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A note on degenerate type 2 Changhee polynomials and numbers

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Abstract.

In this paper, we consider the degenerate type 2 Changhee numbers and polynomials $c_{n,\lambda}(x)$ and derive some new identities involving degenerate type 2 Changhee polynomials, the type 2 Changhee polynomials, the type 2 Euler polynomials, generalized Bell numbers, the Changhee-central numbers of the second kind and Euler polynomials by using the generating function method and the Riordan matrix method.

Keywords: Degenerate type 2 Changhee polynomials; Generating functions; Stirling number of two kind; Riordan matrix method.

1. Introduction

In recent years, many researchers studied various kinds of Changhee numbers and polynomials and degenerate Changhee polynomials. The type 2 Changhee numbers and polynomials are introduced by T.Kim and D.S.Kim in 2019. And many interesting identities of these polynomials have been found by them. In this paper, we define the degenerate type 2 Changhee polynomials and numbers by using type 2 Changhee polynomials and numbers and derive some new identities and properties.

Let p be a fixed prime number with $p \equiv 1(mod2)$. Throughout this paper, Z_p, Q_p and C_p will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of Q_p, respectively. Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on Z_p is defined by Kim as(see[1])

$$\int_{Z_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1}$$

So, we get

$$\int_{Z_p} f(x+1)d\mu_{-1}(x) + \int_{Z_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{2}$$

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the Changhee polynomials $Ch_n(x)$ are defined by the following generating function

$$\int_{Z_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \ (see[1,2,4,7]). \tag{3}$$

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the type 2 Changhee polynomials $c_n(x)$ are defined by the following generating function

$$\int_{Z_p}(1+t)^{x+1+2y}d\mu_{-1}(y)=\frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x=\sum_{n=0}^{\infty}c_n(x)\frac{t^n}{n!},\ (see[1]). \tag{4}$$

In this paper, generating functions of special combinatorial sequence are applied as follows:

Stirling numbers of the second kind $S(n,k)$ have following generating function

$$\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}=\frac{(e^t-1)^k}{k!},\ (see[1,5,10]). \tag{5}$$

Stirling numbers of the first kind $s(n,k)$ have following generating function

$$\sum_{n=k}^{\infty}s(n,k)\frac{t^n}{n!}=\frac{(\ln(1+t))^k}{k!},\ (see[10]). \tag{6}$$

Euler polynomials $E_n(x)$ have following generating function

$$\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt},\ (see[1,3,7]). \tag{7}$$

The type 2 Euler polynomials $E_n^*(x)$ have following generating function

$$\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!}=\frac{2}{e^t+e^{-t}}e^{xt},\ (see[1]). \tag{8}$$

The Changhee-central numbers of the second kind $C(n,k)$ have following generating function

$$\frac{1}{k!}(e^{\frac{t}{2}}+e^{-\frac{t}{2}})^k=\sum_{n=0}^{\infty}C(n,k)\frac{t^n}{n!},\ (see[1]). \tag{9}$$

Generalized Bell numbers of the first kind $\beta(n,k)$ are defined by the generating function

$$\frac{(e^{e^t-1}-1)^k}{k!}=\sum_{n=1}^{\infty}\beta(n,k)\frac{t^n}{n!},\ (see[6]). \tag{10}$$

In this paper, we let $[t^n]f(t)$ denote the coefficient of $[t^n]$ in the formal power series of $f(t)$, where $f(t)=\sum_{n=0}^{\infty}f_nt^n$.

A Riordan array is a pair $(d(t),f(t))$ of formal power series. Where $d(t)=\sum_{n=0}^{\infty}d_nt^n$, $f(t)=\sum_{n=1}^{\infty}f_nt^n$. It defines an infinite lower triangular array $(d_{n,k})_{n,k\in N}$ according to the rule $d_{n,k}=[t^n]d(t)(f(t))^k$. So we set $(d_{n,k})=(d(t),f(t))$.

Lemma 1.1 Let $D=(d(t),f(t))$ be a Riordan array and let $h(t)=\sum_{n=0}^{\infty}h_nt^n$ be the generating function of the sequence $\{h_n\}_{n\in N}$, we have (see[8])

$$\sum_{k=0}^{\infty}d_{n,k}h_k=[t^n]d(t)h(f(t))=[t^n]d(t)[(h(y)|y=f(t)]. \tag{11}$$

2. Some properties of degenerate type 2 Changhee polnomials

In this section, we define the degenerate type 2 Changhee polynomials and arrived at some interesting identities involving degenerate type 2 Changhee polynomials and some special combination sequence. The connections of them are obtained by generating functions method and Riordan matrix method.

