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A Closed Formula for the Sums of Squares of Generalized Tribonacci numbers

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Abstract.

In this paper, closed forms of the sum formulas for the squares of generalized Tribonacci numbers are presented. As special cases, we give summation formulas of the squares of Tribonacci, Tribonacci Lucas, Padovan, Perrin, Narayana and some other third order linear recurrence sequences.

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1. Introduction

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

(1.1) $$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1,2,6,7,13,16,19,20,21,22,24,25,26,27,29].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1 A few special case of generalized Tribonacci sequences.

Sequences (Numbers)	Notation	OEIS [23]
Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	-
Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{D_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328
Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012
third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=1}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+2}T_{n+3} + 2T_{n+1}T_{n+3} + 1)$$

and

$$\sum_{k=1}^{n} N_k^2 = \frac{1}{3}(-N_{n+3}^2 - 4N_{n+2}^2 - 4N_{n+1}^2 + 4N_{n+2}N_{n+3} + 2N_{n+1}N_{n+3} + 2N_{n+1}N_{n+2} + 1).$$

In this work, we derive expressions for sums of second powers of generalized Tribonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[3,4,10,11,12]	[9,28]	[5,8,14]
Generalized Tribonacci	[17]		
Generalized Tetranacci	[15,18]		

2. Main Result

Let

$$\Delta = (s + rt - t^2 + 1)(r + s + t - 1)(r - s + t + 1).$$

THEOREM 2.1. *If* $\Delta \neq 0$ *then*

(a):

$$\sum_{k=1}^{n} W_k^2 = \frac{\Delta_1}{\Delta},$$

(b):

$$\sum_{k=1}^{n} W_{k+1} W_k = \frac{\Delta_2}{\Delta},$$

(c):

$$\sum_{k=1}^{n} W_{k+2} W_k = \frac{\Delta_3}{\Delta},$$

where

$$\begin{aligned}
\Delta_1 &= -(t^2+rt+s-1)W_{n+3}^2 - (r^3t+r^2t^2+r^2s+r^2+t^2+2rst+rt+s-1)W_{n+2}^2 \\
&\quad -(r^3t+r^2t^2+s^2t^2-rs^2t-s^3+r^2s+4rst+r^2+s^2+t^2+rt+s-1)W_{n+1}^2 \\
&\quad +2\,(r+t)\,(s+rt)\,W_{n+3}W_{n+2} + 2t\,(r+st)\,W_{n+3}W_{n+1} - 2t\,(s-1)\,(s+rt)\,W_{n+2}W_{n+1} \\
&\quad +(2rst+2r^2+t^2+rt+s-1)W_3^2 + (r^3t+r^2t^2+r^2s+2rst+r^2+t^2+rt+s-1)W_2^2 \\
&\quad +(r^3t+r^2t^2+s^2t^2-rs^2t-s^3+r^2s+4rst+r^2+s^2+t^2+rt+s-1)W_1^2 \\
&\quad -2\,(r+st)\,W_4W_3 - 2t(r^2-s^2+rt+s)W_3W_2 + 2t\,(s-1)\,(s+rt)\,W_2W_1
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2 &= (r+st)\,W_{n+3}^2 + (s+rt)\,(t+rs)\,W_{n+2}^2 + t^2\,(r+st)\,W_{n+1}^2 \\
&\quad -(2rst+r^2+s^2+t^2-1)W_{n+3}W_{n+2} + t(r^2-s^2-t^2+1)W_{n+3}W_{n+1} \\
&\quad -(r^3t-rt^3-rs^2t+r^2s-s^3-st^2+2rst+r^2+s^2+t^2+rt+s-1)W_{n+2}W_{n+1} \\
&\quad +(r^3-rs^2-rt^2-st)W_3^2 - (t+rs)\,(s+rt)\,W_2^2 - t^2\,(r+st)\,W_1^2 \\
&\quad -(r^2-s^2-t^2+1)W_4W_3 + (r^2s-st^2-s^3+2rst+r^2+s^2+t^2+s-1)W_3W_2 \\
&\quad +(-rt^3+r^3t-rs^2t+r^2s-st^2-s^3+r^2+s^2+t^2+2rst+rt+s-1)W_2W_1
\end{aligned}$$

