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Matrix Sequences of Third-Order Pell and Third-Order Pell -Lucas Numbers

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Abstract. In this paper, we define third-order Pell and third-order Pell-Lucas matrix sequences and investigate their properties.

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1. Introduction and Preliminaries

In [8], third-order Pell sequence $\{P_n^{(3)}\}_{n\geq 0}$ and third-order Pell-Lucas sequence $\{Q_n^{(3)}\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$P_{n+3}^{(3)} = 2P_{n+2}^{(3)} + P_{n+1}^{(3)} + P_n^{(3)}, \qquad P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2, \tag{1.1}$$

and

$$Q_{n+3}^{(3)} = 2Q_{n+2}^{(3)} + Q_{n+1}^{(3)} + Q_n^{(3)}, \qquad Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6 \tag{1.2}$$

respectively. In the rest of the paper, for easy writing, we drop the superscripts and write P_n and Q_n for $P_n^{(3)}$ and $Q_n^{(3)}$, respectively. Note that P_n is the sequence A077939 in [6] associated with the expansion of $1/(1-2x-x^2-x^3)$, Q_n is the sequence A276225 in [6]

Basic properties of third-order Pell and third-order Pell-Lucas sequences are given in [8]. The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - 2P_{-(n-2)} + P_{-(n-3)}$$

and

$$Q_{-n}^{(3)} = -Q_{-(n-1)}^{(3)} - 2Q_{-(n-2)}^{(3)} + Q_{-(n-3)}^{(3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

Table 1. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	13	33	84	214	545	1388	3535	9003	22929	58396
P_{-n}	0	0	1	−1	−1	4	−3	−6	16	−7	−31	61	−6	−147
Q_n	3	2	6	17	42	107	273	695	1770	4508	11481	29240	74469	189659
Q_{-n}	3	−1	−3	8	−3	−16	30	−1	−75	107	42	−331	354	350

For all integers n, third-order Pell and Pell-Lucas numbers (using initial conditions in (1.1) and (1.2)) can be expressed using Binet's formulas as

$$P_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.3}$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n, \tag{1.4}$$

respectively. Here, α, β and γ are the roots of the cubic equation $x^3 - 2x^2 - x - 1 = 0$. Moreover

$$\begin{aligned} \alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\ \beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\ \gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

The generating functions for the third-order Pell sequence $\{P_n\}_{n\geq 0}$ and third-order Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2-x^3} \quad \text{and} \quad \sum_{n=0}^{\infty} Q_n x^n = \frac{3-4x-x^2}{1-2x-x^2-x^3}. \tag{1.5}$$

Note that the Binet form of a sequence satisfying (1.3) and (1.4) for non-negative integers is valid for all integers n. This result of Howard and Saidak [5] is even true in the case of higher-order recurrence relations as the following theorem shows.

THEOREM 1.1. *[5]Let $\{w_n\}$ be a sequence such that*

$$\{w_n\} = a_1 w_{n-1} + a_2 w_{n-2} + \ldots + a_k w_{n-k}$$

for all integers n, with arbitrary initial conditions $w_0, w_1, \ldots, w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$f(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_{k-1} x - a_k \tag{1.6}$$
$$= (x - \alpha_1)^{d_1} (x - \alpha_2)^{d_2} \ldots (x - \alpha_h)^{d_h}$$

with $d_1 + d_2 + \ldots + d_h = k$, and $\alpha_1, \alpha_2, \ldots, \alpha_k$ distinct. Then

(a): *For all n,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n \tag{1.7}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + \ldots + A_{r_m}^{(m)} n^{r_m - 1} = \sum_{u=0}^{r_m - 1} A_{u+1}^{(m)} n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (1.7) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0) \neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (1.6) are distinct, as they are in our examples, then

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \ldots + A_k(\alpha_k)^n.$$

(b): *The Binet form for $\{w_n\}$ is valid for all integers n.*

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized third-order Pell numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art.

On the other hand, the matrix sequences have taken so much interest for different type of numbers. For matrix sequences of generalized Horadam type numbers, see for example [2,3,4,11,12,13,15,18], for matrix sequences of generalized Tribonacci type numbers, see for instance [1,9,10,16,17] and for matrix sequence of generalized Tetranacci type numbers, see for example [7].

In this paper, the matrix sequences of third-order Pell and third-order Pell-Lucas numbers will be defined for the first time in the literature. Then, by giving the generating functions, the Binet formulas, and summation formulas over these new matrix sequences, we will obtain some fundamental properties on third-order Pell and third-order Pell-Lucas numbers. Also, we will present the relationship between these matrix sequences.

2. The Matrix Sequences of third-order Pell and third-order Pell-Lucas Numbers

In this section we define third-order Pell and third-order Pell-Lucas matrix sequences and investigate their properties.

