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The Regularization q-Homotopy Analysis Method for (1 and 2) –Dimensional Non-linear First Kind Fredholm Integral Equations

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Abstract

In this paper, the solutions of(1 and 2)- dimensional non-linear first kind Fredholm integral equations are studied by combine the q-homotopy analysis method (q-HAM) [2-11] and the regularization method [16,17]. The utilization of this technique depends on converting the first kind Fredholm Integral Equations to the second kind of equations by applying the regularization method. Then q-HAM is employed to the resulting second kind of equations to obtain a solution. Some illustrative examples are given to demonstrate the validity and applicability of this technique.

Keywords: Regularization method ; q-homotopy analysis method; Fredholm integral equations.

1. Introduction

Integral equations show up in numerous scientific applications with a very wide range from physical sciences to engineering. An A tremendous amount of work has been done on solving them. First kind Fredholm integral equations are normally considered to be ill-posed problems. That means, solutions may not exist and in the event that it exists, it may not be unique.

In this paper, we consider the (1 and 2)-dimensional nonlinear Fredholm integral equations of the first kind:

$$f(x) = \int_a^b K(x,t)F\big(u(t)\big)dt, \qquad (1)$$

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)F(u(y,z))dy\,dz, \qquad (2)$$

where f is a known function,a, b, c and d are constants**,** K is the kernel of the integral equations**,** F is a continuous function which has continuous inverse and finally $u$is the unknown function of the equations to be find. Obviously, if $F$is linear then the equations (1) and (2) will be linear. In general, integral equations are classified as either first or second kind relying upon where the unknown function $u(x)$ shows up. If it shows up only inside the integral sign, it is called an integral equation of the first kind, else, it is called an integral equation of the second kind. The presence of the unknown function only inside the integral sign presents some difficulties. These, for instance, include applying known valuable techniques presented for solving the second kind of equations to the first kind. To overcome this, one needs either to modify the existing techniques, transform the integral equation, or develop another technique in the event that it is possible.

In this article, first we will employ the regularization method [12,13].This method transforms the first kind equation to the second kind equation then we employ the q-homotopy analysis method that will used to handle the linear and nonlinear Fredholm integral equation.

2. The Regularization Method

The regularization method which was established firstly by Phillips [12] and Tikhonov [13] transforms the (1 and 2)-dimensional linear first kind Fredholm integral equation

$$f(x) = \int_a^b K(x,t)u(t)dt, , \quad (3)$$

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)u(y,z)dy\, dz, \quad (4)$$

into approximation the (1 and 2)-dimensional linear second kind Fredholm integral equation

$$\mu u_\mu(x) = f(x) - \int_a^b K(x,t)u_\mu(t)dt \quad (5)$$

$$\mu u_\mu(x,t) = f(x,t) - \int_a^b \int_c^d K(x,t,y,z)u_\mu(y,z)dy\, dz, \quad (6)$$

respectively , where μ is a small positive parameter. It is obvious that equations (5) and (6)can express as

$$u_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu}\int_a^b K(x,t)u_\mu(t)dt \quad (7)$$

$$u_\mu(x,t) = \frac{1}{\mu} f(x,t) - \frac{1}{\mu}\int_a^b \int_c^d K(x,t,y,z)u_\mu(y,z)dy\, dz, \quad (8)$$

Moreover it was proved in [12-14] that the solutionsu_μ of equations (7) and (8) converge to the solution u of equations (3) and (4) as $\mu \to 0$. In other word

$u = \lim_{\mu \to 0} u_\mu$(9)

In nonlinear case, that is the (1 and 2)-dimensional first kind Fredholm integral equations have the forms as in equations (1) and (2) firstly we convert the equations to a linear first kind Fredholm integral equations by use the transformation

$v = F(u),$ (10)

Hence the (1 and 2)-dimensional nonlinear first kind Fredholm integral equations (1) and (2) take the following linear forms:

$f(x) = \int_a^b K(x,t)v(t)dt,$,(11)
$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)v(y,z)dy\,dz,$(12)

Moreover, as above the method of regularization also can be transforms the first kind equations (11) and (12) to the second kind equations:

