



## On an elliptic equation with singular cylindrical growth

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### Abstract.

In the present paper, an elliptic equation with singular cylindrical growth is considered. By using the Nehari manifold and mountain pass theorem, the existence of at least four distinct solutions is obtained. The result depends crucially on the parameters  $k$ ,  $\lambda$ ,  $g$  and  $\mu$ .

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### 1. Introduction

In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following problem (1.1)

$$\begin{cases} -\Delta u - \mu \frac{u}{|y|^2} = |u|^{2^*-2} u + \lambda g(y) |u|^{q-2} u, & \text{in } \Omega, y \neq 0 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

With  $\Omega \subset \mathbb{R}^k \times \mathbb{R}^{N-k}$  where each point  $x$  in  $\mathbb{R}^N$  is written as a pair

$(y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$  where  $k$  and  $N$  are integers such that  $N \geq 3$  and  $k$  belongs  $\{2, \dots, N\}$ ,  $2^* = 2N/(N-2)$  to is the Sobolev critical exponent,  $1 < q < 2$ ,  $-\infty < \mu < \bar{\mu}_k := (k-2)^2/4$ ,  $\lambda$ , is a real parameter and  $g$  is continuous function in  $\bar{\Omega}$ . In recent years, many auteurs have paid much attention to the following singular elliptic problem, i.e., the case  $k = N$ ,  $g \equiv 1$  in (1.1), (1.2):

$$\begin{cases} -\Delta u - \mu|x|^{-2}u = |u|^{p-2}u + \lambda u, & \text{in } \Omega \\ u = 0 & \partial\Omega, \end{cases}$$

$$\mathbb{R}^N \ (N \geq 3), \ 0 \in \Omega, \ \lambda > 0, \ 0 \leq \mu < \bar{\mu}_N := (N-2)^2/4$$

Where  $\Omega$  is a smooth bounded domain in

And  $2^* = 2N/(N-2)$  is the critical Sobolev exponent, see [5,6,8] and references therein. The quasilinear form of (1.2) is discussed in [14]. Some results are already available for (1.1). Wang and Zhou [20] proved that there exist at least two solutions for (1.1) with  $0 < \mu \leq \bar{\mu}_N = (N-2)^2/4$ , Boucekif and Matallah [2] showed the existence of two solutions of (1.1) under certain conditions on a weighted function  $h$ , when  $0 < \mu \leq \bar{\mu}_N, \lambda \in (0, \Lambda_*)$  with  $\Lambda_*$  a positive constant.

Concerning existence results in the case  $k < N$ , we cite [9,10, 11, 12, 13,17] and the references therein. Musina [17] considered (1.1) with  $\lambda = 0$ , also (1.1). She

established the existence of a ground state solution when  $2 < k \leq N$  and

$0 < \mu < \bar{\mu}_k = ((k-2)/2)^2$  for (1.1) with  $\lambda = 0$ . She also showed that (1.1) with  $\lambda = 0$  does not admit ground state solutions. Badiale et al. [1] studied (1.1) with  $\lambda = 0$ .

They proved the existence of at least a nonzero nonnegative weak solution  $u$ , satisfying  $u(y,z) = u(|y|,z)$  when  $2 \leq k < N$  and  $\mu < 0$ . Boucekif and El Mokhtar [3] proved that (1.1) admits two distinct solutions when  $2 < k \leq N, b = N - p(N-2)/2$  with  $p \in (2, 2^*]$ ,  $\mu < \bar{\mu}_k$ , and

$\lambda \in (0, \Lambda_*)$  where  $\Lambda_*$  is a positive constant. Terracini [19] proved that there is no positive solutions of (1.1) with  $\lambda = 0$  when  $\mu < 0$ . The regular problem corresponding to has been considered on a regular bounded domain  $\Omega$  by Tarantello [18]. She proved that, with a nonhomogeneous term  $g \in H^{-1}(\Omega)$ , the dual of, not  $H_0^1(\Omega)$  identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notations.