New we define the degenerate type 2 Changhee polynomials by the generating function:

$$\frac{2}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})+(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{x}=\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!}. \tag{12}$$

When $x=0$, $c_{n,\lambda}=c_{n,\lambda}(0)$ called by the degenerate type 2 Changhee numbers. We can also the generating function of high order degenerate type 2 Changhee numbers

$$\left(\frac{2}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})+(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}\right)^{\alpha}=\sum_{n=0}^{\infty}c_{n,\lambda}^{(\alpha)}\frac{t^n}{n!}. \tag{13}$$

Obviously, we know

$$\lim_{\lambda\to 0}\frac{2}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})+(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{x}=\frac{2}{(1+t)+(1+t)^{-1}}(1+t)^{x}. \tag{14}$$

So that, we get

$$\lim_{\lambda\to 0}c_{n,\lambda}(x)=c_n(x). \tag{15}$$

Theorem 2.1 For nonnegative integer n, we have

$$c_{n,\lambda}(x)=\sum_{l=0}^{n}c_l(x)\lambda^{n-l}s(n,l). \tag{16}$$

Proof Let t be insteated of $\ln(1+\lambda t)^{\frac{1}{\lambda}}$ in (4), we obtion

$$\begin{aligned}
&\frac{2}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})+(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{x}\\
&=\sum_{l=0}^{\infty}c_l(x)\lambda^{-l}\frac{(\ln(1+\lambda t))^l}{l!}\\
&=\sum_{l=0}^{\infty}c_l(x)\lambda^{-l}\sum_{n=l}^{\infty}s(n,l)\lambda^n\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}c_l(x)\lambda^{n-l}s(n,l)\frac{t^n}{n!}.
\end{aligned} \tag{17}$$

From equation (12) and (17), we can get this theorem.

Corollary 2.1 When $x=0$ in (16), we can get the equation of type 2 Changhee numbers

$$c_{n,\lambda}=\sum_{l=0}^{n}c_l\lambda^{n-l}s(n,l). \tag{18}$$

In [9], the following inversion relation

$$f_n=\sum_{k=0}^{n}S(n,k)g_k\Longleftrightarrow g_n=\sum_{k=0}^{n}s(n,k)f_k,\ (see[9]) \tag{19}$$

where f_n and g_n are two sequences, denoted as

$$f=\sum_{m=0}^{\infty} f_n \frac{t^m}{m!},\ g=\sum_{n=0}^{\infty} g_n \frac{t^n}{n!}. \tag{20}$$

Therefore, we can get the inverse relationship between Stirling number of the first kind and Stirling number of the second kind

$$x_n=\sum_{k=0}^{n} S(n,k)(x)_k \Longleftrightarrow (x)_n=\sum_{k=0}^{n} s(n,k)x^n,\ (see[5,10]). \tag{21}$$

Equation (16) use the inversion relation formula, we have the following theorem.

Theorem 2.2 For nonnegative integer n, we have

$$c_n(x)=\sum_{l=0}^{n} c_{l,\lambda}(x)\lambda^{n-l}S(n,l). \tag{22}$$

When $x=0$, $c_n=\sum_{l=0}^{n} c_{n,\lambda}\lambda^{n-l}S(n,l)$.

Proof By (19), we know

$$\frac{c_{n,\lambda}(x)}{\lambda^n}=\sum_{l=0}^{n} s(n,l)\lambda^{-l}c_l(x) \Longleftrightarrow \frac{c_n(x)}{\lambda^n}=\sum_{l=0}^{n} S(n,l)\lambda^{-l}c_{l,\lambda}(x).$$

Theorem 2.3 For nonnegative integer n, we have

$$c_{n,\lambda}(x)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}c_{k,\lambda}\lambda^{n-k-l}(x)_l s(n-k,l). \tag{23}$$