$v_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu}\int_a^b K(x,t)v_\mu(t)dt$(13)

$v_\mu(x,t) = \frac{1}{\mu} f(x,t) - \frac{1}{\mu}\int_a^b \int_c^d K(x,t,y,z)v_\mu(y,z)dy\,dz,$(14)

Therefore the solutions v_μ of equations (13) and (14) converge to the solutions$v$ of equations (11) and (12) as $\mu \to 0$. That is

$v = \lim_{\mu \to 0} v_\mu,$(15)

then we can set

$u = F^{-1}(v)$(16)

3. The Regularization q-Homotopy Analysis Method (RqHAM)

Consider the (1 and 2)-dimensional nonlinear first kind Fredholm integral equations(1) and (2). Under the regularization method after setting, $v = F(u)$ we obtain the following (1 and 2)-dimensional linear second kind Fredholm integral equations

$v_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu}\int_a^b K(x,t)v_\mu(t)dt$ (17)

$v_\mu(x,t) = \frac{1}{\mu} f(x,t) - \frac{1}{\mu}\int_a^b \int_c^d K(x,t,y,z)v_\mu(y,z)dy\,dz,$ (18)

Constructs the so called the 0^{th} order deformation equations

$$(1-\gamma q)L\left[\theta_\mu(x,q)-v_\mu^{[0]}(x)\right]=$$
$$q\,\Psi(x,\gamma)\left[\theta_\mu(x,q)-\frac{1}{\mu}f(x)+\frac{1}{\mu}\int_a^b K(x,t)\theta_\mu(t,q)dt\right] \quad (19)$$

$$(1-\gamma q)L\left[\theta_\mu(x,t,q)-v_\mu^{[0]}(x,t)\right]=q\,\Psi(x,t,\gamma)\left[\theta_\mu(x,t,q)-\frac{1}{\mu}f(x,t)+\right.$$
$$1\mu abcdKx,t,y,z\theta\mu y,z,qdydz \quad (20)$$

Where $\gamma\neq 0$, $q$varies from 0 to $\frac{1}{\gamma}$is the embedding parameter,L is alinear operator,$\Psi$is a non -zerofunction,$v_\mu^{[0]}$ is an initial guess of v_μ.

It is clear when $q=0$ and $q=\frac{1}{\gamma}$ equations (19) and (20) becomes:

$$\theta_\mu(x,0)=v_\mu^{[0]}(x) \quad ,\theta_\mu\left(x,\frac{1}{\gamma}\right)=v_\mu(x) \quad (21)$$

$$\theta_\mu(x,t,0)=v_\mu^{[0]}(x,t) \quad , \quad \theta_\mu\left(x,t,\frac{1}{\gamma}\right)=v_\mu(x,t) \quad (22)$$

So as$q$varies from 0 to $\frac{1}{\gamma}$, the solutionsθ_μ deformsfrom the initial guesses $v_\mu^{[0]}$ to the solutionsv_μ.

Taking the Taylor series of$\theta_\mu(x,q)$and $\theta_\mu(x,t,q)$with respect to q we obtain:

$$\theta_\mu(x,q)=v_\mu^{[0]}(x)+\sum_{m=1}^{\infty}v_\mu^{[m]}(x)q^m \quad (23)$$

$$\theta_\mu(x,t,q)=v_\mu^{[0]}(x,t)+\sum_{m=1}^{\infty}v_\mu^{[m]}(x,t)q^m, \quad (24)$$

where

$$v_\mu^{[m]}(x)=\frac{1}{m!}\frac{\partial^m\theta_\mu(x,q)}{\partial q^m}\Big|_{q=0} \quad (25)$$

$$v_\mu^{[m]}(x,t)=\frac{1}{m!}\frac{\partial^m\theta_\mu(x,t,q)}{\partial q^m}\Big|_{q=0} \quad (26)$$