and

$$\begin{aligned}
\Delta_3 &= (r^2-s^2+rt+s)W_{n+3}^2 - (rs^2t-rt^3-r^2t^2+r^2s+s^2-s)W_{n+2}^2 + t^2(r^2-s^2+rt+s)W_{n+1}^2 \\
&\quad -(r+t)\left(r^2-s^2+t^2-1\right)W_{n+3}W_{n+2} - (r^2s-st^2-s^3+2rst+r^2+s^2+t^2+s-1)W_{n+3}W_{n+1} \\
&\quad +t\,(s-1)\left(r^2-s^2+t^2-1\right)W_{n+2}W_{n+1} + (rs^2t+r^4-r^2s^2-r^2t^2+2r^2s-rt^3+r^3t+s^2-s)W_3^2 \\
&\quad +(rs^2t-rt^3-r^2t^2+r^2s+s^2-s)W_2^2 - t^2(r^2-s^2+rt+s)W_1^2 \\
&\quad -(r^3-t^3-rs^2-rt^2+r^2t+s^2t+2rs+r+t)W_4W_3 \\
&\quad +(r^3s-st^3+s^3t-rst^2-rs^3+r^2st+rs^2+rt^2+r^2t-s^2t+r^3+t^3+rs+st-r-t)W_3W_2 \\
&\quad +\left(s+rt-t^2+1\right)(r-s+t+1)\,(r+s+t-1)\,W_3W_1 - t\,(s-1)\,(r^2-s^2+t^2-1)W_2W_1
\end{aligned}$$

Proof. First, we obtain $\sum_{k=1}^{n} W_k^2$. Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n$$

or

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

we obtain

$$\begin{aligned}
t^2W_n^2 &= W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 - 2rW_{n+3}W_{n+2} - 2sW_{n+3}W_{n+1} + 2rsW_{n+2}W_{n+1} \\
t^2W_{n-1}^2 &= W_{n+2}^2 + r^2W_{n+1}^2 + s^2W_n^2 - 2rW_{n+2}W_{n+1} - 2sW_{n+2}W_n + 2rsW_{n+1}W_n \\
&\vdots \\
t^2W_2^2 &= W_5^2 + r^2W_4^2 + s^2W_3^2 - 2rW_5W_4 - 2sW_5W_3 + 2rsW_4W_3 \\
t^2W_1^2 &= W_4^2 + r^2W_3^2 + s^2W_2^2 - 2rW_4W_3 - 2sW_4W_2 + 2rsW_3W_2.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
t^2\sum_{k=1}^{n} W_k^2 &= \sum_{k=4}^{n+3} W_k^2 + r^2\sum_{k=3}^{n+2} W_k^2 + s^2\sum_{k=2}^{n+1} W_k^2 - 2r\sum_{k=3}^{n+2} W_{k+1}W_k \\
&\quad -2s\sum_{k=2}^{n+1} W_{k+2}W_k + 2rs\sum_{k=2}^{n+1} W_{k+1}W_k.
\end{aligned} \tag{2.1}$$

Note that if we replace the followings into (2.1),

$$\begin{aligned}
\sum_{k=4}^{n+3} W_k^2 &= -W_1^2 - W_2^2 - W_3^2 + W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 + \sum_{k=1}^{n} W_k^2, \\
\sum_{k=3}^{n+2} W_k^2 &= -W_1^2 - W_2^2 + W_{n+1}^2 + W_{n+2}^2 + \sum_{k=1}^{n} W_k^2, \\
\sum_{k=2}^{n+1} W_k^2 &= -W_1^2 + W_{n+1}^2 + \sum_{k=1}^{n} W_k^2, \\
\sum_{k=3}^{n+2} W_{k+1}W_k &= -W_2W_1 - W_3W_2 + W_{n+2}W_{n+1} + W_{n+3}W_{n+2} + \sum_{k=1}^{n} W_{k+1}W_k, \\
\sum_{k=2}^{n+1} W_{k+1}W_k &= -W_2W_1 + W_{n+2}W_{n+1} + \sum_{k=1}^{n} W_{k+1}W_k, \\
\sum_{k=2}^{n+1} W_{k+2}W_k &= -W_3W_1 + W_{n+3}W_{n+1} + \sum_{k=1}^{n} W_{k+2}W_k.
\end{aligned}$$