Proof From (12), we note that

$$\sum_{n=0}^{\infty} c_{n,\lambda}(x)\frac{t^n}{n!}=(\sum_{n=0}^{\infty} c_{n,\lambda}\frac{t^n}{n!})(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x, \tag{24}$$

while

$$\begin{aligned}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x&=\sum_{l=0}^{\infty}(x)_l\frac{(\frac{1}{\lambda}\ln(1+\lambda t))^l}{l!}\\ &=\sum_{l=0}^{\infty}(x)_l\lambda^{-l}\sum_{n=l}^{\infty} s(n,l)\lambda^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}(x)_l\lambda^{n-l}s(n,l)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Therefore, by (24) and (25), we get

$$\begin{aligned}\sum_{n\geq 0} c_{n,\lambda}(x)\frac{t^n}{n!}&=(\sum_{n=0}^{\infty} c_{n,\lambda}\frac{t^n}{n!})(\sum_{n\geq 0}\sum_{l=0}^{n}(x)_l\lambda^{n-l}s(n,l)\frac{t^n}{n!})\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}c_{k,\lambda}\sum_{l=0}^{n-k}(x)_l\lambda^{n-k-l}s(n-k,l)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}c_{k,\lambda}\lambda^{n-k-l}(x)_l s(n-k,l)\frac{t^n}{n!}.\end{aligned} \tag{26}$$

By the same way, we get the following property

Theorem 2.4 For nonnegative integer n, we have

$$c_{n,\lambda}(x+y)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}c_{k,\lambda}(x)\lambda^{n-k-l}(y)_l s(n-k,l). \tag{27}$$

Proof

$$\begin{aligned}\sum_{n=0}^{\infty}c_{n,\lambda}(x+y)\frac{t^n}{n!}&=(\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!})(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^y\\&=(\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!})(\sum_{n=0}^{\infty}\sum_{l=0}^{n}(y)_l\lambda^{n-l}s(n,l)\frac{t^n}{n!})\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}c_{k,\lambda}(x)\sum_{l=0}^{n-k}(y)_l\lambda^{n-k-l}s(n-k,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}c_{k,\lambda}(x)\lambda^{n-k-l}(y)_l s(n-k,l)\frac{t^n}{n!}.\end{aligned}$$

Theorem 2.5 For nonnegative integer n, we have

$$c_{n,\lambda}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}E_l^*(x)\lambda^{n-m}s(n,m)s(m,l). \tag{28}$$

Proof By (12), we have

$$\begin{aligned}\sum_{n\geq 0}c_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{2}{e^{\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))}+e^{-\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))}}(e^{\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))})^x\\&=\sum_{l=0}^{\infty}E_l^*(x)\frac{(\ln(1+\frac{1}{\lambda}\ln(1+\lambda t)))^l}{l!}\\&=\sum_{l=0}^{\infty}E_l^*(x)\sum_{m=l}^{\infty}s(m,l)\lambda^{-m}\sum_{n=m}^{\infty}s(n,m)\lambda^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}s(n,m)\lambda^{n-m}\sum_{l=0}^{m}E_l^*(x)s(m,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}E_l^*(x)\lambda^{n-m}s(n,m)s(m,l)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of equation (29), we can easily get this identities.

Corollary 2.2 Let $x=0$ in (28),we can get

$$c_{n,\lambda}=\sum_{m=0}^{n}\sum_{l=0}^{m}E_l^*\lambda^{n-m}s(n,m)s(m,l). \tag{30}$$

Fron (28) we used the inversion formula repeatedly to obtain the following theorem.

Theorem 2.6 For nonnegative integer n, we have

$$E_n^*(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}c_{l,\lambda}(x)\lambda^{m-l}S(n,m)S(m,l). \tag{31}$$

When $x=0$, $E_n^*=\sum_{m=0}^{n}S(n,m)\sum_{l=0}^{m}c_{l,\lambda}\lambda^{m-l}S(m,l)$.