We denote by  $\mathcal{D}_0^{1,2} = \mathcal{D}_0^{1,2}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$  and,  $\mathcal{H}_\mu = \mathcal{H}_\mu(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$  the closure of  $C_0^\infty(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$  with respect to the norms and

$$\|u\|_\mu = \left( \int_\Omega (|\nabla u|^2 - \mu|y|^{-2}|u|^2) dx \right)^{1/2}, \quad \|u\| = \left( \int_\Omega |\nabla u|^2 dx \right)^{1/2}$$

respectively, with  $\mu < \bar{\mu}_k = ((k-2)/2)^2$  for  $k \neq 2$ .

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm  $\|u\|_\mu$  is equivalent to  $\|u\|$ . More explicitly, we have

$$\left( 1 - \left( \sqrt{\bar{\mu}_k} \right)^{-2} \mu^+ \right)^{1/2} \|u\| \leq \|u\|_\mu \leq \left( 1 - \left( \sqrt{\bar{\mu}_k} \right)^{-2} \mu^- \right)^{1/2} \|u\|,$$

with  $\mu^+ = \max(\mu, 0)$  and  $\mu^- = \min(\mu, 0)$  for all  $u \in \mathcal{H}_\mu$ .

We list here a few integral inequalities.

The starting point for studying (1.1), is the Hardy inequality with cylindrical weights [17]. It states that

$$\bar{\mu}_k \int_{\Omega} |y|^{-2} v^2 dx \leq \int_{\Omega} |\nabla v|^2 dx, \text{ for all } v \in \mathcal{H}_{\mu},$$

Since our approach is variational, we define the functional I on  $\mathcal{H}_{\mu}$  by

$$I(u) := (1/2) \|u\|_{\mu}^2 - (1/2^*) \int_{\Omega} |u|^{2^*} dx - (\lambda/q) \int_{\Omega} g|u|^q dx,$$

A point  $u \in \mathcal{H}_{\mu}$  is a weak solution of the equation (1.1) if it satisfies

$$\begin{aligned} \langle I'(u), \varphi \rangle &:= \int_{\Omega} ((\nabla u \nabla \varphi) - \mu |y|^{-2} (u \varphi)) - \int_{\Omega} |u|^{2^*-2} u \varphi dx \\ &\quad - \lambda \int_{\Omega} g |u|^{q-2} u \varphi dx = 0, \text{ for all } \varphi \in \mathcal{H}_{\mu} \end{aligned}$$

denotes the product in the duality  $\mathcal{H}'_{\mu}, \mathcal{H}_{\mu}$  ( $\mathcal{H}'_{\mu}$  dual of  $\mathcal{H}_{\mu}$ ). Let here  $\langle \cdot, \cdot \rangle$

$$S_{\mu} := \inf_{u \in \mathcal{H}_{\mu} \setminus \{0\}} \frac{\|u\|_{\mu}^2}{\left( \int_{\Omega} |u|^p dx \right)^{2/p}}$$

From [15],  $S_{\mu}$  is achieved. Now we consider the following assumption:

G)  $g$  is a continuous function defined in  $\bar{\Omega}$  and there exist  $g_0$  and  $\varrho_0$  positive )  
(such that  $g(x) \geq g_0$  for all  $x \in B(0, 2\varrho_0)$ ).

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our system.

Let  $\lambda_0$  be positive number such that

$$\lambda_0 := (S_{\mu})^{2(2-q)/2^*(2^*-2)} \left( \frac{2^*-2}{2^*-q} \right) \left[ \left( \frac{2-q}{(2^*-1)} \right) \right]^{(2-q)/(2^*-2)} \|g\|_{\mathcal{H}_{\mu}^{-1}}^{-1}.$$

Now we can state our main results.

**Theorem 1:** Assume that,  $-\infty < \mu < \bar{\mu}_k$  and  $\lambda$  verifying  $0 < \lambda < \lambda_0$ , then the system (1.1) has at least one positive solution.

**Theorem 2:** In addition to the assumptions of the Theorem 1, there exists  $\lambda_1 - \frac{q}{2} \lambda_0$  such that if satisfying  $0 < \lambda < \lambda_1$ , then (1.1) has at least two positive solutions.

**Theorem 3:** In addition to the assumptions of the Theorem 2, assuming  $N \geq 6$ , there exists a positive real  $\lambda_2$  such that, if  $\lambda$  satisfy  $0 < \lambda < \min(\lambda_1, \lambda_2)$ , then (1.1) has at least two positive solutions and at least one pair of sign-changing solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem 3.