we get

$$(2.2) \qquad t^2\sum_{k=1}^{n}W_k^2 = (-r^2W_1^2 - r^2W_2^2 + r^2W_{n+1}^2 + r^2W_{n+2}^2 - s^2W_1^2 + s^2W_{n+1}^2$$
$$-W_1^2 - W_2^2 - W_3^2 + W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 + (1+r^2+s^2)\sum_{k=1}^{n}W_k^2)$$
$$+(2rW_1W_2 - 2rW_{n+1}W_{n+2} - 2rW_{n+2}W_{n+3} + 2rW_2W_3$$
$$+2rsW_{n+1}W_{n+2} - 2rsW_1W_2 + (-2r+2rs)\sum_{k=1}^{n}W_kW_{k+1})$$
$$-2s(-W_3W_1 + W_{n+3}W_{n+1} + \sum_{k=1}^{n}W_{k+2}W_k)$$

Next we obtain $\sum_{k=1}^{n}W_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+1} we get

$$tW_{n+1}W_n = W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned} tW_{n+1}W_n &= W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2 \\ tW_nW_{n-1} &= W_{n+2}W_n - rW_{n+1}W_n - sW_n^2 \\ &\vdots \\ tW_3W_2 &= W_5W_3 - rW_4W_3 - sW_3^2 \\ tW_2W_1 &= W_4W_2 - rW_3W_2 - sW_2^2. \end{aligned}$$

If we add the equations by side by, we get

$$t\sum_{k=1}^{n}W_{k+1}W_k = \sum_{k=2}^{n+1}W_{k+2}W_k - r\sum_{k=2}^{n+1}W_{k+1}W_k - s\sum_{k=2}^{n+1}W_k^2.$$

Now it follows that

$$(2.3) \quad t\sum_{k=1}^{n}W_{k+1}W_k = (-W_3W_1 + W_{n+3}W_{n+1} + \sum_{k=1}^{n}W_{k+2}W_k)$$
$$-r(-W_2W_1 + W_{n+2}W_{n+1} + \sum_{k=1}^{n}W_{k+1}W_k) - s(-W_1^2 + W_{n+1}^2 + \sum_{k=1}^{n}W_k^2).$$

Now, we obtain $\sum_{k=2}^{n}W_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+2} we get

$$tW_{n+2}W_n = W_{n+3}W_{n+2} - rW_{n+2}W_{n+2} - sW_{n+2}W_{n+1}.$$

Then using last recurrence relation, we obtain

$$\begin{aligned} tW_{n+2}W_n &= W_{n+3}W_{n+2} - rW_{n+2}^2 - sW_{n+2}W_{n+1} \\ tW_{n+1}W_{n-1} &= W_{n+2}W_{n+1} - rW_{n+1}^2 - sW_{n+1}W_n \\ &\vdots \\ tW_5W_3 &= W_6W_5 - rW_5^2 - sW_5W_4 \\ tW_4W_2 &= W_5W_4 - rW_4^2 - sW_4W_3. \end{aligned}$$

If we add the equations by side by, we get

$$t\sum_{k=2}^{n} W_{k+2}W_k = \sum_{k=4}^{n+2} W_{k+1}W_k - r\sum_{k=4}^{n+2} W_k^2 - s\sum_{k=3}^{n+1} W_{k+1}W_k.$$

