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Optimal route of a moving service vehicle on the plane with stochastic demands

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Abstract. In this paper, I examine the problem of finding the optimal routethat minimizes the sum of the weighted expected distances between the moving servicevehicle and the stochastic demand points. The problem is investigatedusing a quadratic function for the route. Different distance measures and probability distributions of thestochasticdemand points are considered.

Keywords: Facility location; Moving Service Vehicle; Random demand; planar location.

1 Introduction:

Facility location has attracted much research in discrete and continuous optimization over nearly four decades. Investigators have focused on both algorithms and formulations in diverse settings. Facility location analysis refers to the modeling, formulation, and solution of a class ofproblems that can best be described as locating facilities in some given space. The expressions deployment, positioning, and locating are frequently used as synonyms.The location decisions must often be made considering different types of performance measures. Choices for the best location(s) differ for various types of objectives. For example for a company that wants to build a warehouse for its retailers, it may be important to find a location that minimizes the sum of the distances from the warehouse to the retailers. However, for the location of an emergency facility such as a fire station, the most suitable objective could be to minimize the maximum distance from the facility to the demand points in order for the fire station to respond quickly enough to the farthest point. Another example might be the location of a waste incinerator for a local municipality. Residents might want that the facility be located as far as possible from residential areas, while the municipality wants it to be close enough to transport the waste. In that case, an objective that maximizes the minimum distance of the facility from the residential areas would be more appropriate. Surveys on facility location can be found in literature for the interested reader ([1], [5], [6], and [14]). There are four components that characterize location problems; these are (1) customers, who are presumed to bealready located at points or on routes, (2) facilities that will be located, (3) a space in which customers and facilities are located, and (4) a metric that indicates distances or times between customers and facilities. On the macro scale, they involve location of airports, waste disposal sites, manufacturing and distribution facilities. Depending on the application being modeled, the facilities and demand points may be nodes in a network or points in a planar region. Facility analysis involvesas well the problem of determining a path of a moving service vehicle, which during its journey provides service to a set of demand points.

In this paper, we focus on the problem of finding the optimal travel route for a service vehicle, which moves through a planar region and interacts with a number of potential demand points and it is not known which particular ones will request service. In several instances the assumption of known fixed demands points does not hold. Consider for example, the case of determining a route for a patrol car maintaining radio contacts with potential stations and it is not known which particular ones will request service or the case of determining a trajectory for a surveillance aircraft or submarine moving in enemy territory and threatened by several unknown missile sites. Our objective is the minimization of the weighted expected distances between the facility and the existing facilities over all instants of time during the travel period.

There is a scarce research work on moving vehicle location problemwith deterministic demands points, but to the best of our knowledge, research on moving vehicle location with random demand points remains largely unexplored. The problem assessed here is relatively new and has been treated barely in the literature. In a previous paper [7], I investigated the problem when the route is a straight line. I showed that the optimal path is reduced to a line parallel to the axis. Here is some of the current literature on movingvehicle location problem with deterministic demands points. Sherali and Kim [10] have introduced a new class of problems involving the determination of an optimal constrained path for a moving service vehicle that interacts with a set of fixed existing facilities. Using weighted-distance related cost function, they have analyzed both the total cost and the average cost problems. Later, Kim and Choi [12] extended the model of Sherali and Kim [10] to a larger class that includes a general cost structure. Then they showed that an optimal path can be easily obtained to the model with a specific form of nontrivial cost function. In [13], Kim and Choi have formulated the problem of finding an optimal path of a moving vehicle on a sphere as a variational problem under the assumption that the demands were generated from an independent Poisson process. The perturbation technique, coupled with dynamic programming procedure, was suggested to solve the variational problem.

In ([3], [4], [8], [9], and [11]), a class of moving vehicle location problems, called "Transit path problems", has been investigated. Such problems arise when an object needs to traverse between two points through a specific region. The path must optimize a prescribed criterion such as risk, reliability, or cost and satisfy a number of constraints such as total travel time.

The remaining of the paper is organized as follows. In Section 2, the problem is analyzed and main results are described. Illustrative examples are provided.