## 1. Preliminaries

**Definition1:** Let  $c \in \mathbb{R}$ ,  $E$  a Banach space and  $I \in C^1(E, \mathbb{R})$

i)  $(u_n)_n$  is a Palais-Smale sequence at level  $c$  ( in short  $(PS)_c$ ) in  $E$  for  $I$  if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1),$$

where  $o_n(1)$  tends to 0 as  $n$  goes at infinity.

ii) We say that  $I$  satisfies the  $(PS)_c$  condition if any  $(PS)_c$  sequence in  $E$  for  $I$  has a convergent subsequence.

**Lemma1:** Let  $X$  Banach space, and  $J \in C^1(X, \mathbb{R})$  verifying the Palais -Smale condition. Suppose that  $J(0)=0$  and that:

i) there exist  $R > 0, r > 0$  such that if  $\|u\|=R$ , then  $J(u) \geq r$

ii) there exist  $(u_0) \in X$  such that  $\|u_0\| > R$  and  $J(u_0) \leq 0$

let  $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} (J(\gamma(t)))$  where

$$\Gamma = \left\{ \gamma \in C([0, 1]; X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = u_0 \right\},$$

then  $c$  is critical value of  $J$  such that  $c \geq r$ .

### Nehari manifold

It is well known that  $I$  is of class  $C^1$  in  $\mathcal{H}_\mu$  and the solutions of (1.1) are the critical points of  $I$  which is not bounded below on  $\mathcal{H}_\mu$ . Consider the following Nehari manifold

$$\mathcal{N} = \left\{ u \in \mathcal{H}_\mu \setminus \{0\} : \langle I'(u), u \rangle = 0 \right\},$$

Thus,  $u \in \mathcal{N}$  if and only if

$$(1) \quad \|u\|_\mu^2 - \int_\Omega |u|^{2^*} dx - \lambda \int_\Omega g|u|^q dx = 0.$$

Note that  $\mathcal{N}$  contains every nontrivial solution of the problem (1.1). Moreover, we have the following results.

**Lemma 2:**  $I$  is coercive and bounded from below on  $\mathcal{N}$ .

Proof: If  $u \in \mathcal{N}$ , then by (1) and the Hölder inequality, we deduce that

$$\begin{aligned} I(u) &= ((2^* - 2)/2^* 2) \|u\|_\mu^2 - \lambda((2^* - q)/2^* q) \int_\Omega g|u|^q dx \\ &\geq ((2^* - 2)/2^* 2) \|u\|_\mu^2 - \lambda((2^* - q)/2^* q) \|u\|_\mu^q \|g\|_{\mathcal{H}_\mu^{-1}}. \end{aligned} \quad (2)$$

Thus,  $I$  is coercive and bounded from below on  $\mathcal{N}$ .

Define

$$\varphi(u) = \langle I'(u), u \rangle.$$

Then, for  $u \in \mathcal{N}$

$$\begin{aligned}
 \langle \phi'(u), u \rangle &= 2\|u\|_{\mu}^2 - 2^* \int_{\Omega} |u|^{2^*} dx - \lambda q \int_{\Omega} g|u|^q dx \\
 &= (2 - q)\|u\|_{\mu}^2 - (2^* - 1) \int_{\Omega} |u|^{2^*} dx \\
 &= \lambda(2^* - q) \int_{\Omega} g|u|^q dx - (2^* - 2)\|u\|_{\mu}^2.
 \end{aligned}$$

(3)

Now, we split  $N$  in three parts:

$$\begin{aligned}
 \mathcal{N}^+ &= \{u \in \mathcal{N} : \langle \phi'(u), u \rangle > 0\}, \\
 \mathcal{N}^0 &= \{u \in \mathcal{N} : \langle \phi'(u), u \rangle = 0\}, \\
 \mathcal{N}^- &= \{u \in \mathcal{N} : \langle \phi'(u), u \rangle < 0\}.
 \end{aligned}$$

We have the following results.

**Lemma3:** Suppose that  $u_0$  is a local minimizer for  $I$  on  $N$ . Then, if  $u_0 \notin N^0$ ,  $u_0$  is a critical point of  $I$ .