Now it follows that

$$\begin{aligned} (2.4)\quad t(-W_3W_1 + \sum_{k=1}^{n} W_{k+2}W_k) &= (-W_4W_3 - W_3W_2 - W_2W_1 + W_{n+3}W_{n+2} + W_{n+2}W_{n+1} \\ &\quad + \sum_{k=1}^{n} W_{k+1}W_k) - r(-W_1^2 - W_2^2 - W_3^2 + W_{n+1}^2 + W_{n+2}^2 \\ &\quad + \sum_{k=1}^{n} W_k^2) - s(-W_3W_2 - W_2W_1 + W_{n+2}W_{n+1} + \sum_{k=1}^{n} W_{k+1}W_k) \end{aligned}$$

bulunur. Solving the system (2.2)-(2.3)-(2.4), the results in (a), (b) and (c) follow.

3. Specific Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} W_k^2$, $\sum_{k=1}^{n} W_{k+1}W_k$ and $\sum_{k=1}^{n} W_{k+2}W_k$ for the specific case of sequence $\{W_n\}$.

Taking $r = s = t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.1. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

- **(a):** $\sum_{k=1}^{n} W_k^2 = \frac{1}{4}(-W_{n+3}^2 - 4W_{n+2}^2 - 5W_{n+1}^2 + 4W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + 3W_3^2 + 4W_2^2 + 5W_1^2 - 2W_4W_3 - 2W_2W_3)$.
- **(b):** $\sum_{k=1}^{n} W_{k+1}W_k = \frac{1}{4}(W_{n+3}^2 + 2W_{n+2}^2 + W_{n+1}^2 - 2W_{n+2}W_{n+3} - 2W_{n+1}W_{n+2} - W_3^2 - 2W_2^2 - W_1^2 + 2W_2W_3 + 2W_1W_2)$.
- **(c):** $\sum_{k=1}^{n} W_{k+2}W_k = \frac{1}{4}(W_{n+3}^2 + W_{n+1}^2 - 2W_{n+1}W_{n+3} + W_3^2 - W_1^2 - 2W_3W_4 + 2W_2W_3 + 4W_1W_3)$.

From the above proposition, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 3.2. *For $n \geq 1$, Tribonacci numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+2}T_{n+3} + 2T_{n+1}T_{n+3} + 1)$.
- **(b):** $\sum_{k=1}^{n} T_{k+1}T_k = \frac{1}{4}(T_{n+3}^2 + 2T_{n+2}^2 + T_{n+1}^2 - 2T_{n+2}T_{n+3} - 2T_{n+1}T_{n+2} - 1)$.

(c): $\sum_{k=1}^{n} T_{k+2}T_k = \frac{1}{4}(T_{n+3}^2 + T_{n+1}^2 - 2T_{n+1}T_{n+3} - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 3.3. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} K_k^2 = \frac{1}{4}(-K_{n+3}^2 - 4K_{n+2}^2 - 5K_{n+1}^2 + 4K_{n+2}K_{n+3} + 2K_{n+1}K_{n+3} - 8)$.

(b): $\sum_{k=1}^{n} K_{k+1}K_k = \frac{1}{4}(K_{n+3}^2 + 2K_{n+2}^2 + K_{n+1}^2 - 2K_{n+2}K_{n+3} - 2K_{n+1}K_{n+2} - 20)$.

(c): $\sum_{k=1}^{n} K_{k+2}K_k = \frac{1}{4}(K_{n+3}^2 + K_{n+1}^2 - 2K_{n+1}K_{n+3} - 36)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.4. *If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} W_k^2 = \frac{1}{9}(-W_{n+3}^2 - 9W_{n+2}^2 - 10W_{n+1}^2 + 6W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + 5W_3^2 + 9W_2^2 + 10W_1^2 - 2W_4W_3 - 4W_2W_3)$.