DEFINITION 2.1. *For any integer $n \geq 0$, the third-order Pell matrix $(\mathcal{P}_n)$ and third-order Pell-Lucas matrix $(\mathcal{Q}_n)$ are defined by*

$$\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2} + \mathcal{P}_{n-3}, \tag{2.1}$$

$$\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2} + \mathcal{Q}_{n-3}, \tag{2.2}$$

respectively, with initial conditions

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{P}_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{P}_2 = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{Q}_0 = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -4 & -1 \\ -1 & 5 & -3 \end{pmatrix}, \ \mathcal{Q}_1 = \begin{pmatrix} 6 & 5 & 2 \\ 2 & 2 & 3 \\ 3 & -4 & -1 \end{pmatrix}, \mathcal{Q}_2 = \begin{pmatrix} 17 & 8 & 6 \\ 6 & 5 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

The sequences $\{\mathcal{P}_n\}_{n\geq 0}$ and $\{\mathcal{Q}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{P}_{-n} = -\mathcal{P}_{-(n-1)} - 2\mathcal{P}_{-(n-2)} + \mathcal{P}_{-(n-3)}$$

and

$$\mathcal{Q}_{-n} = -\mathcal{Q}_{-(n-1)} - 2\mathcal{Q}_{-(n-2)} + \mathcal{Q}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.1) and (2.2) hold for all integers n.

The following theorem gives the nth general terms of the third-order Pell and third-order Pell-Lucas matrix sequences.

THEOREM 2.2. *For any integer $n \geq 0$, we have the following formulas of the matrix sequences:*

$$\mathcal{P}_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} & P_{n-2} \end{pmatrix} \tag{2.3}$$

$$\mathcal{Q}_n = \begin{pmatrix} Q_{n+1} & Q_n + Q_{n-1} & Q_n \\ Q_n & Q_{n-1} + Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_{n-2} + Q_{n-3} & Q_{n-2} \end{pmatrix}. \tag{2.4}$$

Proof. We prove (2.3) by strong mathematical induction on n. (2.4) can be proved similarly.

If $n=0$ then, since $P_1=1, P_2=2, P_0=P_{-1}=0$, $P_{-2}=1$, $P_{-3}=-1$, we have

$$\mathcal{P}_0=\begin{pmatrix} P_1 & P_0+P_{-1} & P_0 \\ P_0 & P_{-1}+P_{-2} & P_{-1} \\ P_{-1} & P_{-2}+P_{-3} & P_{-2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true and

$$\mathcal{P}_1=\begin{pmatrix} P_2 & P_1+P_0 & P_1 \\ P_1 & P_0+P_{-1} & P_0 \\ P_0 & P_{-1}+P_{-2} & P_{-1} \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is true. Assume that the equality holds for $n\leq k$. For $n=k+1$, we have

$$\begin{aligned}
\mathcal{P}_{k+1} &= 2\mathcal{P}_k+\mathcal{P}_{k-1}+\mathcal{P}_{k-2} \\
&= \begin{pmatrix} 2P_{k+1} & 2P_k+2P_{k-1} & 2P_k \\ 2P_k & 2P_{k-1}+2P_{k-2} & 2P_{k-1} \\ 2P_{k-1} & 2P_{k-2}+2P_{k-3} & 2P_{k-2} \end{pmatrix}+\begin{pmatrix} P_k & P_{k-1}+P_{k-2} & P_{k-1} \\ P_{k-1} & P_{k-2}+P_{k-3} & P_{k-2} \\ P_{k-2} & P_{k-3}+P_{k-4} & P_{k-3} \end{pmatrix} \\
&\quad +\begin{pmatrix} P_{k-1} & P_{k-2}+P_{k-3} & P_{k-2} \\ P_{k-2} & P_{k-3}+P_{k-4} & P_{k-3} \\ P_{k-3} & P_{k-4}+P_{k-5} & P_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} 2P_{k+1}+P_k+P_{k-1} & 2P_k+3P_{k-1}+2P_{k-2}+P_{k-3} & 2P_k+P_{k-1}+P_{k-2} \\ 2P_k+P_{k-1}+P_{k-2} & 2P_{k-1}+3P_{k-2}+2P_{k-3}+P_{k-4} & 2P_{k-1}+P_{k-2}+P_{k-3} \\ 2P_{k-1}+P_{k-2}+P_{k-3} & 2P_{k-2}+3P_{k-3}+2P_{k-4}+P_{k-5} & 2P_{k-2}+P_{k-3}+P_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} P_{k+2} & P_k+P_{k+1} & P_{k+1} \\ P_{k+1} & P_k+P_{k-1} & P_k \\ P_k & P_{k-1}+P_{k-2} & P_{k-1} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on n, this proves (2.3).

We now give the Binet formulas for the third-order Pell and third-order Pell-Lucas matrix sequences.