Proof: If  $u_0$  is a local minimizer for  $I$  on  $N$ , then  $u_0$  is a solution of the optimization problem

$$\min_{\{u \mid \phi(u)=0\}} I(u).$$

Hence, there exists a Lagrange multipliers  $\theta \in \mathbb{R}$  such that

$$I'(u_0) = \theta \phi'(u_0) \text{ in } \mathcal{H}'$$

Thus,

$$\langle J'(u_0), u_0 \rangle = \theta \langle \phi'(u_0), u_0 \rangle.$$

But  $\langle \phi'(u_0), u_0 \rangle \neq 0$ , since  $u_0 \notin N^0$ . Hence  $\theta = 0$

This completes the proof.

**Lemma 4:** There exists a positive number  $\lambda_0$  such that for all  $\lambda$ , verifying

$$0 < \lambda < \lambda_0,$$

we have  $N^0 \neq \emptyset$ .

.Proof: Let us reason by contradiction

Suppose  $N^0 = \emptyset$  such that  $\lambda < \lambda_0$ . Then, by (3) and for  $u \in N^0$ , we have

$$\begin{aligned}
 (2 - q)\|u\|_{\mu}^2 - (2^* - 1) \int_{\Omega} |u|^{2^*} dx &= 0 \\
 \lambda(2^* - q) \int_{\Omega} g|u|^q dx - (2^* - 2)\|u\|_{\mu}^2 &= 0
 \end{aligned}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$(4) \|u\|_{\mu} \geq (S_{\mu})^{2/2^{*}(2^{*}-2)} [(2-q)/(2^{*}-1)]^{1/(2^{*}-2)}$$

And

$$(5) \|u\|_{\mu} \leq \left[ \left( \frac{2^{*}-q}{2^{*}-2} \right)^{1/(2-q)} (\lambda^{1/(2-q)}) \right].$$

From (4) and (5), we obtain  $\lambda \geq \lambda_0$ , which contradicts an hypothesis.

Thus  $N=N^{+} \cup N^{-}$ . Define

$$c := \inf_{u \in N^{+}} I(u), \quad c^{+} := \inf_{u \in N^{+}} I(u) \quad \text{and} \quad c^{-} := \inf_{u \in N^{-}} I(u).$$

For the sequel, we need the following Lemma.

**Lemma 5:**

i) For all  $\lambda$  such that  $0 < \lambda < \lambda_0$ , one has  $c \leq c^{+} < 0$

ii) For all  $\lambda$  such that  $0 < \lambda < \frac{q}{2} \lambda_0$ , one has

$$c^{-} > C_0 = C_0(\lambda, S_{\mu}, |h^{+}|_{\infty}, q)$$

Proof: (i) Let  $u \in N^{+}$ . By (3), we have

$$\left( \frac{2-q}{2^{*}-1} \right) \|u\|_{\mu}^2 > \int_{\Omega} |u|^{2^{*}} dx$$

And so

$$\begin{aligned} I(u) &= [(q-2)/2q] \|u\|_{\mu}^2 + [(2^{*}-q)/2^{*}q] \int_{\Omega} |u|^{2^{*}} dx \\ &< -(2-q) \left[ \frac{2^{*}(2^{*}-1) - 2(2^{*}-q)}{2q2^{*}(2^{*}-1)} \right] \|u\|_{\mu}^2 < 0 \end{aligned}$$

We conclude that  $c \leq c^{+} < 0$ .

ii) Let  $u \in N^{-}$ . By (3), we get

$$\left( \frac{2-q}{2^{*}-1} \right) \|u\|_{\mu}^2 < \int_{\Omega} |u|^{2^{*}} dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\Omega} |u|^{2^{*}} dx \leq (S_{\mu})^{-2^{*}/2} \|u\|_{\mu}^{2^{*}}.$$

This implies

$$(6) \|u\|_{\mu} > (S_{\mu})^{\frac{2^{*}}{2(2^{*}-2)}} \left[ \frac{(2-q)}{(2^{*}-1)} \right]^{\frac{1}{(2^{*}-2)}}, \text{ for all } u \in N^{-}.$$