(b): $\sum_{k=1}^{n} W_{k+1}W_k = \frac{1}{9}(W_{n+3}^2 + 3W_{n+2}^2 + W_{n+1}^2 - 3W_{n+2}W_{n+3} + W_{n+1}W_{n+3} - 6W_{n+1}W_{n+2} + W_3^2 - 3W_2^2 - W_1^2 - W_4W_3 + 4W_2W_3 + 6W_1W_2)$.

(c): $\sum_{k=1}^{n} W_{k+2}W_k = \frac{1}{9}(2W_{n+3}^2 + 2W_{n+1}^2 - 3W_{n+2}W_{n+3} - 4W_{n+1}W_{n+3} + 8W_3^2 - 2W_1^2 - 5W_3W_4 + 8W_2W_3 + 9W_1W_3)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n^{(3)}$ with $P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 1$).

COROLLARY 3.5. *For $n \geq 1$, third-order Pell numbers have the following properties:*

(a): $\sum_{k=1}^{n} P_k^{(3)2} = \frac{1}{9}(-P_{n+3}^{(3)2} - 9P_{n+2}^{(3)2} - 10P_{n+1}^{(3)2} + 6P_{n+2}^{(3)}P_{n+3}^{(3)} + 2P_{n+1}^{(3)}P_{n+3}^{(3)} + 1)$.

(b): $\sum_{k=1}^{n} P_{k+1}^{(3)}P_k^{(3)} = \frac{1}{9}(P_{n+3}^{(3)2} + 3P_{n+2}^{(3)2} + P_{n+1}^{(3)2} - 3P_{n+2}^{(3)}P_{n+3}^{(3)} + P_{n+1}^{(3)}P_{n+3}^{(3)} - 6P_{n+1}^{(3)}P_{n+2}^{(3)} - 1)$.

(c): $\sum_{k=1}^{n} P_{k+2}^{(3)}P_k^{(3)} = \frac{1}{9}(2P_{n+3}^{(3)2} + 2P_{n+1}^{(3)2} - 3P_{n+2}^{(3)}P_{n+3}^{(3)} - 4P_{n+1}^{(3)}P_{n+3}^{(3)} - 2)$.

Taking $W_n = Q_n^{(3)}$ with $Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6$ in the last Proposition, we have the following Corollary which presents sum formulas of third-order Pell-Lucas numbers.

COROLLARY 3.6. *For $n \geq 1$, third-order Pell-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} Q_k^{(3)2} = \frac{1}{9}(-Q_{n+3}^{(3)2} - 9Q_{n+2}^{(3)2} - 10Q_{n+1}^{(3)2} + 6Q_{n+2}^{(3)}Q_{n+3}^{(3)} + 2Q_{n+1}^{(3)}Q_{n+3}^{(3)} - 27)$.

(b): $\sum_{k=1}^{n} Q_{k+1}^{(3)}Q_k^{(3)} = \frac{1}{9}(Q_{n+3}^{(3)2} + 3Q_{n+2}^{(3)2} + Q_{n+1}^{(3)2} - 3Q_{n+2}^{(3)}Q_{n+3}^{(3)} + Q_{n+1}^{(3)}Q_{n+3}^{(3)} - 6Q_{n+1}^{(3)}Q_{n+2}^{(3)} - 57)$.

(c): $\sum_{k=1}^{n} Q_{k+2}^{(3)}Q_k^{(3)} = \frac{1}{9}(2Q_{n+3}^{(3)2} + 2Q_{n+1}^{(3)2} - 3Q_{n+2}^{(3)}Q_{n+3}^{(3)} - 4Q_{n+1}^{(3)}Q_{n+3}^{(3)} - 144)$.

From the last Proposition, we have the following Corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n^{(3)}$ with $E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1$).

COROLLARY 3.7. *For $n \geq 1$, third-order modified Pell numbers have the following properties:*

(a): $\sum_{k=1}^{n} E_k^{(3)2} = \frac{1}{9}(-E_{n+3}^{(3)2} - 9E_{n+2}^{(3)2} - 10E_{n+1}^{(3)2} + 6E_{n+2}^{(3)}E_{n+3}^{(3)} + 2E_{n+1}^{(3)}E_{n+3}^{(3)} + 4)$.