Suppose that the linear operators L , the initial guesses $v_\mu^{[0]}$, and the function $\Psi$are so properly chosen such that the series (23) and (24) converges at $q=\frac{1}{\gamma}$then we obtain

$$v_\mu(x)=\theta_\mu\left(x,\frac{1}{\gamma}\right)=v_\mu^{[0]}(x)+\sum_{i=1}^{+\infty}v_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i \quad (27)$$

$v_\mu(x,t) = \theta_\mu\left(x,t,\frac{1}{\gamma}\right) = v_\mu^{[0]}(x,t) + \sum_{i=1}^{+\infty} v_\mu^{[i]}(x,t)(\frac{1}{\gamma})^i$ (28)

Define the vector$v_\mu^{[r]} = \left\{v_\mu^{[0]}, v_\mu^{[1]} \dots, v_\mu^{[r]}\right\}$. Differentiating equations (19) and (20)m times with respect to q and then setting $q = 0$ and dividing them by $m!$, we get the m^{th} -order deformation equations

$L\left[v_\mu^{[m]}(x) - x_m v_\mu^{[m-1]}(x)\right] = \Psi(x,\gamma)R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right]$(29)

$L\left[v_\mu^{[m]}(x,t) - x_m v_\mu^{[m-1]}(x,t)\right] = \Psi(x,t,\gamma)R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]$,(30)

where

$R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right] = \frac{1}{(m-1)!}\frac{\partial^{m-1}[v_\mu(x) - \frac{1}{\mu}f(x) + \frac{1}{\mu}\int_a^b K(x,t)v_\mu(t)dt]}{\partial q^{m-1}}|_{q=0}$(31)

$R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right] = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left[v_\mu(x,t) - \frac{1}{\mu}f(x,t) + \frac{1}{\mu}\int_a^b\int_c^d K(x,t,y,z)v_\mu(y,z)dydz\right]}{\partial q^{m-1}}|_{q=0}$(32) And

$x_m = \begin{cases} 0, & m \le 1 \\ \gamma, & m > 1 \end{cases}$(33)

Applying L^{-1} on both side of equations (29) and (30) we obtain,

$v_\mu^{[m]}(x) = x_m v_\mu^{[m-1]}(x) + L^{-1}\left[\Psi(x,\gamma)R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right]\right]$(34)

$v_\mu^{[m]}(x,t) = x_m v_\mu^{[m-1]}(x,t) + L^{-1}\left[\Psi(x,t,\gamma)R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right]$(35)

Now it is easily to obtain $v_\mu^{[m]}$ for $m \ge 1$, and the exact solutions of equations (17) and (18) can be obtain by

$v(x) = \lim_{\mu\to 0}\sum_{i=0}^{+\infty} v_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i$(36)

$v(x,t) = \lim_{\mu\to 0}\sum_{i=0}^{+\infty} v_\mu^{[i]}(x,t)\left(\frac{1}{\gamma}\right)^i$(37)

Hence, the exact solutions of equations (1) and (2) can be obtain by setting $u = F^{-1}(v)$.

4. Applications

Example 1. Consider the following1-dimensional linear first kind Fredholm integral equation [15]

$\frac{1}{4}x = \int_0^1 xt\, u(t)dt$ (38)

The regularization method transform Eq.(38) to

$$\mu u_\mu(x) = \frac{1}{4}x - \int_0^1 xt\, u_\mu(t)dt$$

or ,equivalently

$$u_\mu(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, u_\mu(t)dt \qquad (39)$$

For RqHAM solution, we choose the linear operator:

$L[\theta_\mu(x,q)] = \theta_\mu(x,q)$(40)

Using initial approximation$u_\mu^{[0]}(x) = 0$ (41)

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,q) - u_\mu^{[0]}(x)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,q) - \frac{1}{4\mu}x + \frac{1}{\mu}\int_0^1 xt\,\theta_\mu(t,q)dt)\right]$$

Take$\Psi(x,\gamma) = -1$ and the m^{th}order deformation equation is

$L\left[u_\mu^{[m]}(x) - x_m u_\mu^{[m-1]}(x)\right] = -R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right]$,(42)

wherex_m as define by (33) and

$$R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right] = u_\mu^{[m-1]} - \left(\frac{1}{4\mu}x\right)\left(1 - \frac{1}{\gamma}x_m\right) + \frac{1}{\mu}\int_0^1 xt\, u_\mu^{[m-1]}(t)dt$$