Proof From theorem 2.5, we know

$$\frac{c_{n,\lambda}(x)}{\lambda^n} = \sum_{m=0}^{n} s(n,m)\lambda^{-m} \sum_{l=0}^{m} E_l^*(x)s(m,l).$$

by (19), we get

$$\sum_{l=0}^{n} s(n,l)E_l^*(x) = \lambda^{n-m} \sum_{m=0}^{n} S(n,m)c_{m,\lambda}(x),$$

$$E_n^*(x) = \sum_{l=0}^{n} S(n,l) \sum_{m=0}^{l} \lambda^{l-m} S(l,m)c_{m,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l} c_{m,\lambda}(x)\lambda^{l-m} S(n,l)S(l,m).$$

Theorem 2.7 For nonnegative integer n, $k \geq 1$, we have

$$\frac{k!}{2^{k-n}}C(n,k) = \sum_{m=0}^{n}\sum_{l=0}^{m} c_{l,\lambda}^{(-k)} \lambda^{m-l} S(n,m)S(m,l). \tag{32}$$

Proof By replacing t with $\frac{\exp(\lambda(e^{\frac{t}{2}}-1)-1)}{\lambda}$ in (13) and $\alpha = -k$

$$\begin{aligned} Right &= \sum_{l=0}^{\infty} c_{l,\lambda}^{(-k)} \frac{(\frac{\exp(\lambda(e^{\frac{t}{2}}-1)-1)}{\lambda})^l}{l!} \\ &= \sum_{l=0}^{\infty} c_{l,\lambda}^{(-k)} \lambda^{-l} \sum_{m=l}^{\infty} S(m,l)\lambda^m \frac{(e^{\frac{t}{2}}-1)^m}{m!} \\ &= \sum_{l=0}^{\infty} c_{l,\lambda}^{(-k)} \lambda^{-l} \sum_{m=l}^{\infty} S(m,l)\lambda^m \sum_{n=m}^{\infty} S(n,m)2^{-n}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{m=0}^{n}\sum_{l=0}^{m} c_{l,\lambda}^{(-k)} \lambda^{m-l} S(m,l)S(n,m)\frac{t^n}{n!}. \end{aligned} \tag{33}$$

$$Left = (\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}})^{-k} = \frac{k!}{2^k}\frac{(e^{\frac{t}{2}} + e^{-\frac{t}{2}})^k}{k!} = \frac{k!}{2^k}\sum_{n=0}^{\infty} C(n,k)\frac{t^n}{n!}. \tag{34}$$

By comparing the coefficients of $\frac{t^n}{n!}$ of (34) and (35), Theorem2.7 is proved.

Theorem 2.8 For nonnegative integer n, we have

$$\sum_{k=0}^{n} \lambda^{-k}\beta(n,k)c_{k,\lambda}(x) = \sum_{l=0}^{n} c_l(x)\lambda^{-l}S(n,l). \tag{35}$$

Proof From (10) and the Riordon matrix method, we have

$$\sum_{k=0}^{n}\lambda^{-k}\beta(n,k)c_{k,\lambda}(x)=n!\sum_{k=0}^{n}\frac{k!\lambda^{-k}\beta(n,k)}{n!}\frac{c_{k,\lambda}(x)}{k!}$$
$$=n![t^n][\frac{2}{(1+\frac{1}{\lambda}\ln(1+\lambda y))+(1+\frac{1}{\lambda}\ln(1+\lambda y)^{-1}}(1+\frac{1}{\lambda}\ln(1+\lambda y)^x|y=\frac{\exp(e^t-1)-1}{\lambda}]$$
$$=n![t^n]\frac{2}{(1+\frac{1}{\lambda}(e^t-1))+(1+\frac{1}{\lambda}(e^t-1))^{-1}}(1+\frac{1}{\lambda}(e^t-1))^x \tag{36}$$
$$=n![t^n]\sum_{l=0}^{\infty}c_l(x)\lambda^{-l}\frac{(e^t-1)^l}{l!}=n![t^n]\sum_{l=0}^{\infty}c_l(x)\lambda^{-l}\sum_{n\geq l}S(n,l)\frac{t^n}{n!}$$
$$=n![t^n]\sum_{n=0}^{\infty}\sum_{l=0}^{n}c_l(x)\lambda^{-l}S(n,l)\frac{t^n}{n!}=\sum_{l=0}^{n}\lambda^{-l}c_l(x)S(n,l).$$

Corollary 2.3 Let $x=0$ in (35), we can get

$$\sum_{k=0}^{n}\lambda^{-k}\beta(n,k)c_{k,\lambda}=\sum_{l=0}^{n}c_l\lambda^{-l}S(n,l). \tag{37}$$

Corollary 2.4 By (19) and (35), we can get

$$\sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{n-k}\beta(l,k)s(n,l)c_{k,\lambda}(x)=c_n(x). \tag{38}$$

When $x=0$, we have $\sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{n-k}\beta(l,k)s(n,l)c_{k,\lambda}=c_n$.