THEOREM 2.3. *For every integer n, the Binet formulas of the third-order Pell and third-order Pell-Lucas matrix sequences are given by*

$$\mathcal{P}_n = A_1\alpha^n+B_1\beta^n+C_1\gamma^n, \tag{2.5}$$

$$\mathcal{Q}_n = A_2\alpha^n+B_2\beta^n+C_2\gamma^n. \tag{2.6}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha\mathcal{P}_2+\alpha(\alpha-2)\mathcal{P}_1+\mathcal{P}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1=\frac{\beta\mathcal{P}_2+\beta(\beta-2)\mathcal{P}_1+\mathcal{P}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_1=\frac{\gamma\mathcal{P}_2+\gamma(\gamma-2)\mathcal{P}_1+\mathcal{P}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)} \\
A_2 &= \frac{\alpha\mathcal{Q}_2+\alpha(\alpha-2)\mathcal{Q}_1+\mathcal{Q}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_2=\frac{\beta\mathcal{Q}_2+\beta(\beta-2)\mathcal{Q}_1+\mathcal{Q}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2=\frac{\gamma\mathcal{Q}_2+\gamma(\gamma-2)\mathcal{Q}_1+\mathcal{Q}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.
\end{aligned}$$

Proof. We prove the theorem only for $n \geq 0$ because of Theorem 1.1. We prove (2.5). By the assumption, the characteristic equation of (2.1) is $x^3 - 2x^2 - x - 1 = 0$ and the roots of it are α, β and γ. So it's general solution is given by

$$\mathcal{P}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n.$$

Using initial condition which is given in Definition 2.1, and also applying linear algebra operations, we obtain the matrices A_1, B_1, C_1 as desired. This gives the formula for $\mathcal{P}_n$.

Similarly we have the formula (2.6).

The well known Binet formulas for third-order Pell and third-order Pell-Lucas numbers are given in (1.3) and (1.4) respectively. But, we will obtain these functions in terms of third-order Pell and third-order Pell-Lucas matrix sequences as a consequence of Theorems 2.2 and 2.3. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 2.4. *For every integers n, the Binet's formulas for third-order Pell and third-order Pell-Lucas numbers are given as*

$$\begin{aligned} P_n &= \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\beta)(\gamma-\alpha)}, \\ Q_n &= \alpha^n + \beta^n + \gamma^n. \end{aligned}$$

Proof. From Theorem 2.3, we have

$$\begin{aligned} \mathcal{P}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n \\ &= \frac{\alpha\mathcal{P}_2 + \alpha(\alpha-2)\mathcal{P}_1 + \mathcal{P}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{P}_2 + \beta(\beta-2)\mathcal{P}_1 + \mathcal{P}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\ &\quad + \frac{\gamma\mathcal{P}_2 + \gamma(\gamma-2)\mathcal{P}_1 + \mathcal{P}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)} \begin{pmatrix} 2\alpha^2+\alpha+1 & \alpha(\alpha+1) & \alpha^2 \\ \alpha^2 & \alpha+1 & \alpha \\ \alpha & \alpha(\alpha-2) & 1 \end{pmatrix} \\ &\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)} \begin{pmatrix} 2\beta^2+\beta+1 & \beta(\beta+1) & \beta^2 \\ \beta^2 & \beta+1 & \beta \\ \beta & \beta(\beta-2) & 1 \end{pmatrix} \\ &\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)} \begin{pmatrix} 2\gamma^2+\gamma+1 & \gamma(\gamma+1) & \gamma^2 \\ \gamma^2 & \gamma+1 & \gamma \\ \gamma & \gamma(\gamma-2) & 1 \end{pmatrix} \end{aligned}$$

By Theorem 2.2, we know that

$$\mathcal{P}_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} & P_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned} P_n &= \frac{\alpha^{n-1}\alpha^2}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}\beta^2}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}\gamma^2}{(\gamma-\beta)(\gamma-\alpha)} \\ &= \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

From Theorem 2.3, we obtain

$$\begin{aligned} \mathcal{Q}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n \\ &= \frac{\alpha Q_2 + \alpha(\alpha-2)Q_1 + Q_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta Q_2 + \beta(\beta-2)Q_1 + Q_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\ &\quad + \frac{\gamma Q_2 + \gamma(\gamma-2)Q_1 + Q_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)} \begin{pmatrix} 6\alpha^2+5\alpha+2 & 5\alpha^2-2\alpha+2 & 2\alpha^2+2\alpha+3 \\ 2\alpha^2+2\alpha+3 & 2\alpha^2+\alpha-4 & 3\alpha^2-4\alpha-1 \\ 3\alpha^2-4\alpha-1 & -4\alpha^2+10\alpha+5 & -\alpha^2+5\alpha-3 \end{pmatrix} \\ &\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)} \begin{pmatrix} 6\beta^2+5\beta+2 & 5\beta^2-2\beta+2 & 2\beta^2+2\beta+3 \\ 2\beta^2+2\beta+3 & 2\beta^2+\beta-4 & 3\beta^2-4\beta-1 \\ 3\beta^2-4\beta-1 & -4\beta^2+10\beta+5 & -\beta^2+5\beta-3 \end{pmatrix} \\ &\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)} \begin{pmatrix} 6\gamma^2+5\gamma+2 & 5\gamma^2-2\gamma+2 & 2\gamma^2+2\gamma+3 \\ 2\gamma^2+2\gamma+3 & 2\gamma^2+\gamma-4 & 3\gamma^2-4\gamma-1 \\ 3\gamma^2-4\gamma-1 & -4\gamma^2+10\gamma+5 & -\gamma^2+5\gamma-3 \end{pmatrix}. \end{aligned}$$