(b): $\sum_{k=1}^{n} E_{k+1}^{(3)}E_{k}^{(3)} = \frac{1}{9}(E_{n+3}^{(3)2} + 3E_{n+2}^{(3)2} + E_{n+1}^{(3)2} - 3E_{n+2}^{(3)}E_{n+3}^{(3)} + E_{n+1}^{(3)}E_{n+3}^{(3)} - 6E_{n+1}^{(3)}E_{n+2}^{(3)} - 1)$.

(c): $\sum_{k=1}^{n} E_{k+2}^{(3)}E_{k}^{(3)} = \frac{1}{9}(2E_{n+3}^{(3)2} + 2E_{n+1}^{(3)2} - 3E_{n+2}^{(3)}E_{n+3}^{(3)} - 4E_{n+1}^{(3)}E_{n+3}^{(3)} + 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.8. *If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} W_k^2 = -2W_{n+1}^2 - W_{n+3}^2 - W_{n+2}^2 + 2W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + W_3^2 + W_2^2 + 2W_1^2 - 2W_4W_3$.

(b): $\sum_{k=1}^{n} W_{k+1}W_k = W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+2}W_{n+3} - W_{n+1}W_{n+3} - W_3^2 - W_2^2 - W_1^2 + W_4W_3$.

(c): $\sum_{k=1}^{n} W_{k+2}W_k = W_{n+2}W_{n+3} - W_3W_4 + W_1W_3$.

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

COROLLARY 3.9. *For $n \geq 1$, Padovan numbers have the following properties:*

(a): $\sum_{k=1}^{n} P_k^2 = -P_{n+3}^2 - P_{n+2}^2 - 2P_{n+1}^2 + 2P_{n+2}P_{n+3} + 2P_{n+1}P_{n+3} - 1$.

(b): $\sum_{k=1}^{n} P_{k+1}P_k = P_{n+3}^2 + P_{n+2}^2 + P_{n+1}^2 - P_{n+2}P_{n+3} - P_{n+1}P_{n+3} - 2$.

(c): $\sum_{k=1}^{n} P_{k+2}P_k = P_{n+2}P_{n+3} - 2$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Perrin numbers.

COROLLARY 3.10. *For $n \geq 1$, Perrin numbers have the following properties:*

(a): $\sum_{k=1}^{n} E_k^2 = -E_{n+3}^2 - E_{n+2}^2 - 2E_{n+1}^2 + 2E_{n+2}E_{n+3} + 2E_{n+1}E_{n+3} + 1$.

(b): $\sum_{k=1}^{n} E_{k+1}E_k = E_{n+3}^2 + E_{n+2}^2 + E_{n+1}^2 - E_{n+2}E_{n+3} - E_{n+1}E_{n+3} - 7$.

(c): $\sum_{k=1}^{n} E_{k+2}E_k = E_{n+2}E_{n+3} - 6$.

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

COROLLARY 3.11. *For $n \geq 1$, Padovan-Perrin numbers have the following properties:*

(a): $\sum_{k=1}^{n} S_k^2 = -S_{n+3}^2 - S_{n+2}^2 - 2S_{n+1}^2 + 2S_{n+2}S_{n+3} + 2S_{n+1}S_{n+3} + 1$.

(b): $\sum_{k=1}^{n} S_{k+1}S_k = S_{n+3}^2 + S_{n+2}^2 + S_{n+1}^2 - S_{n+2}S_{n+3} - S_{n+1}S_{n+3} - 1$.

(c): $\sum_{k=1}^{n} S_{k+2}S_k = S_{n+2}S_{n+3}$.

Taking $r = 0, s = 1, t = 2$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.12. *If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} W_k^2 = \frac{1}{2}(W_{n+3}^2 + W_{n+2}^2 + 2W_{n+1}^2 - W_{n+2}W_{n+3} - 2W_{n+1}W_{n+3} - W_3^2 - W_2^2 - 2W_1^2 + W_4W_3)$.