By (2), we get

$$\begin{aligned} I(u) &\geq \|u\|_{\mu}^q \left( \frac{(2^{*}-2)}{2^{*}2} \right) \left[ \frac{(2-q)}{(2^{*}-1)} \right]^{\frac{(2-q)}{(2^{*}-2)}} (S_{\mu})^{\frac{2^{*}(2-q)}{2(2^{*}-2)}} + \\ &\quad - \lambda \|u\|_{\mu}^q \left( \frac{(2^{*}-q)}{2^{*}q} \right) \|g\|_{\mathcal{H}_{\mu}^{-1}} \end{aligned}$$

Thus, for all  $\lambda$  such that

$$0 < \lambda < \lambda_1 = \left( \frac{(2^* - 2)}{2^{*2}} \right) \left[ \frac{(2 - q)}{(2^* - 1)} S_{\mu}^{\frac{2^*}{2}} \right]^{\frac{(2-q)}{(2^*-2)}} \left( \frac{(2^* - q)}{2^* q} \right)^{-1} \|g\|_{\mathcal{H}_{\mu}^{-1}}^{-1} = \frac{q}{2} \lambda_0,$$

we have  $I(u) \geq C_0$ .

**Proposition1:** (see [4])

i) For all  $\lambda$  such that  $0 < \lambda < \lambda_0$ , there exists a  $(PS)_{C^+}$  sequence in  $\mathcal{N}^+$

ii) For all  $\lambda$  such that  $0 < \lambda < \frac{q}{2} \lambda_0$ , there exists a  $(PS)_{C^-}$  sequence in  $\mathcal{N}^-$

We write

$$t_M := t_{\max}(u) = \left[ \frac{(2-q)\|u\|_{\mu}^2}{(2^*-q) \int_{\Omega} |u|^{2^*} dx} \right]^{1/(2^*-2)} > 0.$$

**Lemma6:** Let real parameters such that  $0 < \lambda < \lambda_0$ . For each  $u \in \mathcal{H}_{\mu}$ , there exist unique  $t^+$  and  $t^-$  such that  $0 < t^+ < t_m < t^-$ ,  $(t^+u) \in \mathcal{N}^+$ ,  $(t^-u) \in \mathcal{N}^-$

$$I(t^+u) = \inf_{0 \leq t \leq t_m} I(tu) \text{ and } I(t^-u) = \sup_{t \geq 0} I(tu).$$

Proof: With minor modifications, we refer to [4].

## 2. Proofs

### Proof of Theorem1

Now, taking as a starting point the work of Tarantello [16], we establish the existence of a local minimum for  $I$  on  $\mathcal{N}^+$ .

**Proposition 2:** For all  $\lambda$  such that  $0 < \lambda < \lambda_0$ , the functional  $I$  has a minimizer  $u_0^+ \in \mathcal{N}^+$  and it satisfies

i)  $I(u_0^+) = c = c^+$

ii)  $(u_0^+)$  is a nontrivial solution of (1.1).

Proof: If  $0 < \lambda < \lambda_0$ , then by Proposition1(i) there exists a  $(u_n)_n$   $(PS)_{c^+}$  sequence in  $\mathcal{N}^+$ , thus it bounded by **Lemma2**. Then, there exists  $u_0^+ \in \mathcal{H}$  and we can extract a subsequence

which will denoted by  $(u_n)_n$  such that

$$\begin{aligned}
 &u_n \rightharpoonup u_0^+ \text{ weakly in } \mathcal{H} \\
 &u_n \rightharpoonup u_0^+ \text{ weakly in } L^{2^*}(\Omega) \\
 &u_n \rightarrow u_0^+ \text{ strongly in } L^q(\Omega) \\
 &u_n \rightarrow u_0^+ \text{ a.e in } \Omega
 \end{aligned}
 \tag{7}$$

Thus, by (7),  $u_0^+$  is a weak nontrivial solution of (1.1). Now, we show that  $u_n$  converges to  $u_0^+$  strongly in H. Suppose otherwise. By the lower semi-continuity of

and we obtain  $\|u_0^+\|_\mu < \liminf_{n \rightarrow \infty} \|u_n\|_\mu$  the norm, then either

$$\begin{aligned}
 c &\leq I(u_0^+) = ((2^* - 2)/2^*) \|u_0^+\|_\mu^2 - \lambda((2^* - q)/q2^*) \int_\Omega g|u_0^+|^q dx \\
 &< \liminf_{n \rightarrow \infty} I(u_n) = c.
 \end{aligned}$$