By Theorem 2.2, we know that

$$\mathcal{Q}_n = \begin{pmatrix} Q_{n+1} & Q_n + Q_{n-1} & Q_n \\ Q_n & Q_{n-1} + Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_{n-2} + Q_{n-3} & Q_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above last two equations, then we obtain

$$Q_n = \frac{\alpha^{n-1}(2\alpha^2+2\alpha+3)}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}(2\beta^2+2\beta+3)}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}(2\gamma^2+2\gamma+3)}{(\gamma-\beta)(\gamma-\alpha)}.$$

Using the relations, $\alpha+\beta+\gamma=2$, $\alpha\beta\gamma=1$ and considering α,β and γ are the roots the equation $x^3-2x^2-x-1=0$, we obtain

$$\begin{aligned}
\frac{2\alpha^2+2\alpha+3}{(\alpha-\gamma)(\alpha-\beta)} &= \frac{2\alpha^2+2\alpha+3}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma}=\frac{\alpha}{\alpha}\frac{(2\alpha^2+2\alpha+3)}{\alpha^2+\alpha(-\beta-\gamma)+\beta\gamma} \\
&= \frac{(2\alpha^2+2\alpha+3)\alpha}{\alpha^3+\alpha^2(\alpha-2)+1}=\frac{(2\alpha^2+2\alpha+3)}{2\alpha^3-2\alpha^2+1}\alpha \\
&= \frac{(2\alpha^2+2\alpha+3)}{2(2\alpha^2+\alpha+1)-2\alpha^2+1}\alpha=\alpha \\
\frac{2\beta^2+2\beta+3}{(\beta-\gamma)(\beta-\alpha)} &= \beta, \\
\frac{2\gamma^2+2\gamma+3}{(\gamma-\beta)(\gamma-\alpha)} &= \gamma.
\end{aligned}$$

Finally, we conclude that

$$Q_n=\alpha^n+\beta^n+\gamma^n$$

as required.

Now, we present summation formulas for third-order Pell and third-order Pell-Lucas matrix sequences.

THEOREM 2.5. *Let m and j be integers. Then we have*

$$\sum_{i=0}^{n-1}\mathcal{P}_{mi+j}=\frac{\mathcal{P}_{mn+m+j}+\mathcal{P}_{mn-m+j}+(1-Q_m)\mathcal{P}_{mn+j}}{Q_m-Q_{-m}}-\frac{\mathcal{P}_{m+j}+\mathcal{P}_{j-m}+(1-Q_m)\mathcal{P}_j}{Q_m-Q_{-m}} \tag{2.7}$$

and

$$\sum_{i=0}^{n-1}\mathcal{Q}_{mi+j}=\frac{\mathcal{Q}_{mn+m+j}+\mathcal{Q}_{mn-m+j}+(1-Q_m)\mathcal{Q}_{mn+j}}{Q_m-Q_{-m}}-\frac{\mathcal{Q}_{m+j}+\mathcal{Q}_{j-m}+(1-Q_m)\mathcal{Q}_j}{Q_m-Q_{-m}}. \tag{2.8}$$

Proof. Note that

$$\begin{aligned}
\sum_{i=0}^{n-1}\mathcal{P}_{mi+j} &= \sum_{i=0}^{n-1}(A_1\alpha^{mi+j}+B_1\beta^{mi+j}+C_1\gamma^{mi+j}) \\
&= A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right)+B_1\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right)+C_1\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{n-1}\mathcal{Q}_{mi+j} &= \sum_{i=0}^{n-1}(A_2\alpha^{mi+j}+B_2\beta^{mi+j}+C_2\gamma^{mi+j}) \\
&= A_2\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right)+B_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right)+C_2\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right).
\end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (2.7) and (2.8) as required.

As in Corollary 2.4, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

Corollary 2.6. *Let m and j be integers. Then we have*

$$\sum_{i=0}^{n-1} P_{mi+j} = \frac{P_{mn+m+j} + P_{mn-m+j} + (1-Q_m)P_{mn+j}}{Q_m - Q_{-m}} - \frac{P_{m+j} + P_{j-m} + (1-Q_m)P_j}{Q_m - Q_{-m}} \tag{2.9}$$

and

$$\sum_{i=0}^{n-1} Q_{mi+j} = \frac{Q_{mn+m+j} + Q_{mn-m+j} + (1-Q_m)Q_{mn+j}}{Q_m - Q_{-m}} - \frac{Q_{m+j} + Q_{j-m} + (1-Q_m)Q_j}{Q_m - Q_{-m}} \tag{2.10}$$

Note that using the above Corollary we obtain the following well known formulas (taking $m = 1, j = 0$):

$$\sum_{i=0}^{n-1} P_i = \frac{P_{n+1} - P_n + P_{n-1} - 1}{3} \Rightarrow \sum_{i=0}^{n} P_i = \frac{P_{n+1} + 2P_n + P_{n-1} - 1}{3}$$

and

$$\sum_{i=0}^{n-1} Q_i = \frac{Q_{n+1} - Q_n + Q_{n-1} + 2}{3} \Rightarrow \sum_{i=0}^{n} Q_i = \frac{Q_{n+1} + 2Q_n + Q_{n-1} + 2}{3}.$$

We now give generating functions of $\mathcal{P}$ and Q.