2 Analysis:

Suppose there is a set of m demand destinations$\{Y_i = (U_i, V_i) : i = 1,2, \dots, m\}$ randomly distributed over some rectangle $[0, s] \times [0, t]$in R^2 $(s, t > 0)$. Assume that U_i (resp.V_i) has probability density function $f_{U_i}(u)$(resp.$f_{V_i}(v)$) and cumulative distribution function $F_{U_i}(u)$ (resp.$F_{V_i}(v)$). Let $w_i > 0$ be the demand for service at destination $i= 1,2, \dots, m$ .Suppose that there is a vehicle moving in this plane along the route $z(t) = (x(t), y(t))$at a constant velocity v, starting from some origin S and arriving at a destination located at D. The problem we address here is to determine a route for the moving vehicle that minimizes the expected sum of the weighted distances between the moving vehicle and the demand destinations over some time framework T. The problem can be stated as

$$\underset{z(t)}{\text{Min}}\, J(z(t)) = \int_0^T \left\{\sum_{i=1}^{m} w_i E[d(z(t), Y_i)]\right\} dt \quad (1)$$

Where $d(. \ , \ .)$ designates either the rectilinear or the squared Euclidean distance between the position of the vehicle $z(t)$ and the destination Y_i and E [.] denotes expected value.

Assume that the path traced by the moving vehicle is represented by $z(x) = \left(x, y(x)\right)$, where $y(x)$ is a quadratic function. Then the feasible set of paths from the origin S to the destination D, denoted by Z, may be described by the set

$$Z = \{z(x) = \left(x, y(x)\right) : \ y(x) = \alpha x^2 + \beta x + \gamma, \ \ x \in I = [0\,,\theta]\}$$

Where θ is a positive real number, α, β, γ are nonnegative real nmbers. $S = (0, y(0))$, $D = (\theta\,, y(\theta))$.

Note that $v=\frac{ds}{dt}$, where ds is the incremental distance travelled in the incremental time dt, we obtain that $ds = \sqrt{(dx)^2 + \ (dy)^2} = \sqrt{1 + \ (y')^2}.\,dx$. The problem may then be stated as follows:

$$\underset{z(x)\in Z}{\text{Min}} \, J\left(z(x)\right) = \int_0^\theta \left\{\sum_{i=1}^{m} w_i E[d(z(x), Y_i)]\right\} \sqrt{1 + y'(x)^2}\, dx \ (2)$$

In the coming sub-sections, we study Problem (2) for different distance measures $d(z(x), Y_i)$ and different probability distributions for the random demand points Y_i.

2.1 Case when $d(z(x), Y_i)$ is the rectilinear distance:

The rectilinear distance between the demand point $Y_i = (U_i\,, V_i)$ and moving vehicle $z(x) = \left(x, y(x)\right)$ is given by:

$$d(Y_i\,, (x, y(x))\,) = |x - U_i| + |y(x) - V_i|$$

and its expected value by:

$$E\left[d\left(Y_i\,, (x, y(x)\right)\right] = E[|x - U_i|] + \ E[|y(x) - V_i|]$$

Now, let $f_i(x) = \ E[|x - U_i|]$ and $g_i(y(x)) = \ E[|y(x) - V_i|]$. Then, we have:

$$\text{and } f_i(x) = E(U_i) - \ x\left[1 - \ 2F_{U_i}(x)\right] - \ 2\int_{-\infty}^{x} u f_{U_i}(u) du$$

$$g_i(y(x)) = E(V_i) - \ y(x)\left[1 - \ 2F_{V_i}(y(x))\right] - \ 2\int_{-\infty}^{y(x)} v f_{V_i}(v) dv$$

Then problem (2) can be stated as:

$$\underset{(x,y(x))\in Z}{\text{Min}} \, J\left(x, y(x)\right) = \int_0^\theta \left\{\sum_{i=1}^{m} w_i[f_i(x) + \quad g_i(y(x))]\sqrt{1 + (y'(x))^2}\right\} dx \quad (3)$$