We get a contradiction. Therefore,  $u_n$  converge to  $u_0^+$  strongly in H. Moreover, we have  $u_0^+ \in N^+$ . If not, then by **Lemma6**, there are two numbers  $t_0^+$  and  $t_0^-$ , uniquely defined so that  $(t_0^+ u_0^+) \in N^-$  and  $(t_0^- u_0^+) \in N^+$ . In particular, we have  $t_0^- < t_0^+ = 1$ . Since

$$\frac{d}{dt} I(tu_0^+) \Big|_{t=t_0^+} = 0 \text{ and } \frac{d^2}{dt^2} I(tu_0^+) \Big|_{t=t_0^+} > 0,$$

there exists  $t_0^- < t^- \leq t_0^+$  such that  $I(t_0^- u_0^+) < I(t^+ u_0^+)$ . By **Lemma6**, we get

$$I(t_0^- u_0^+) < I(t^- u_0^+) < I(t_0^+ u_0^+) = I(u_0^+),$$

which contradicts the fact that  $I(u_0^+) = c^+$ . Since  $I(u_0^+) = I(|u_0^+|)$  and  $|u_0^+| \in N^+$ , then by **Lemma 3**, we may assume that  $u_0^+$  is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that  $u_0^+ > 0$ , see for example [7].

### Proof of Theorem2

Next, we establish the existence of a local minimum for I on  $N^-$ . For this, we require the following Lemma.

**Lemma7:** For all  $\lambda$  such that  $0 < \lambda < \frac{q}{2} \lambda_0$ , the functional I has a minimizer  $u_0^-$  in  $N^-$  and it satisfies:

- i)  $I(u_0^-) = c^- > 0$
- ii)  $u_0^-$  is a nontrivial solution of (1.1) in H.

Proof: If  $0 < \lambda < \frac{q}{2} \lambda_0$ , then by Proposition1 (ii) there exists a  $(u_n)_n$ ,  $(PS)_{c^-}$  sequence in  $N^-$ , thus it bounded by **Lemma2**. Then, there exists  $u_0^- \in H$  and we can extract a subsequence which will denoted by  $(u_n)_n$  such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \text{ weakly in } \mathcal{H} \\ u_n &\rightharpoonup u_0^- \text{ weakly in } L^{2^*}(\Omega) \\ u_n &\rightarrow u_0^- \text{ strongly in } L^q(\Omega) \\ u_n &\rightarrow u_0^- \text{ a.e in } \Omega \end{aligned}$$

This implies

$$\int_{\Omega} |u_n|^{2^*} dx \rightarrow \int_{\Omega} |u_0^-|^{2^*} dx,$$

Moreover, by (3) we obtain

$$(8) \int_{\Omega} |u_n|^{2^*} dx > \left( \frac{2-q}{2^*-1} \right) \|u\|_{\mu}^2,$$

By (4) and (8) there exists a positive number

$$C_1 := \left[ \frac{(2-q)}{(2^*-1)} \right]^{(2^*-1)/(2^*-2)} (S_{\mu})^{\frac{2}{2^*(2^*-2)}},$$

such that

$$(9) \int_{\Omega} |u_n|^{2^*} dx > C_1.$$

This implies that

$$\int_{\Omega} |u_0^-|^{2^*} dx \geq C_1.$$

Now, we prove that  $(u_n)_n$  converges to  $u_0^-$  strongly in  $H$ . Suppose otherwise. Then, either  $\|u_0^-\|_{\mu} < \liminf_{n \rightarrow \infty} \|u_n\|_{\mu}$ . By **Lemma6** there is a unique  $t_0^-$  such that  $(t_0^- u_0^-) \in N^-$ . Since

$$u_n \in N^-, I(u_n) \geq I(tu_n), \text{ for all } t \geq 0,$$

we have

$$I(t_0^- u_0^-) < \liminf_{n \rightarrow \infty} I(t_0^- u_n) \leq \lim_{n \rightarrow \infty} I(u_n) = c^-,$$

and this is a contradiction. Hence,

$$(u_n)_n \rightarrow u_0^- \text{ strongly in } \mathcal{H}.$$

Thus

$$J(u_n) \text{ converges to } J(u_0^-) = c^- \text{ as } n \text{ tends to } +\infty.$$

Since  $I(u_0^-) = I(|u_0^-|)$  and  $u_0^- \in N^-$ , then by (9) and **Lemma6**, we may assume that  $u_0^-$  is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that  $u_0^- > 0$ .