Theorem 2.7. *The generating function for the third-order Pell and third-order Pell-Lucas matrix sequences are given as*

$$\sum_{n=0}^{\infty} \mathcal{P}_n x^n = \frac{1}{1-2x-x^2-x^3} \begin{pmatrix} 1 & x^2+x & x \\ x & 1-2x & x^2 \\ x^2 & x-2x^2 & -x^2-2x+1 \end{pmatrix}$$

and

$$\sum_{n=0}^{\infty} \mathcal{Q}_n x^n = \frac{1}{1-2x-x^2-x^3} \begin{pmatrix} 3x^2+2x+2 & -4x^2+x+2 & -x^2-4x+3 \\ -x^2-4x+3 & 5x^2+10x-4 & -3x^2+5x-1 \\ -3x^2+5x-1 & 5x^2-14x+5 & 8x^2+5x-3 \end{pmatrix}$$

respectively.

Proof. We prove the third-order Pell case. Using the Definition 2.1 and substracting $2x\sum_{n=0}^{\infty}\mathcal{P}_n x^n$, $x^2\sum_{n=0}^{\infty}\mathcal{P}_n x^n$ and $x^3\sum_{n=0}^{\infty}\mathcal{P}_n x^n$ from $\sum_{n=0}^{\infty}\mathcal{P}_n x^n$ we obtain

$$\begin{aligned}
(1-2x-x^2-x^3)\sum_{n=0}^{\infty}\mathcal{P}_n x^n &= \sum_{n=0}^{\infty}\mathcal{P}_n x^n - 2x\sum_{n=0}^{\infty}\mathcal{P}_n x^n - x^2\sum_{n=0}^{\infty}\mathcal{P}_n x^n - x^3\sum_{n=0}^{\infty}\mathcal{P}_n x^n \\
&= \sum_{n=0}^{\infty}\mathcal{P}_n x^n - 2\sum_{n=0}^{\infty}\mathcal{P}_n x^{n+1} - \sum_{n=0}^{\infty}\mathcal{P}_n x^{n+2} - \sum_{n=0}^{\infty}\mathcal{P}_n x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{P}_n x^n - 2\sum_{n=1}^{\infty}\mathcal{P}_{n-1} x^n - \sum_{n=2}^{\infty}\mathcal{P}_{n-2} x^n - \sum_{n=3}^{\infty}\mathcal{P}_{n-3} x^n \\
&= (\mathcal{P}_0 + \mathcal{P}_1 x + \mathcal{P}_2 x^2) - 2(\mathcal{P}_0 x + \mathcal{P}_1 x^2) - \mathcal{P}_0 x^2 \\
&\quad + \sum_{n=3}^{\infty}(\mathcal{P}_n - 2\mathcal{P}_{n-1} - \mathcal{P}_{n-2} - \mathcal{P}_{n-3})x^n \\
&= \mathcal{P}_0 + \mathcal{P}_1 x + \mathcal{P}_2 x^2 - 2\mathcal{P}_0 x - 2\mathcal{P}_1 x^2 - \mathcal{P}_0 x^2 \\
&= \mathcal{P}_0 + (\mathcal{P}_1 - 2\mathcal{P}_0)x + (\mathcal{P}_2 - 2\mathcal{P}_1 - \mathcal{P}_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} \mathcal{P}_n x^n = \frac{\mathcal{P}_0 + (\mathcal{P}_1 - 2\mathcal{P}_0)x + (\mathcal{P}_2 - 2\mathcal{P}_1 - \mathcal{P}_0)x^2}{1 - 2x - x^2 - x^3}.$$

This completes the proof. Third-order Pell-Lucas case can be proved similarly.

The well known generating functions for third-order Pell and third-order Pell-Lucas numbers are as in (1.5). However, we will obtain these functions in terms of third-order Pell and third-order Pell-Lucas matrix sequences as a consequence of Theorem 2.7. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.7. Thus we have the following corollary.

COROLLARY 2.8. *The generating functions for the third-order Pell sequence* $\{P_n\}_{n\geq 0}$ *and third-order Pell-Lucas sequence* $\{Q_n\}_{n\geq 0}$ *are given as*

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2 - x^3} \text{ and } \sum_{n=0}^{\infty} Q_n x^n = \frac{3 - 4x - x^2}{1 - 2x - x^2 - x^3}.$$

respectively.

3. Relation Between third-order Pell and third-order Pell-Lucas Matrix Sequences

We can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

LEMMA 3.1. *The following equalities are true:*

$$Q_n = 8P_{n+4} - 19P_{n+3} - 3P_{n+2},$$

(3.1) $$Q_n = -3P_{n+3} + 5P_{n+2} + 8P_{n+1},$$

(3.2) $$Q_n = -P_{n+2} + 5P_{n+1} - 3P_n,$$

(3.3) $$Q_n = 3P_{n+1} - 4P_n - P_{n-1},$$

(3.4) $$Q_n = 2P_n + 2P_{n-1} + 3P_{n-2},$$

and

(3.5) $$87P_n = 2Q_{n+4} - 18Q_{n+3} + 37Q_{n+2},$$

(3.6) $$87P_n = -14Q_{n+3} + 39Q_{n+2} + 2Q_{n+1},$$

(3.7) $$87P_n = 11Q_{n+2} - 12Q_{n+1} - 14Q_n,$$

(3.8) $$87P_n = 10Q_{n+1} - 3Q_n + 11Q_{n-1},$$

(3.9) $$87P_n = 17Q_n + 21Q_{n-1} + 10Q_{n-2},$$

Proof: It is given in [8].