Now the solution ofm^{th}order deformation equation (42)

$$u_\mu^{[m]}(x) = x_m u_\mu^{[m-1]}(x) - L^{-1}\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right]\right]$$

Consequently, we obtain the components:

$$u_\mu^{[0]}(x) = 0$$

$$u_\mu^{[1]}(x) = \frac{1}{4\mu}\,x$$

$$u_\mu^{[2]}(x) = -\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}$$

$u_{\mu}^{[3]}(x) = \gamma(-\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}) + \frac{x}{12\mu^2} + \frac{x}{4\mu} - \frac{x\gamma}{4\mu} - \frac{x(-1+3(-1+\gamma)\mu)}{36\mu^3} u_{\mu}^{[4]}(x) =$
$-\gamma(-\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}) - \frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu} - \frac{x(1-3(-1+\gamma)\mu)^2}{108\mu^4} + \frac{x(-1+3(-1+\gamma)\mu)}{36\mu^3} +$
$\gamma(\gamma(-\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}) + \frac{x}{12\mu^2} + \frac{x}{4\mu} - \frac{x\gamma}{4\mu} - \frac{x(-1+3(-1+\gamma)\mu)}{36\mu^3}),$

$u_{\mu}^{[m]}(x), (m = 5,6, \ldots)$ can be calculated similarly. Then the approximate solution obtaining by RqHAM can be written as the following series:

$$u_{\mu}(x,\gamma,\mu) = u_{\mu}^{[0]}(x) + \sum_{i=1}^{+\infty} u_{\mu}^{[i]}(x) \left(\frac{1}{\gamma}\right)^{i}$$

$$= \frac{-\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}}{\gamma^2} + \frac{x}{4\gamma\mu} + \frac{\gamma(-\frac{x}{12\mu^2} - \frac{x}{4\mu} + \frac{x\gamma}{4\mu}) + \frac{x}{12\mu^2} + \frac{x}{4\mu} - \frac{x\gamma}{4\mu} - \frac{x(-1+3(-1+\gamma)\mu)}{36\mu^3}}{\gamma^3} + \ldots$$

When $\gamma = 1$ we have

$$u_{\mu}(x,\mu) = -\frac{x}{108\mu^4} + \frac{x}{36\mu^3} - \frac{x}{12\mu^2} + \frac{x}{4\mu} - \cdots$$

Based on this we get the approximate solution

$$u_{\mu}(x,\mu) = \frac{3}{4(1+3\mu)}\, x$$

Hence, the exact solution$u(x)$ of equation (38) can be obtain by

$u(x) = \lim_{\mu \to 0} u_{\mu}(x) = \frac{3}{4}x.$

Example 2: Consider the following 1-dimensional linear first kind Fredholm integral equation[16]

$(e-1)e^{3x} = \int_0^1 e^{3x-4t}\, u(t)dt$(43)

By the regularization method, Eq. (43) can be transformed to

$$\mu u_{\mu}(x) = (e-1)e^{3x} - \int_0^1 e^{3x-4t}\, u(t)dt$$

or ,equivalently

$u_{\mu}(x) = \frac{1}{\mu}(e-1)e^{3x} - \frac{1}{\mu}\int_0^1 e^{3x-4t}\, u(t)dt$ (44)

For RqHAM solution, we choose the linear operator $L\left[\theta_\mu(x,q)\right] = \theta_\mu(x,q)$ and initial approximation $u_\mu^{[0]}(x) = \frac{1}{\mu}(e-1)e^{3x}$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,q) - u_\mu^{[0]}(x)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,q) - \frac{1}{\mu}(e-1)e^{3x} + \frac{1}{\mu}\int_0^1 e^{3x-4t}\,\theta_\mu(t,q)dt\right]$$