By substituting $y(x) = \alpha x^2 + \beta x + \gamma$ into the above problem and denoting the real function $\sum_{i=1}^{m} w_i[f_i(x) + g_i(\alpha x^2 + \beta x + \gamma)]$ in the variables $\alpha, \beta,$ and γ by $G_x(\alpha, \beta, \gamma)$. Problem (3) then becomes :

$$\min_{(\alpha,\beta,\gamma)} J(\alpha, \beta, \gamma) = \int_0^{\theta} G_x(\alpha, \beta, \gamma)\sqrt{1 + (2\alpha x + \beta)^2}dx \qquad (4)$$

In the sequel, $(\alpha^*, \beta^*, \gamma^*)$ denotes an optimal solution to Problem (4), and

$y^*(x) = \alpha^* x^2 + \beta^* x + \gamma^*$the corresponding optimal quadratic path.

Now let's consider the following sub-problem:

$$\min_{(\alpha,\beta,\gamma)} G_x(\alpha, \beta, \gamma) \qquad (5)$$

Optimal solutions to Problem (5) are characterized by the following theorem:

Theorem 1. For arbitrary bivariate distribution of the random demand point $Y_i = (U_i , V_i)$, any solution(α, β , γ) to the following equation :

$$\sum_{i=1}^{m} w_i(2F_{V_i}(\alpha x^2 + \beta x + \gamma) - 1) = 0$$

is a global minimum to Problem (5).

Proof.

By applying first order optimality conditions to the unconstrained Problem (5), we have that:

$$\nabla G_x(\alpha, \beta, \gamma) = \begin{bmatrix} x^2 \sum_{i=1}^{m} w_i(2F_{V_i}(\alpha x^2 + \beta x + \gamma) - 1) \\ x \sum_{i=1}^{m} w_i(2F_{V_i}(\alpha x^2 + \beta x + \gamma) - 1) \\ \sum_{i=1}^{m} w_i(2F_{V_i}(\alpha x^2 + \beta x + \gamma) - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$<===> \sum_{i=1}^{m} w_i(2F_{V_i}(\alpha x^2 + \beta x + \gamma) - 1) = 0 \qquad (6)$$

satisfying Equation (6) is a critical point to Problem (5). To Then clearly any value of $\alpha, \beta,$ and γ determine the kind of these critical points, we use second order optimality conditions.Since the Hessian matrix of $G_x(\alpha, \beta, \gamma)$, given by

$$H(\alpha,\beta,\gamma) = 2\sum_{i=1}^{m} w_i f_{V_i}(\alpha x^2 + \beta x + \gamma)\begin{bmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{bmatrix},$$

is positive semi-definite, it follows that the function $G_x(\alpha,\beta,\gamma)$ is convex and therefore each critical point is a global minimum.

Corollary 1. For arbitrary bivariate distribution of the random demand point $Y_i = (U_i\,, V_i)$. In particular, if $(0\,,\beta^*\,,\gamma^*)$ solves equation (6), then$(0\,,\beta^*,\gamma^*)$ is a global minimum to Problem (4).

Proof. Let $(0\,,\beta^*\,,\gamma^*)$ solves equation (6), and thenby definition we have:

$$(0,\beta^*,\gamma^*), \forall\beta \in R^+, \forall\gamma \in R^+.\, G_x(\alpha,\beta,\gamma) \geq G_x$$

$$\text{Since } \alpha \text{ and } x \text{ are nonnegative (by previous assumptions), we have that}$$

$$\sqrt{1+(2\alpha x+\beta)^2} \geq \sqrt{1+\beta^2}, \forall\beta R^+.$$

$\text{Combining the previous two inequalities, we get :}$

$$\int_0^\theta G_x(\alpha,\beta,\gamma)\sqrt{1+(2\alpha x+\beta)^2}dx \geq \int_0^\theta G_x(0,\beta^*,\gamma^*)\sqrt{1+\beta^2}dx$$

which implies that $J(\alpha,\beta,\gamma) \geq J(0,\beta^*,\gamma^*), \quad \forall\, \beta,\gamma \in R.$

This shows that $(\,0,\beta^*,\gamma^*)$ is a global minimum to Problem (4).