The following theorem shows that there always exist interrelation between third-order Pell and third-order Pell-Lucas matrix sequences.

THEOREM 3.2. *For the matrix sequences* $\{\mathcal{P}_n\}_{n\geq 0}$ *and* $\{\mathcal{Q}_n\}_{n\geq 0}$*, we have the following identities.*

$$\mathcal{Q}_n = 8\mathcal{P}_{n+4} - 19\mathcal{P}_{n+3} - 3\mathcal{P}_{n+2},$$

$$\mathcal{Q}_n = -3\mathcal{P}_{n+3} + 5\mathcal{P}_{n+2} + 8\mathcal{P}_{n+1}, \tag{3.10}$$

$$\mathcal{Q}_n = -\mathcal{P}_{n+2} + 5\mathcal{P}_{n+1} - 3\mathcal{P}_n, \tag{3.11}$$

$$\mathcal{Q}_n = 3\mathcal{P}_{n+1} - 4\mathcal{P}_n - \mathcal{P}_{n-1}, \tag{3.12}$$

$$\mathcal{Q}_n = 2\mathcal{P}_n + 2\mathcal{P}_{n-1} + 3\mathcal{P}_{n-2}, \tag{3.13}$$

and

$$87\mathcal{P}_n = 2\mathcal{Q}_{n+4} - 18\mathcal{Q}_{n+3} + 37\mathcal{Q}_{n+2}, \tag{3.14}$$

$$87\mathcal{P}_n = -14\mathcal{Q}_{n+3} + 39\mathcal{Q}_{n+2} + 2\mathcal{Q}_{n+1}, \tag{3.15}$$

$$87\mathcal{P}_n = 11\mathcal{Q}_{n+2} - 12\mathcal{Q}_{n+1} - 14\mathcal{Q}_n, \tag{3.16}$$

$$87\mathcal{P}_n = 10\mathcal{Q}_{n+1} - 3\mathcal{Q}_n + 11\mathcal{Q}_{n-1}, \tag{3.17}$$

$$87\mathcal{P}_n = 17\mathcal{Q}_n + 21\mathcal{Q}_{n-1} + 10\mathcal{Q}_{n-2}, \tag{3.18}$$

Proof. Proofs of the identities follow from Lemma 3.1.

LEMMA 3.3. *For all non-negative integers* m *and* n*, we have the following identities.*

(a): $\mathcal{Q}_0\mathcal{P}_n = \mathcal{P}_n\mathcal{Q}_0 = \mathcal{Q}_n,$
(b): $\mathcal{P}_0\mathcal{Q}_n = \mathcal{Q}_n\mathcal{P}_0 = \mathcal{Q}_n.$

Proof. Identities can be established easily. Note that to show (a) we need to use some of the relations which is given in Lemma 3.1.

To prove the following Theorem we need the next Lemma.

LEMMA 3.4. *Let* $A_1, B_1, C_1; A_2, B_2, C_2$ *as in Theorem 2.3. Then the following relations hold:*

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,. \end{aligned}$$

Proof. Using $\alpha+\beta+\gamma = 2$, $\alpha\beta+\alpha\gamma+\beta\gamma = -1$ and $\alpha\beta\gamma = 1$, required equalities can be established by matrix calculations.

THEOREM 3.5. *For all non-negative integers* m *and* n*, we have the following identities.*

(a): $P_mP_n = P_{m+n} = P_nP_m.$
(b): $P_m\mathcal{Q}_n = \mathcal{Q}_nP_m = \mathcal{Q}_{m+n}.$
(c): $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = 9\mathcal{P}_{m+n+2} - 24\mathcal{P}_{m+n+1} + 10\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + \mathcal{P}_{m+n-2}.$

(d): $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = 4\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + 16\mathcal{P}_{m+n-2} + 12\mathcal{P}_{m+n-3} + 9\mathcal{P}_{m+n-4}$.

(e): $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = \mathcal{P}_{m+n+4} - 10\mathcal{P}_{m+n+3} + 31\mathcal{P}_{m+n+2} - 30\mathcal{P}_{m+n+1} + 9\mathcal{P}_{m+n}$.

Proof.

(a): Using Lemma 3.4 we obtain

$$\begin{aligned}
\mathcal{P}_m\mathcal{P}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n) \\
&= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\alpha^n\beta^m \\
&\quad +A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n + C_1B_1\beta^n\gamma^m \\
&= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} \\
&= \mathcal{P}_{m+n}.
\end{aligned}$$

(b): By Lemma 3.3, we have

$$\mathcal{P}_m\mathcal{Q}_n = \mathcal{P}_m\mathcal{P}_n\mathcal{Q}_0.$$

Now from (a) and again by Lemma 3.3 we obtain $\mathcal{P}_m\mathcal{Q}_n = \mathcal{P}_{m+n}\mathcal{Q}_0 = \mathcal{Q}_{m+n}$.