Take $\Psi(x,\gamma) = -1$ and the m^{th} order deformation equation is

$$L\left[u_\mu^{[m]}(x) - x_m u_\mu^{[m-1]}(x)\right] = -R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right], \tag{45}$$

where

$$R_\mu^{[m]}(u_\mu^{[m-1]}(x)) = u_\mu^{[m-1]} - \left(\frac{1}{\mu}(e-1)e^{3x}\right)\left(1 - \frac{1}{\gamma}x_m\right) + \frac{1}{\mu}\int_0^1 \quad e^{3x-4t}\,u_\mu^{[m-1]}(t)dt$$

Now the solution of m^{th} order deformation equation (45) become:

$$u_\mu^{[m]}(x) = x_m u_\mu^{[m-1]}(x) - L^{-1}\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right]\right]$$

Consequently, we obtain the components:

$$u_\mu^{[1]}(x) = -\frac{(-1+e)^2 e^{-1+3x}}{\mu^2}$$

$$u_\mu^{[2]}(x) = \frac{(-1+e)^3 e^{-2+3x}}{\mu^3} + \frac{(-1+e)^2 e^{-1+3x}}{\mu^2} - \frac{(-1+e)^2 e^{-1+3x}\gamma}{\mu^2}$$

$$u_\mu^{[3]}(x) = \gamma\left(\frac{(-1+e)^3 e^{-2+3x}}{\mu^3} + \frac{(-1+e)^2 e^{-1+3x}}{\mu^2} - \frac{(-1+e)^2 e^{-1+3x}\gamma}{\mu^2}\right) - \frac{(-1+e)^3 e^{-2+3x}}{\mu^3} - \frac{(-1+e)^2 e^{-1+3x}}{\mu^2} + \frac{(-1+e)^2 e^{-1+3x}\gamma}{\mu^2} + \frac{(-1+e)^3 e^{-3+3x}(1+e(-1+(-1+\gamma)\mu))}{\mu^4}$$

$\vdots$

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$u_\mu(x,\gamma,\mu)=u_\mu^{[0]}(x)+\sum_{i=1}^{+\infty}u_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i$$

$$=\frac{\frac{(-1+e)^3e^{-2+3x}}{\mu^3}+\frac{(-1+e)^2e^{-1+3x}}{\mu^2}-\frac{(-1+e)^2e^{-1+3x}\gamma}{\mu^2}}{\gamma^2}-\frac{(-1+e)^2e^{-1+3x}}{\gamma\mu^2}+\frac{(-1+e)e^{3x}}{\mu}+$$
$$\frac{1}{\gamma^3}\left(\gamma\left(\frac{(-1+e)^3e^{-2+3x}}{\mu^3}+\frac{(-1+e)^2e^{-1+3x}}{\mu^2}-\frac{(-1+e)^2e^{-1+3x}\gamma}{\mu^2}\right)-\frac{(-1+e)^3e^{-2+3x}}{\mu^3}-\frac{(-1+e)^2e^{-1+3x}}{\mu^2}+\right.$$
$$\left.\frac{(-1+e)^2e^{-1+3x}\gamma}{\mu^2}+\frac{(-1+e)^3e^{-3+3x}(1+e(-1+(-1+\gamma)\mu))}{\mu^4}\right)+\dots$$

When $\gamma=1$ we have

$$u_\mu(x,\mu)=\frac{(1-e)(-1+e)^3e^{-3+3x}}{\mu^4}+\frac{(-1+e)^3e^{-2+3x}}{\mu^3}-\frac{(-1+e)^2e^{-1+3x}}{\mu^2}+\frac{(-1+e)e^{3x}}{\mu}+\dots$$

$$=\frac{1}{\mu}(e-1)e^{3x}\left(1-\frac{e-1}{e\mu}+\frac{(e-1)^2}{e^2\mu^2}+\frac{(e-1)^3}{e^3\mu^3}+\cdots\right)$$