Theorem 2.9 For nonnegative integer n, we have

$$c_{n,\lambda}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}2^lE_l(\frac{x+1}{2})\lambda^{n-m}s(m,l)s(n,m). \tag{39}$$

Proof By equation (12), we can also get

$$\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!}=\frac{2}{(1+\frac{1}{\lambda}\ln(1+\lambda t))^2+1}(1+\frac{1}{\lambda}\ln(1+\lambda t))^{x+1}$$
$$=\frac{2}{e^{\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))^2}+1}e^{\frac{x+1}{2}\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))^2}$$
$$=\sum_{l=0}^{\infty}E_l(\frac{x+1}{2})\frac{(\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))^2)^l}{l!} \tag{40}$$
$$=\sum_{l=0}^{\infty}E_l(\frac{x+1}{2})2^l\sum_{m=l}^{\infty}s(m,l)\lambda^{-m}\sum_{n=m}^{\infty}s(n,m)\lambda^n\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}E_l(\frac{x+1}{2})2^l\lambda^{n-m}s(m,l)s(n,m)\frac{t^n}{n!}.$$

Corollary 2.5 Let $x=0$ in (39), we can get

$$c_{n,\lambda}=\sum_{m=l}^{n}\sum_{l=0}^{m}E_l(\frac{1}{2})2^l\lambda^{n-m}s(m,l)s(n,m). \tag{41}$$

Now we given the degenerate type 2 Euler polynomials defined by

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}^{*}(x)\frac{t^n}{n!}. \tag{42}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n=\prod_{k=0}^{n-1}(x-\lambda k),\ (see[11]), \tag{43}$$

for positive integer n, with the convention $(x|\lambda)_0=1$. So we can get the following formula by expanding the binomial theorem: for a variable x,

$$(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{44}$$

Therefore, we get

$$(1+\lambda t)^{\frac{1}{\lambda}}-1=\sum_{n=1}^{\infty}(1|\lambda)_n\frac{t^n}{n!}. \tag{45}$$

In order to prove the following theorem we have to introduce some definitions of Bell polynomials

$$\frac{1}{k!}(\sum_{m\geq 1}x_m\frac{t^m}{m!})^k=\sum_{n\geq k}B_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!},\ (see[9,12]). \tag{46}$$

Therefor, we get the following identities:

$$\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k=\frac{1}{k!}(\sum_{m=1}^{\infty}(1|\lambda)_m\frac{t^m}{m!})^k=\sum_{n\geq k}B_{n,m}((1|\lambda)_1,(1|\lambda)_2,\cdots,(1|\lambda)_{n-m+1})\frac{t^n}{n!}. \tag{47}$$

Theorem 2.10 For nonnegative integer n, we have

$$E_{n,\lambda}^{*}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}c_{l,\lambda}(x)\lambda^{m-l}S(m,l)B_{n,m}((1|\lambda)_1,(1|\lambda)_2,\cdots,(1|\lambda)_{n-m+1}). \tag{48}$$

Proof By replacing t with $\frac{e^{\lambda[(1+\lambda t)^{\frac{1}{\lambda}}-1]}-1}{\lambda}$ in (12),we get

$$\begin{aligned}
\sum_{n=0}^{\infty}E_{n,\lambda}^{*}(x)\frac{t^n}{n!}&=\sum_{l=0}^{\infty}c_{l,\lambda}(x)\lambda^{-l}\frac{(e^{\lambda[(1+\lambda t)^{\frac{1}{\lambda}}-1]}-1)^l}{l!}\\
&=\sum_{l=0}^{\infty}c_{l,\lambda}(x)\lambda^{-l}\sum_{m=l}^{\infty}S(m,l)\lambda^m\frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^m}{m!}\\
&=\sum_{l=0}^{\infty}c_{l,\lambda}(x)\lambda^{-l}\sum_{m=l}^{\infty}S(m,l)\lambda^m\frac{(\sum_{k\geq 1}(1|\lambda)_k\frac{t^k}{k!})^m}{m!}\\
&=\sum_{l=0}^{\infty}c_{l,\lambda}(x)\lambda^{-l}\sum_{m=l}^{\infty}S(m,l)\lambda^m\sum_{n=m}^{\infty}B_{n,m}((1|\lambda)_1,(1|\lambda)_2,\cdots,(1|\lambda)_{n-m+1})\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}c_{l,\lambda}(x)\lambda^{m-l}S(m,l)B_{n,m}((1|\lambda)_1,(1|\lambda)_2,\cdots,(1|\lambda)_{n-m+1})\frac{t^n}{n!}.
\end{aligned} \tag{49}$$