Case 1: U_i (resp.V_i) is a random variable that follows the uniform distribution over $[a_i\,, b_i]$ (resp. $[c_i\,, d_i]$)

Consider the case whereU_i(resp.V_i) is a random variable that follows the uniform distribution over $[a_i\,, b_i]$ (resp. $[c_i\,, d_i]$). From previous theorem, the function $J(0,\beta,\gamma)$ has a global minimum $(0,\beta^*,\gamma^*)$. $J(0,\beta,\gamma)$can be expressed as

$$J(0,\beta,\gamma) = \sqrt{1+\beta^2}\left\{\int_0^\theta G_x(0,\beta,\gamma)dx\right\} = (a\beta^2 + b\gamma^2 + c\beta\gamma - c_1\beta - c_2\gamma + c_3)\sqrt{1+\beta^2}$$

Where $a = \frac{\theta^3}{3}\sum_{i=1}^{m}\frac{w_i}{(d_i-c_i)}$, $b = \theta\sum_{i=1}^{m}\frac{w_i}{(d_i-c_i)}$, $c = \theta^2\sum_{i=1}^{m}\frac{w_i}{(d_i-c_i)}$, $c_1 = \theta^2\sum_{i=1}^{m}\frac{w_i(c_i+d_i)}{2(d_i-c_i)}$, $c_2 = 2\theta\sum_{i=1}^{m}\frac{w_i(c_i+d_i)}{d_i-c_i}$, and$c_3 = \sum_{i=1}^{m}\frac{w_i(c_i+d_i)}{2(d_i-c_i)}$.

Let us illustrate this case with the following example.

Example 1. Consider the routing of a military vehicle through explosives detection field. Assume that each explosive is a point $Y_i = (U_i\,, V_i)$ that could blast according to a bivariate uniform distribution over the rectangular region $[a_i\,, b_i]$x$[c_i\,, d_i]$, i = 1,2, 3,4. Let $\theta = 8$.Table 1 gives the data for this example.

Table 1

i	w_i	a_i	b_i	c_i	d_i
1	1	13	3	5	
2	2	26	1	4	
3	2	4	7	0	2
4	1	4	8	3	6

Using Mathematica software, we found that β^* =0.59, $\gamma^* = 1.13$ and hence

s the optimal route from$S = (0,\ 1.13\ )\ to\ D = (\,8,\ \ 5.77).y^*(x) = 0.59x + 1.13$ i

Case 2: U_i (resp.V_i) is a random variable that follows follows an exponential distribution with parameter ρ_i (resp. τ_i)

Consider the case whereU_i (resp.V_i) is a random variable that follows follows an exponential distribution with parameter ρ_i (resp. τ_i). From previous theorem, the function $J(0, \beta, \gamma)$has a global minimum$(0, \beta^*, \gamma^*)$.$J(0, \beta, \gamma)$ can be expressed as

$$J(0, \beta, \gamma) = \sqrt{1 + \beta^2}\left\{C + \sum_{i=1}^{m} \int_0^{\theta} (\beta x + \gamma + 2\tau_i e^{\frac{\beta x + \gamma}{\tau_i}} - \tau_i) dx\right\} =$$

$$= \sqrt{1 + \beta^2}\left\{C + \sum_{i=1}^{m} \frac{2w_i\tau_i^2}{\beta} e^{-\frac{\beta\theta + \gamma}{\tau_i}}(-1 + e^{\frac{\beta\theta}{\tau_i}}) + (-\tau_i + \gamma)\theta + \frac{\beta\theta^2}{2} dx\right\}$$

Let us illustrate this case with the following example.

Example 2. Suppose that we have three demand points $Y_i = (U_i\,, V_i)$, *i=1,2,3* distributed according to a bivariate exponential distribution.$\theta = 10$.

Table 2 gives the data for this example.

