{m=0}^{n}\binom{n}{m}E(u\Gamma')^m E[\lambda(u_1X_1+\cdots+u_mX_m)]^{n-m}. \tag{36}$$

$$\beta_{n,\lambda} = \sum_{m=0}^{n}\binom{n}{m}E(u\Gamma_1+\cdots+u\Gamma_{\lambda-n})^m E[\lambda(u_1X_1+\cdots+u_mX_m)]^{n-m}. \tag{37}$$

Proof According to Eq.(7), we have

$$\begin{aligned}
\sum_{n\geq 0}\beta_{n,\lambda}\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \frac{\frac{1}{\lambda}((1+(1+\lambda t)^{\frac{1}{\lambda}}-1)^{\lambda}-1)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\\
&= \sum_{m\geq 0} D_{\lambda,3}(m)\frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^m}{m!} = \sum_{m\geq 0} D_{\lambda,3}(m)\sum_{n\geq m} S_{2,\lambda}(n,m)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\sum_{m=0}^{n} E(u\Gamma')^m\lambda^{n-m}\binom{n}{m}E(u_1X_1+\cdots+u_mX_m)^{n-m}\frac{t^n}{n!}.
\end{aligned}$$

Thus, comparing the coefficients $\frac{t^n}{n!}$ gives the Eq.(36). Eq.(37) can be directly obtained by Theorem 2.5.

We investigate the relationship of the degenerate Bernoulli numbers $\beta_{n,\lambda}$ and the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$, we arrive at the fifth moment representation of the degenerate Daehee numbers of the third kind $D_{\lambda,3}(n)$ in the following theorem.

Theorem 2.14 Suppose that r.v $u, u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma', \Gamma_1', \Gamma_2', \cdots$, $i.i.d \sim \Gamma(\lambda-n,1)$ and $\lambda-n>0$, $X_1, X_2, \cdots, i.i.d \sim \Gamma(\frac{1}{\lambda}-n,1)$ and $\frac{1}{\lambda}-n>0$ then for $n\in\mathbb{N}$ we have:

$$\begin{aligned}
D_{\lambda,3}(n) = &\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}E(u\Gamma')^m E[\lambda(u_1X_1+\cdots+u_mX_m)]^{l-m}\\
&\times E(u_1\Gamma_1'+\cdots+u_l\Gamma_l')^{n-l}.
\end{aligned} \tag{38}$$

Proof According to Eq.(6), we have

$$\begin{aligned}
\sum_{n\geq 0}D_{\lambda,3}(n)\frac{t^n}{n!} &= \frac{\frac{1}{\lambda}((1+t)^{\lambda}-1)}{t} = \frac{\frac{1}{\lambda}((1+t)^{\lambda}-1)}{(1+\lambda(\frac{1}{\lambda}((1+t)^{\lambda}-1)))^{\frac{1}{\lambda}}-1}\\
&= \sum_{l\geq 0}\beta_{l,\lambda}\frac{(\frac{1}{\lambda}((1+t)^{\lambda}-1))^l}{l!} = \sum_{l\geq 0}\beta_{l,\lambda}\sum_{n\geq l}S_{1,\lambda}(n,l)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}E(u\Gamma')^m E[\lambda(u_1X_1+\cdots+u_mX_m)]^{l-m}\\
&\quad\times\binom{n}{l}E(u_1\Gamma_1'+\cdots+u_l\Gamma_l')^{n-l}\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients $\frac{t^n}{n!}$, therefore we derive the Eq.(38).

Theorem 2.15 Suppose that r.v $u, u_1, u_2, \cdots, i.i.d \sim U[0,1]$, $\Gamma' \sim \Gamma(\lambda - n, 1)$ and $\lambda - n > 0$, $X_1, X_2, \cdots, i.i.d \sim \Gamma(\frac{1}{\lambda} - n, 1)$ and $\frac{1}{\lambda} - n > 0$, $\Gamma_1, \Gamma_2, \cdots$, $i.i.d \sim \Gamma(1,1)$, then for $n \in \mathbb{N}$ we obtain the moment representation of the degenerate Daehee numbers of the second kind $D_{\lambda,2}(n)$:

$$D_{\lambda,2}(n) = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \binom{m}{l} (-1)^{n-m} E(u\Gamma')^l E(\sum_{i=1}^{l} \lambda u_i X_i)^{m-l} E(\sum_{j=1}^{m} u_j \Gamma_j)^{n-m} \tag{39}$$

Proof For Eq.(6), we replace t to $(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1$,

$$\begin{aligned}
&\sum_{l \geq 0} D_{\lambda,3}(l) \frac{((1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1)^l}{l!} \\
&= \frac{\frac{1}{\lambda}((1 + ((1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1))^{\lambda} - 1)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \\
&= \frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} = \sum_{n \geq 0} D_{\lambda,2}(n) \frac{t^n}{n!}.
\end{aligned} \tag{40}$$

Then we have

$$\begin{aligned}
&\sum_{l \geq 0} D_{\lambda,3}(l) \frac{((1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1)^l}{l!} \\
&= \sum_{l \geq 0} D_{\lambda,3}(l) \sum_{m \geq l} S_{2,\lambda}(m,l) \frac{(\log(1+t))^m}{m!} \\
&= \sum_{l \geq 0} D_{\lambda,3}(l) \sum_{m \geq l} S_{2,\lambda}(m,l) \sum_{n \geq m} S_1(n,m) \frac{t^n}{n!} \\
&= \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{l=0}^{m} D_{\lambda,3}(l) S_{2,\lambda}(m,l) S_1(n,m) \frac{t^n}{n!} \\
&= \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{l=0}^{m} E(u\Gamma')^l \binom{m}{l} E[\lambda(u_1 X_1 + \cdots + u_l X_l)]^{m-l} \\
&\quad \times (-1)^{n-m} \binom{n}{m} E(u_1 \Gamma_1 + \cdots + u_m \Gamma_m)^{n-m} \frac{t^n}{n!}.
\end{aligned} \tag{41}$$

By comparing the coefficients $\frac{t^n}{n!}$ of Eq.(40) and Eq.(41), we arrive at the Eq.(39).

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Conflict of interest

The authors declare that they have no conflict of interest.

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