Theorem 2.11 For nonnegative integer n, we have

$$c_{n,\lambda}(x)=\sum_{n_1=0}^{n}\sum_{m=0}^{n_1}\sum_{l=0}^{m}E^{*}_{l,\lambda}(x)\lambda^{n+m-l-n_1}s(n,n_1)s(n_1,m)S(m,l). \tag{50}$$

Proof By replacing t with $\frac{e^{\lambda\ln[1+\frac{1}{\lambda}\ln(1+\lambda t)]}-1}{\lambda}$ in (42), we get

$$\begin{aligned}\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!}&=\sum_{l=0}^{\infty}E^{*}_{l,\lambda}(x)\frac{(\frac{e^{\lambda\ln[1+\frac{1}{\lambda}\ln(1+\lambda t)]}-1}{\lambda})^l}{l!}\\&=\sum_{l=0}^{\infty}E^{*}_{l,\lambda}(x)\lambda^{-l}\sum_{m=l}^{\infty}S(m,l)\lambda^m\frac{[\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))]^m}{m!}\\&=\sum_{l=0}^{\infty}E^{*}_{l,\lambda}(x)\lambda^{-l}\sum_{m=l}^{\infty}S(m,l)\lambda^m\sum_{n_1=m}^{\infty}s(n_1,m)\lambda^{-n_1}\sum_{n=n_1}^{\infty}s(n,n_1)\lambda^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{n_1=0}^{n}\sum_{m=0}^{n_1}\sum_{l=0}^{m}E^{*}_{l,\lambda}(x)\lambda^{n+m-l-n_1}S(m,l)s(n_1,m)s(n,n_1)\frac{t^n}{n!}.\end{aligned} \tag{51}$$

The conclusion of the theorem can be obtained by comparing the coefficient of $\frac{t^n}{n!}$ between the first term and the last term.

The Degenerate-stirling numbers of the first kind defined by

$$\sum_{n=k}^{\infty}s(n,k|\lambda)\frac{t^n}{n!}=\frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^k}{k!},\ (see[13]). \tag{52}$$

It can also be expressed as follows

$$\sum_{n=k}^{\infty}\frac{k!\lambda^{-k}s(n,k|\lambda)}{n!}\frac{t^n}{n!}=(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\lambda})^k. \tag{53}$$

Therefore, we have the following identity involving the Degenerate-stirling numbers of the first kind and the degenerate type 2 Changhee polynomials

Theorem 2.12 For nonnegative integer n, we have

$$\sum_{k=0}^{n}\lambda^{n-k}s(n,k|\lambda)c_{k,\lambda}(x)=c_{n,\lambda^2}(x). \tag{54}$$

Proof

$$\begin{aligned}&\sum_{k=0}^{n}\lambda^{-k}s(n,k|\lambda)c_{k,\lambda}(x)=n!\sum_{k=0}^{n}\frac{k!\lambda^{-k}s(n,k|\lambda)}{n!}\frac{c_{k,\lambda}(x)}{k!}\\&=n![t^n][\frac{2}{(1+\frac{1}{\lambda}\ln(1+\lambda y))+(1+\frac{1}{\lambda}\ln(1+\lambda y)^{-1}}(1+\frac{1}{\lambda}\ln(1+\lambda y)^x|y=\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\lambda}]\\&=n![t^n]\frac{2}{1+\ln(1+\lambda^2\frac{t}{\lambda})^{\frac{1}{\lambda^2}}+(1+\ln(1+\lambda^2\frac{t}{\lambda})^{\frac{1}{\lambda^2}})^{-1}}(1+\ln(1+\lambda^2\frac{t}{\lambda})^{\frac{1}{\lambda^2}})^x\\&=n![t^n]\sum_{n=0}^{\infty}c_{n,\lambda^2}(x)\frac{(\frac{t}{\lambda})^n}{n!}=n![t^n]\sum_{n=0}^{\infty}c_{n,\lambda^2}(x)\lambda^{-n}\frac{t^n}{n!}=\lambda^{-n}c_{n,\lambda^2}(x).\end{aligned} \tag{55}$$