Now, we complete the proof of **Theorem2**. By **Propositions2** and **Lemma7**, we obtain that (1.1) has two positive solutions  $u_0^+ \in N^+$  and  $u_0^- \in N^-$ . Since  $N^+ \cap N^- = \emptyset$ , this implies that  $u_0^+$  and  $u_0^-$  are distinct.

**Proof of Theorem 3**

In this section, we consider the following Nehari submanifold of N.

$$\mathcal{N}_r = \left\{ u \in \mathcal{H} \setminus \{0\} : \langle I'(u), u \rangle = 0 \text{ and } \|u\| \geq r > 0 \right\}.$$

Thu  $u \in \mathcal{N}_r$  if and only if

$$\|u\|_{\mu}^2 - \int_{\Omega} |u|^{2^*} dx - \lambda \int_{\Omega} g|u|^q dx = 0 \text{ and } \|u\|_{\mu} \geq r > 0.$$

Firsly, we need the following Lemmas

**Lemma 8:** Under the hypothesis of **Theorem 3**, there exist  $r_0, \lambda_2 > 0$  such that  $\mathcal{N}_r$  is nonempty for any  $\lambda \in (0, \lambda_2)$  and  $r \in (0, r_0)$ .

**Proof:** Fix  $u_0 \in \mathcal{H} \setminus \{0\}$  and let

$$\begin{aligned} g(t) &= \langle I'(tu_0), tu_0 \rangle \\ &= t^2 \|u_0\|_{\mu}^2 - t^{2^*} \int_{\Omega} |u_0|^{2^*} dx - t^q \lambda \int_{\Omega} g|u_0|^q dx. \end{aligned}$$

Clearly  $g(0)=0$  and  $g(t) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Moreover, we have

$$\begin{aligned} g(1) &= \|u_0\|_{\mu}^2 - \int_{\Omega} |u_0|^{2^*} dx - \lambda \int_{\Omega} g|u_0|^q dx \\ &\geq \left[ \|u_0\|_{\mu}^2 - (S_{\mu})^{-2^*/2} \|u_0\|_{\mu}^{2^*} \right] - \lambda \|u_0\|_{\mu}^q \|g\|_{\mathcal{H}_{\mu}^{-1}}. \end{aligned}$$

If  $\|u_0\|_{\mu} \geq r > 0$  for, then there exist  $0 < r < r_0 = (S_{\mu})^{2^*/2(2^*-2)}$

$$\lambda_2 := r^{2-q} \left( 1 - r^{2^*-2} (S_{\mu})^{-2^*/2} \right) \|g\|_{\mathcal{H}_{\mu}^{-1}}^{-1},$$

and  $t_0 > 0$  such that  $g(t_0)=0$ . Thus,  $(t_0 u_0) \in \mathcal{N}_r$  and  $\mathcal{N}_r$  is nonempty for any  $\lambda \in (0, \lambda_2)$ .

**Lemma 9:** There exist  $\varrho, \lambda_3$  positive reals such that  $\langle \varphi'(u), u \rangle < -\varrho < 0$  for  $u \in \mathcal{N}_r$  and any  $\lambda$  verifying

$$0 < \lambda < \min(\lambda_2, \lambda_3).$$

Let  $u \in \mathcal{N}_r$ , then by (1), (3) and the Holder inequality, allows us to write

$$\begin{aligned} \langle \varphi'(u), u \rangle &= \lambda(2^* - q) \int_{\Omega} g|u|^q dx - (2^* - 2) \|u\|_{\mu}^2 \\ &\leq \lambda(2^* - q) \|u\|_{\mu}^q \|g\|_{\mathcal{H}_{\mu}^{-1}} - (2^* - 2) \|u\|_{\mu}^2 \\ &\leq \|u_n\|_{\mu}^q \left[ \lambda(2^* - q) \|g\|_{\mathcal{H}_{\mu}^{-1}} - (2^* - 2)r^{2-q} \right], \end{aligned}$$