Table 2

i	$w_i\tau_i$	
1	2	1
2	1	5
3	2	2

Using Mathematica software, we found that $\beta^* = 0.68$, $\gamma^* = 1.83$ and hence theand hence the optimal route from $S = (0,\ 1.83\ )\ to\ D = (\ 8,\ \ 7.27)$ is

$$y^*(x) = 0.47x + 3.83\ .$$

2.2 Case when $d(z(x), Y_i)$ is the squared Euclidean distance:

The squared Euclidean distance between the demand point $Y_i = (U_i\ , V_i)$ and moving vehicle$z(x) = \left(\ x, y(x)\right)$ is given by:

$$d(Y_i\ , (x, y(x)\ ) = (x - U_i)^2 + (y(x) - V_i)^2$$

and its expected value by:

$$E\left[d\left(Y_i\ , (x, y(x)\right)\right] = E[(x - U_i)^2] +\ E[(y(x) - V_i)^2]$$

Now, let $f_i(x) =\ E[(x - U_i)^2]$ and $g_i(y(x)) = E[(y(x) - V_i)^2]$. Then, $f_i(x)$ and $g_i(y(x))$ can be expressed as:

$$f_i(x) = (x - E[U_i])^2 +\ Var[U_i]$$

$$\text{and}\ \ g_i(y(x)) = (y(x) - E[V_i])^2 +\ Var[V_i]$$

where $Var[U_i]$(resp. $Var[V_i]$) denotes the variance of the random variable U_i(resp.V_i).

The following theorem is a restatement of Theorem1 in case$d(z(x), Y_i)$ is the squared Euclidean distance.

Theorem 2. For arbitrary bivariate distribution of the random demandpoint $Y_i = (U_i\ , V_i)$, any solution $(\ \alpha, \beta, \gamma)$ to the following equation :

$$\sum_{i=1}^{m} w_i(\alpha x^2 + \beta x + \gamma - E[V_i]) \quad = 0$$

isa global minimum to Problem (5).

Proof. Followingthe proof steps of Theorem1 and equalizing the gradient vector $G_x(\alpha, \beta, \gamma)$ to zero, we obtain :

<==>

$$\nabla G_x \alpha, \beta, \gamma) = \begin{bmatrix} 2x^2 \sum_{i=1}^{m} w_i(\alpha x^2 + \beta x + \gamma - E[V_i]) \\ 2x \sum_{i=1}^{m} w_i(\alpha x^2 + \beta x + \gamma - E[V_i]) \\ 2 \sum_{i=1}^{m} w_i(\alpha x^2 + \beta x + \gamma - E[V_i]) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<==>

$$\sum_{i=1}^{m} w_i(\alpha x^2 + \beta x + \gamma - E[V_i]) = 0 \qquad (7)$$

Clearly any value of α , β, and γ that satisfies (7) is a critical point.

To determine the kind of these critical points, we use second order optimality conditions for unconstrained problems. Since the Hessian matrix of $G_x(\alpha, \beta)$, given by

$$H(\alpha, \beta, \gamma) = 2 \sum_{i=1}^{m} w_i \begin{bmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{bmatrix},$$

is positive semi-definite, it follows that the function $G_x(\alpha, \beta, \gamma)$ is convex and therefore each critical point is a global minimum.

We now prove that the function $J(0, \beta, \gamma)$has a general and unique expression for all types of bivariate distribution of the random demand point $Y_i = (U_i, V_i)$. Let's denote by $\mu_i, \bar{\mu}_i, \sigma_i, \bar{\sigma}_i$the expected values and variances $E[V_i]$, $E[V_i]$, $Var[U_i]$, $Var[V_i]$, respectively.

After some calculations, $J(0, \beta, \gamma)$ can be expressed as:

$$,J(0, \beta, \gamma) = \sqrt{1 + \beta^2}[a\beta^2 + b\gamma^2 + c\beta\gamma + d\beta + e\gamma + f]$$

where $= \frac{\theta^3}{3}$, $b = -\theta$, $c = -\theta^2$, $d = \theta^2 \sum_{i=1}^{m} \bar{\mu}_i$, $e = 2\theta \sum_{i=1}^{m} \bar{\mu}_i$, and

$$f = \sum_{i=1}^{m} \left(\frac{\theta^3}{3} - \theta\sigma_i^2 - \theta^2\sigma_i - \theta\bar{\mu}_i \right)$$

Example 3. Suppose that we have three demand points $Y_i = (U_i, V_i)$, *i=1,2,3*distributed according to a bivariate beta distribution (See [2]) with parameters $(\alpha_i, \beta_i)$and $(\bar{\alpha}_i, \bar{\beta}_i)$, component wise.$\theta = 10$.