Logarithmic polynomials are defined as L_n

$$\log(\sum_{n=0}^{\infty} g_n \frac{t^n}{n!}) = \log(1 + g_1 t + g_2 \frac{t^2}{2!} + \cdots) = \sum_{n=1}^{\infty} L_n \frac{t^n}{n!}, (g_0 = 1), \ (see[9]) \tag{56}$$

While

$$L_n = L_n(g_1, g_2, \cdots, g_n) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}(g_1, g_2, \cdots), (L_0 = 0). \tag{57}$$

Theorem 2.13 For $n \geq 1$, we have

$$c_{n,\lambda}(x+1) - c_{n,\lambda}(x) = \sum_{l=0}^{n-1} \sum_{k=1}^{n-l} \binom{n}{l} c_{l,\lambda}(x) (-1)^{k-1} (k-1)! B_{n-l,k}, \tag{58}$$

$$\frac{c_{n,\lambda}(x+1) - c_{n,\lambda}(x)}{n!} = \sum_{l=0}^{n-1} \frac{c_{l,\lambda}(x)}{l!} \frac{(-\lambda)^{n-l-1}}{n-l}, \tag{59}$$

While

$$B_{n-l,k} = B_{n-l,k}((1|\lambda)_1, (1|\lambda)_2, \cdots, (1|\lambda)_{n-m+1}). \tag{60}$$

Proof

$$\begin{aligned} &\sum_{n=0}^{\infty} c_{n,\lambda}(x+1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!} \\ &= \frac{2}{(1 + \ln(1+\lambda t)^{\frac{1}{\lambda}}) + (1 + \ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}} (1 + \ln(1+\lambda t)^{\frac{1}{\lambda}})^x [(1 + \ln(1+\lambda t)^{\frac{1}{\lambda}}) - 1] \\ &= \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!} \ln(1+\lambda t)^{\frac{1}{\lambda}}. \end{aligned} \tag{61}$$

From the above definition of logarithmic polynomials

$$\begin{aligned} \ln(1+\lambda t)^{\frac{1}{\lambda}} &= \ln(\sum_{n=0}^{\infty} (1|\lambda)_n \frac{t^n}{n!}) \\ &= \ln(1 + (1|\lambda)_1 t + (1|\lambda)_2 \frac{t^2}{2!} + \cdots) \\ &= \sum_{n=1}^{\infty} L_n((1|\lambda)_1, (1|\lambda)_2, \cdots) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \{\sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}((1|\lambda)_1, (1|\lambda)_2, \cdots)\}. \end{aligned} \tag{62}$$

Thus, we know

$$\begin{aligned} &\sum_{n=0}^{\infty} c_{n,\lambda}(x+1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!} \sum_{n=1}^{\infty} \{\sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}((1|\lambda)_1, (1|\lambda)_2, \cdots)\} \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \binom{n}{l} c_{l,\lambda}(x) \sum_{k=1}^{n-l} (-1)^{k-1} (k-1)! B_{n-l,k}((1|\lambda)_1, (1|\lambda)_2, \cdots) \frac{t^n}{n!}. \end{aligned} \tag{63}$$

On the other hand

$$\ln(1+\lambda t)^{\frac{1}{\lambda}}=\frac{1}{\lambda}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\lambda t)^n}{n}=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}}{n}t^n. \tag{64}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}c_{n,\lambda}(x+1)\frac{t^n}{n!}-\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}c_{n,\lambda}(x)\frac{t^n}{n!}\ln(1+\lambda t)^{\frac{1}{\lambda}}\\
&=\sum_{n=0}^{\infty}\frac{c_{n,\lambda}(x)}{n!}t^n\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}}{n}t^n\\
&=\sum_{n=1}^{\infty}\{\sum_{l=0}^{n-1}\frac{c_{l,\lambda}(x)}{l!}\frac{(-\lambda)^{n-l-1}}{n-l}\}t^n.
\end{aligned} \tag{65}$$

In (63) and (65), compare the ooefficients of the first term with the last term $\frac{t^n}{n!}$. By comparing the coefficients of $\frac{t^n}{n!}$ on both sides the (63) and (65), Theorem2.13 is proved.