(b): $\sum_{k=1}^{n} W_{k+1}W_k = \frac{1}{4}(-W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 + 2W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + W_3^2 + W_2^2 + 4W_1^2 - 2W_4W_3)$.

(c): $\sum_{k=1}^{n} W_{k+2}W_k = \frac{1}{2}(W_{n+2}W_{n+3} + 2W_1W_3 - W_3W_4)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

COROLLARY 3.13. *For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties:*

(a): $\sum_{k=1}^{n} Q_k^2 = \frac{1}{2}(Q_{n+3}^2 + Q_{n+2}^2 + 2Q_{n+1}^2 - Q_{n+2}Q_{n+3} - 2Q_{n+1}Q_{n+3} - 3)$.

(b): $\sum_{k=1}^{n} Q_{k+1}Q_k = \frac{1}{4}(-Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 2Q_{n+2}Q_{n+3} + 4Q_{n+1}Q_{n+3} - 4)$.

(c): $\sum_{k=1}^{n} Q_{k+2}Q_k = \frac{1}{2}(Q_{n+2}Q_{n+3} - 3)$.

Taking $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Jacobsthal-Perrin numbers.

COROLLARY 3.14. *For $n \geq 1$, Jacobsthal-Perrin numbers have the following properties:*

(a): $\sum_{k=1}^{n} D_k^2 = \frac{1}{2}(D_{n+3}^2 + D_{n+2}^2 + 2D_{n+1}^2 - D_{n+2}D_{n+3} - 2D_{n+1}D_{n+3} - 28)$.

(b): $\sum_{k=1}^{n} D_{k+1}D_k = \frac{1}{4}(-D_{n+3}^2 - D_{n+2}^2 - 4D_{n+1}^2 + 2D_{n+2}D_{n+3} + 4D_{n+1}D_{n+3} + 16)$.

(c): $\sum_{k=1}^{n} D_{k+2}D_k = \frac{1}{2}(D_{n+2}D_{n+3} - 12)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.15. *If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} W_k^2 = \frac{1}{3}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + 2W_{n+1}W_{n+2} + 3W_3^2 + 4W_2^2 + 4W_1^2 - 2W_4W_3 - 4W_2W_3 - 2W_1W_2)$.

(b): $\sum_{k=1}^{n} W_{k+1}W_k = \frac{1}{3}(W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+2}W_{n+3} + W_{n+1}W_{n+3} - 2W_{n+1}W_{n+2} - W_2^2 - W_1^2 - W_3W_4 + W_3W_2 + 2W_1W_2)$.

(c): $\sum_{k=1}^{n} W_{k+2}W_k = \frac{1}{3}(2W_{n+3}^2 + 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+2}W_{n+3} - W_{n+1}W_{n+2} - W_{n+1}W_{n+3} - 2W_2^2 - 2W_1^2 - 2W_3W_4 + 2W_3W_2 + 3W_3W_1 + W_1W_2)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

COROLLARY 3.16. *For $n \geq 1$, Narayana numbers have the following properties:*

(a): $\sum_{k=1}^{n} N_k^2 = \frac{1}{3}(-N_{n+3}^2 - 4N_{n+2}^2 - 4N_{n+1}^2 + 4N_{n+2}N_{n+3} + 2N_{n+1}N_{n+3} + 2N_{n+1}N_{n+2} + 1)$.

(b): $\sum_{k=1}^{n} N_{k+1}N_k = \frac{1}{3}(N_{n+3}^2 + N_{n+2}^2 + N_{n+1}^2 - N_{n+2}N_{n+3} + N_{n+1}N_{n+3} - 2N_{n+1}N_{n+2} - 1)$.

(c): $\sum_{k=1}^{n} N_{k+2}N_k = \frac{1}{3}(2N_{n+3}^2 + 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+2}N_{n+3} - N_{n+1}N_{n+2} - N_{n+1}N_{n+3} - 2)$.

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