Table 3 gives the data for this example.

Table 3

i	α_i	β_i	$\bar{\alpha}_i$	$\bar{\beta}_i$	μ_i	σ_i	$\bar{\mu}_i$	$\bar{\sigma}_i$
1	0.6	0.5	0.3	0.5	0.54	0.12	0.375	0.13
2	0.4	0.8	0.8	0.2	0.33	0.1	0.8	0.08
3	0.5	0.5	0.6	0.3	0.5	0.125	0.67	0.11

Using Mathematica software, we found that $\beta^* = 0.58$, $\gamma^* = 0.43$ and hence the optimal route from $S = (0,\ 0.73\ )\ to\ D = (\ 1,\ \ 1.01)$ is given by :

$$y^*(x) = 0.47x + 3.83$$

3 The conclusion:

In this paper, I studied a moving service vehicle location problem with random demand destinations. In case of rectilineardistance between the moving servicevehicle and the random demand points, the problem of searching for a quadratic trajectory is reduced naturally to searching for an affine straight-line route. I have shown, that in this former case, an optimal straight-line route exists. In case of squared Euclidean distance,I gave ageneral and unique expression for the objective function for arbitrary bivariate distribution of the random demand points and show the existence of an optimal straight-line route that depends only on the expected values and variances of the random components of the demand points.

References :

[1] A.B. Arabani, R.Z. Farahani, R., " Facility location dynamics: An overview of classifications and applications, " Computers & Industrial Engineering, 62(1)(2012) 408–420.

[2] R.G. Bartle, "The Elements of Real Analysis," John Wiley & Sons (1976), New York.

[3] L. Caccetta, I. Loosen,and V. Rehbock, "Modeling Transit Paths for Military Vehicles," Proceedings of the 16th Congress of the Modelling and Simulation Society of Australia and New Zealand, MODSIM05, Melbourne, Australia, 2005.

[4] L. Caccetta, I. Loosen, and V. Rehbock, "Optimal transit path problem for submarines," In proceedings of the 4th International Conference on Engineering Applications and Computational Algorithms, DCDIS, Guelph, Canada, 2005 .

[5] W. Domschke, Drexl W.A., " Location and Layout Planning: An International Bibliography," Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, 1985, Berlin.

[6] Z. Drezner, "Facility Location: A Survey of Applications and Methods."Springer Series in Operations Research and Financial Engineering. Editors: Drezner, Zvi (Eds)1995.

[7] A. Foul, "Determining optimal straight line route of a moving facility on the plane with random demand points," Journal of King Saud University, 24(1)(2012) 19-23.

[8] C.L. Hallam,"Hierarchical Path Generation: An Application to Submarine Transit Paths," Honours Dissertation, Murdoch University,Western Australia.(1997).

[9] P. Howlett, P. Pudney, and B. Benjamin, "Determination of optimal driving strategies for the control of a train," (in B.J. Noye, B.R Benjamin and L.H. Colgan (Eds)), Proc. Computational Techniques and applications, CTAC 91(1992)241-248,

[10] H.D. Sherali, Seong-in Kim, 1992. Variational Problems for Determining Optimal Paths of a Moving Facility , Transportation Science 26(4)(1992)330-345.

[11] V.L Rehbock, C. Caccetta, C.L. Hallam and R. O’Dowd. Optimal Submarine Transit Paths Through Sonar Fields, Research Report, Department of Mathematics and Statistics, Curtin University of Technology, 2000.

[12] Seong-In Kim, In-Chan Choi, "A simple variational problem for a moving vehicle," Operations Research Letters 16(1994)231-239.

[13] Seong-In Kim, In-Chan Choi, "An optimal path of a moving vehicle on a sphere," IIE Transactions 29(1997)383-389.

[14] Z. Ulukan, E. Demircioğl, "A Survey of Discrete Facility Location Problems," International Scholarly and Scientific Research & Innovation 9(7)(2015)2487-2492.