Theorem 2.14 For nonnegative integer n, suppose $r_i\in\mathbb{N}$, $i\in[m]$, $m\in\mathbb{N}^+$, there are properties about Degenerat type 2 Changhe polynomials $c_{n,\lambda}(x)$ as follows

$$\begin{aligned}
&c_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}(x_1+x_2+\cdots+x_m)\\
&=\sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}c_{n_1,\lambda}^{(r_1)}(x_1)c_{n_2,\lambda}^{(r_2)}(x_2)\cdots c_{n_m,\lambda}^{(r_m)}(x_m).
\end{aligned} \tag{66}$$

Proof

$$\begin{aligned}
&\sum_{n=0}^{\infty}c_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}(x_1+x_2+\cdots+x_m)\frac{t^n}{n!}\\
&=\left(\frac{2}{(1+\frac{1}{\lambda}\ln(1+\lambda t))+(1+\frac{1}{\lambda}\ln(1+\lambda t))^{-1}}\right)^{r_1+r_2+\cdots+r_m}(1+\frac{1}{\lambda}\ln(1+\lambda t))^{x_1+x_2+\cdots+x_m}\\
&=\sum_{n=0}^{\infty}c_{n_1,\lambda}^{(r_1)}(x_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}c_{n_2,\lambda}^{(r_2)}(x_2)\frac{t^n}{n!}\cdots\sum_{n=0}^{\infty}c_{n_m,\lambda}^{(r_m)}(x_m)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}c_{n_1,\lambda}^{(r_1)}(x_1)c_{n_2,\lambda}^{(r_2)}(x_2)\cdots c_{n_m,\lambda}^{(r_m)}(x_m)\frac{t^n}{n!}.
\end{aligned} \tag{67}$$

Corollary 2.6 Let $x_1=x_2=\cdots=x_m=0$ in (66), we get the degenerate type 2 Changhee number property

$$c_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}=\sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}c_{n_1,\lambda}^{(r_1)}c_{n_2,\lambda}^{(r_2)}\cdots c_{n_m,\lambda}^{(r_m)}. \tag{68}$$

Let $r_1=\alpha$, $r_2=\beta$, $r_3=\cdots=r_m=0$, $x_1=x$, $x_2=y$, $x_3=\cdots=x_m=0$ in (66), we get

$$c_{n,\lambda}^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}c_{k,\lambda}^{(\alpha)}(x)c_{n-k,\lambda}^{(\beta)}(y). \tag{69}$$

Let $y = 0$ in (69), we get

$$c_{n,\lambda}^{(\alpha+\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} c_{k,\lambda}^{(\alpha)}(x) c_{n-k,\lambda}^{(\beta)}. \tag{70}$$

Theorem 2.15 For nonnegative integer n, we have

$$\frac{\partial}{\partial x} c_{n,\lambda}(x) = \sum_{l=0}^{n-1} \binom{n}{l} c_{l,\lambda}(x) \sum_{k=1}^{n-l} (-1)^{k-1}(k-1)! \lambda^{n-l-k} s(n,k). \tag{71}$$

Proof We assume

$$f(t,x;\lambda) = \frac{2}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}}) + (1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}} (1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{x} = \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!}. \tag{72}$$

So, we get

$$\frac{\partial}{\partial x} f(t,x;\lambda) = f(t,x;\lambda) \ln(1 + \frac{1}{\lambda} \ln(1+\lambda t)). \tag{73}$$

While

$$\begin{aligned} \ln(1 + \frac{1}{\lambda} \ln(1+\lambda t)) &= \sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! \frac{(\frac{1}{\lambda}\ln(1+\lambda t))^k}{k!} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! \lambda^{-k} \sum_{n=k}^{\infty} s(n,k) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \lambda^{n-k} s(n,k) \frac{t^n}{n!}. \end{aligned} \tag{74}$$

$$\begin{aligned} \frac{\partial}{\partial x} f(t,x;\lambda) &= \sum_{n=0}^{\infty} \frac{\partial}{\partial x} c_{n,\lambda}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} c_{n,\lambda}(x) \frac{t^n}{n!} \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \lambda^{n-k} s(n,k) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \binom{n}{l} c_{l,\lambda}(x) \sum_{k=1}^{n-l} (-1)^{k-1}(k-1)! \lambda^{n-l-k} s(n,k) \frac{t^n}{n!}. \end{aligned} \tag{75}$$

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