It can be shown similarly that $\mathcal{Q}_n\mathcal{P}_m = \mathcal{Q}_{m+n}$.

(c): Using (a) and Theorem 3.2 (a) we obtain

$$\begin{aligned}
\mathcal{Q}_m\mathcal{Q}_n &= (3\mathcal{P}_{n+1} - 4\mathcal{P}_n - \mathcal{P}_{n-1})(3\mathcal{P}_{m+1} - 4\mathcal{P}_m - \mathcal{P}_{m-1}) \\
&= 4\mathcal{P}_n\mathcal{P}_{m-1} - 12\mathcal{P}_n\mathcal{P}_{m+1} + 4\mathcal{P}_m\mathcal{P}_{n-1} - 12\mathcal{P}_m\mathcal{P}_{n+1} \\
&\quad +16\mathcal{P}_m\mathcal{P}_n + \mathcal{P}_{m-1}\mathcal{P}_{n-1} - 3\mathcal{P}_{m-1}\mathcal{P}_{n+1} - 3\mathcal{P}_{m+1}\mathcal{P}_{n-1} + 9\mathcal{P}_{m+1}\mathcal{P}_{n+1} \\
&= 4\mathcal{P}_{m+n-1} - 12\mathcal{P}_{m+n+1} + 4\mathcal{P}_{m+n-1} - 12\mathcal{P}_{m+n+1} + 16\mathcal{P}_{m+n} \\
&\quad +\mathcal{P}_{m+n-2} - 3\mathcal{P}_{m+n} - 3\mathcal{P}_{m+n} + 9\mathcal{P}_{m+n+2} \\
&= 9\mathcal{P}_{m+n+2} - 24\mathcal{P}_{m+n+1} + 10\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + \mathcal{P}_{m+n-2}.
\end{aligned}$$

It can be shown similarly that $\mathcal{Q}_n\mathcal{Q}_m = 9\mathcal{P}_{m+n+2} - 24\mathcal{P}_{m+n+1} + 10\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + \mathcal{P}_{m+n-2}$.

The remaining of identities can be proved by considering again (a) and Theorem 3.2.

Comparing matrix entries and using Teorem 2.2 we have next result.

Corollary 3.6. *For third-order Pell and third-order Pell-Lucas numbers, we have the following identities:*

(a): $P_{m+n} = P_mP_{n+1} + P_n(P_{m-1} + P_{m-2}) + P_{m-1}P_{n-1}$.

(b): $Q_{m+n} = P_mQ_{n+1} + Q_n(P_{m-1} + P_{m-2}) + Q_{n-1}P_{m-1}$.

(c): $Q_mQ_{n+1} + Q_n(Q_{m-1} + Q_{m-2}) + Q_{m-1}Q_{n-1} = 9P_{m+n+2} - 24P_{m+n+1} + 10P_{m+n} + 8P_{m+n-1} + P_{m+n-2}$.

(d): $Q_mQ_{n+1}+Q_n(Q_{m-1}+Q_{m-2})+Q_{m-1}Q_{n-1}=4P_{m+n}+8P_{m+n-1}+16P_{m+n-2}+12P_{m+n-3}+9P_{m+n-4}$.

(e): $Q_mQ_{n+1}+Q_n(Q_{m-1}+Q_{m-2})+Q_{m-1}Q_{n-1}=P_{m+n+4}-10P_{m+n+3}+31P_{m+n+2}-30P_{m+n+1}+9P_{m+n}$.

Proof.

(a): From Theorem 3.5 we know that $\mathcal{P}_m\mathcal{P}_n=\mathcal{P}_{m+n}$. Using Theorem 2.2, we can write this result as

$$\begin{pmatrix} P_{m+1} & P_m+P_{m-1} & P_m \\ P_m & P_{m-1}+P_{m-2} & P_{m-1} \\ P_{m-1} & P_{m-2}+P_{m-3} & P_{m-2} \end{pmatrix}\begin{pmatrix} P_{n+1} & P_n+P_{n-1} & P_n \\ P_n & P_{n-1}+P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2}+P_{n-3} & P_{n-2} \end{pmatrix}$$
$$=\begin{pmatrix} P_{m+n+1} & P_{m+n}+P_{m+n-1} & P_{m+n} \\ P_{m+n} & P_{m+n-1}+P_{m+n-2} & P_{m+n-1} \\ P_{m+n-1} & P_{m+n-2}+P_{m+n-3} & P_{m+n-2} \end{pmatrix}.$$

Now, by multiplying the left-side matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identity in (a).

The remaining of identities can be proved by considering again Theorems 3.5 and 2.2.

The next two theorems provide us the convenience to obtain the powers of third-order Pell and third-er Pell-Lucas matrix sequences.