Thus if

$$0 < \lambda < \lambda_4 = \left[ (2^* - 2)r^{2-q}/(2^* - q) \|g\|_{\mathcal{H}_\mu^{-1}} \right],$$

and choosing  $\lambda_3: -\min(\lambda_2, \lambda_4) > 0$  with  $\lambda_2$  defined in **Lemma9**, then we obtain that

$$(10) \langle \varphi'(u), u \rangle < 0, \text{ for any } u \in \mathcal{N}_r.$$

**Lemma10:** Suppose  $N \geq 6$ . Then, there exist  $\alpha$  and  $\eta$  positive constants such that  
i) we have

$$I(u) \geq \eta > 0 \text{ for } \|u\|_\mu = \varepsilon.$$

ii) there exists  $\varepsilon = \|u\|_\mu$ , when  $\|w\|_\mu > \varepsilon$ , with  $w \in \mathcal{N}_r$  such that  $I(w) \leq 0$ .

Proof: We can suppose that the minima of  $J$  are realized by  $u_0^+$  and  $u_0^-$ . The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have:

i) By (3), (10) and the fact that

$$\int_\Omega |u|^{2^*} dx > \left( \frac{2-q}{2^*-1} \right) \|u\|_\mu^2$$

We get

$$I(u) \geq \left( \frac{1}{2} - \frac{1}{q} \right) \|u\|_\mu^2 + \left( \frac{1}{q} - \frac{1}{2^*} \right) \left( \frac{2-q}{2^*-1} \right) \|u\|_\mu^2,$$

By the fact that  $1 < q < 2$  and  $N \geq 6$ , we obtain that

$$I(u) \geq \eta > 0 \text{ when } \varepsilon = \|u\|_\mu \text{ small.}$$

ii) Let  $t > 0$ , then we have for all  $\psi \in \mathcal{N}_r$

$$I(t\psi) := \frac{t^2}{2} \|\psi\|_\mu^2 - \left( \frac{t^{2^*}}{2^*} \right) \int_\Omega |\psi|^{2^*} dx - \left( \lambda \frac{t^q}{q} \right) \int_\Omega g |\psi|^q dx.$$

Letting  $w = t\psi$  for  $t$  large enough, we obtain  $I(w) \leq 0$ . For  $t$  large enough we can ensure  $\|w\|_\mu > \varepsilon$ . Let  $\Gamma$  and  $c$  defined by

$$\Gamma := \left\{ \gamma : [0, 1] \rightarrow \mathcal{N}_r : \gamma(0) = u_0^- \text{ and } \gamma(1) = u_0^+ \right\}$$

And

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} (I(\gamma(t))).$$

If  $0 < \lambda < \min(\lambda_1, \lambda_2)$  then, by the **Lemma2** and **Proposition1** (ii), I verifying the Palais-Smale condition in  $\mathcal{N}_r$ . Moreover, from the Lemmas 3, 9 and 10, there exists  $u_c$  such that

$$I(u_c) = c \text{ and } u_c \in \mathcal{N}_r.$$

Thus  $u_c$  is the third solution of our system such that  $u_c \neq u_c^+$  and  $u_c \neq u_c^-$ . Since

(1.1) is odd with respect  $u$ , we obtain that  $-u_c$  is also a solution of (1.1).

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وجود حلول متعددة لمعادلة إهليلجية مع معاملات اسطوانية ومحتوية علي جهد مفرد  
أسطواني التزايد

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ملخص

في هذه الورقة، يتم النظر في معادلة إهليلجية مع معاملات اسطوانية ومحتوية علي جهد مفرد أسطواني التزايد. باستخدام متغيرة نيهاري ونظرية المرور من والجبل، يتم الحصول على أربعة حلول متميزة على الأقل. النتيجة تعتمد بشكل حاسم على الوسائط  $k$ ،  $\Delta$ ،  $g$  و  $\mu$ .