THEOREM 3.7. *For non-negatif integers m,n and r with $n\geq r$, the following identities hold:*

(a): $\mathcal{P}_n^m=\mathcal{P}_{mn}$,

(b): $\mathcal{P}_{n+1}^m=\mathcal{P}_1^m\mathcal{P}_{mn}$,

(c): $\mathcal{P}_{n-r}\mathcal{P}_{n+r}=\mathcal{P}_n^2=\mathcal{P}_2^n$.

Proof.

(a): We can write $\mathcal{P}_n^m$ as

$$\mathcal{P}_n^m=\mathcal{P}_n\mathcal{P}_n...\mathcal{P}_n \ (m \text{ times}).$$

Using Theorem 3.5 (a) iteratively, we obtain the required result:

$$\begin{aligned}
\mathcal{P}_n^m &= \underbrace{\mathcal{P}_n\mathcal{P}_n...\mathcal{P}_n}_{m \text{ times}} \\
&= \mathcal{P}_{2n}\underbrace{\mathcal{P}_n\mathcal{P}_n...\mathcal{P}_n}_{m-1 \text{ times}} \\
&= \mathcal{P}_{3n}\underbrace{\mathcal{P}_n\mathcal{P}_n...\mathcal{P}_n}_{m-2 \text{ times}} \\
&\vdots \\
&= \mathcal{P}_{(m-1)n}\mathcal{P}_n \\
&= \mathcal{P}_{mn}.
\end{aligned}$$

(b): As a similar approach in (a) we have

$$\mathcal{P}_{n+1}^m = \mathcal{P}_{n+1}.\mathcal{P}_{n+1}...\mathcal{P}_{n+1} = \mathcal{P}_{m(n+1)} = \mathcal{P}_m\mathcal{P}_{mn} = \mathcal{P}_1\mathcal{P}_{m-1}\mathcal{P}_{mn}.$$

Using Theorem 3.5 (a), we can write iteratively $\mathcal{P}_m = \mathcal{P}_1\mathcal{P}_{m-1}$, $\mathcal{P}_{m-1} = \mathcal{P}_1\mathcal{P}_{m-2}$, ..., $\mathcal{P}_2 = \mathcal{P}_1\mathcal{P}_1$. Now it follows that

$$\mathcal{P}_{n+1}^m = \underbrace{\mathcal{P}_1\mathcal{P}_1...\mathcal{P}_1}_{m \text{ times}}\mathcal{P}_{mn} = \mathcal{P}_1^m\mathcal{P}_{mn}.$$

(c): Theorem 3.5 (a) gives

$$\mathcal{P}_{n-r}\mathcal{P}_{n+r} = \mathcal{P}_{2n} = \mathcal{P}_n\mathcal{P}_n = \mathcal{P}_n^2$$

and also

$$\mathcal{P}_{n-r}\mathcal{P}_{n+r} = \mathcal{P}_{2n} = \underbrace{\mathcal{P}_2\mathcal{P}_2...\mathcal{P}_2}_{n \text{ times}} = \mathcal{P}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{Q}_n$.

THEOREM 3.8. *For non-negatif integers m, n and r with $n \geq r$, the following identities hold:*

(a): $\mathcal{Q}_{n-r}\mathcal{Q}_{n+r} = \mathcal{Q}_n^2$,
(b): $\mathcal{Q}_n^m = \mathcal{Q}_0^m\mathcal{P}_{mn}$.

Proof.

(a): We use Binet's formula of third-order Pell-Lucas matrix sequence which is given in Theorem 2.3. So

$$\begin{aligned}
& \mathcal{Q}_{n-r}\mathcal{Q}_{n+r} - \mathcal{Q}_n^2 \\
= \; & (A_2\alpha^{n-r} + B_2\beta^{n-r} + C_2\gamma^{n-r})(A_2\alpha^{n+r} + B_2\beta^{n+r} + C_2\gamma^{n+r}) \\
& -(A_2\alpha^n + B_2\beta^n + C_2\gamma^n)^2 \\
= \; & A_2B_2\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + A_2C_2\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 \\
& +B_2C_2\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= \; & 0
\end{aligned}$$

since $A_2B_2 = A_2C_2 = C_2B_2$ (see Lemma 3.4). Now we get the result as required.

(b): By Theorem 3.7, we have

$$\mathcal{Q}_0^m\mathcal{P}_{mn} = \underbrace{\mathcal{Q}_0\mathcal{Q}_0...\mathcal{Q}_0}_{m \text{ times}}\underbrace{\mathcal{P}_n\mathcal{P}_n...\mathcal{P}_n}_{m \text{ times}}.$$

When we apply Lemma 3.3 (a) iteratively, it follows that

$$\begin{aligned}
\mathcal{Q}_0^m\mathcal{P}_{mn} &= (\mathcal{Q}_0\mathcal{P}_n)(\mathcal{Q}_0\mathcal{P}_n)...(\mathcal{Q}_0\mathcal{P}_n) \\
&= \mathcal{Q}_n\mathcal{Q}_n...\mathcal{Q}_n = \mathcal{Q}_n^m.
\end{aligned}$$

This completes the proof.

Competing Interests

The author declares that he has no competing interests.

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