Hence, the exact solution $u(x)$ of equation (43) can be obtain by

$$u(x)=\lim_{\mu\to 0}u_\mu(x,\mu)=e^{3x+1}$$

Example 3: Consider the following 1-dimensional linear first kind Fredholm integral equation [16]

$$\frac{\pi}{2}cosx=\int_0^\pi \cos(x-t)\,u(t)dt \qquad (46)$$

By the regularization method, Eq. (46) can be transformed to

$$\mu u_\mu(x)=\frac{\pi}{2}cosx-\int_0^\pi \cos(x-t)\,u(t)dt$$

or ,equivalently

$$u_\mu(x)=\frac{\pi}{2\mu}cosx-\frac{1}{\mu}\int_0^\pi \cos(x-t)\,u(t)dt(47)$$

For RqHAM solution, we choose the linear operator $L[\theta_\mu(x,q)]=\theta_\mu(x,q)$ and initial approximation $u_\mu^{[0]}(x)=\frac{\pi}{2\mu}cosx$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,q)-u_\mu^{[0]}(x)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,q)-\frac{\pi}{2\mu}cosx+\frac{1}{\mu}\int_0^\pi \cos(x-t)\,\theta_\mu(t,q)dt\right]$$

Take $\Psi(x,\gamma)=-1$ and the m^{th} order deformation equation is

$$L\left[u_\mu^{[m]}(x)-x_m u_\mu^{[m-1]}(x)\right]=-R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right], \tag{48}$$

where

$$R_\mu^{[m]}(u_\mu^{[m-1]}(x))=u_\mu^{[m-1]}(x)-\left(\frac{\pi}{2\mu}cosx\right)\left(1-\frac{1}{\gamma}x_m\right)+\frac{1}{\mu}\int_0^\pi \cos(x-t)\,u_\mu^{[m-1]}(t)dt$$

Now the solution of m^{th} order deformation equation (48) become:

$$u_\mu^{[m]}(x)=x_m u_\mu^{[m-1]}(x)-L^{-1}\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x)\right]\right]$$

Consequently, we obtain the components:

$$u_\mu^{[1]}(x)=-\frac{\pi^2\text{Cos}[x]}{4\mu^2}$$

$$u_\mu^{[2]}(x)=\frac{\pi^3\text{Cos}[x]}{8\mu^3}+\frac{\pi^2\text{Cos}[x]}{4\mu^2}-\frac{\pi^2\gamma\text{Cos}[x]}{4\mu^2}$$

$$u_\mu^{[3]}(x)=-\frac{\pi^3\text{Cos}[x]}{8\mu^3}-\frac{\pi^2\text{Cos}[x]}{4\mu^2}+\frac{\pi^2\gamma\text{Cos}[x]}{4\mu^2}-\frac{\pi^3(\pi-2(-1+\gamma)\mu)\text{Cos}[x]}{16\mu^4} +\gamma\left(\frac{\pi^3\text{Cos}[x]}{8\mu^3}+\frac{\pi^2\text{Cos}[x]}{4\mu^2}-\frac{\pi^2\gamma\text{Cos}[x]}{4\mu^2}\right)$$

⋮

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$u_\mu(x,\gamma,\mu)=u_\mu^{[0]}(x)+\sum_{i=1}^{+\infty}u_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i$$

$$=-\frac{\pi^2\text{Cos}[x]}{4\gamma\mu^2}+\frac{\pi\text{Cos}[x]}{2\mu}+\frac{\frac{\pi^3\text{Cos}[x]}{8\mu^3}+\frac{\pi^2\text{Cos}[x]}{4\mu^2}-\frac{\pi^2\gamma\text{Cos}[x]}{4\mu^2}}{\gamma^2}+\frac{1}{\gamma^3}\left(-\frac{\pi^3\text{Cos}[x]}{8\mu^3}-\frac{\pi^2\text{Cos}[x]}{4\mu^2}+\right.$$

*π2γ*Cos*x4μ2−π3π−2−1+γμ*Cos*x16μ4+γπ3*Cos*x8μ3+π2*Cos*x4μ2−π2γ*Cos*x4μ2+*
…

When $\gamma = 1$ we have

$$u_\mu(x,\mu) = \frac{\pi^5\text{Cos}[x]}{32\mu^5} - \frac{\pi^4\text{Cos}[x]}{16\mu^4} + \frac{\pi^3\text{Cos}[x]}{8\mu^3} - \frac{\pi^2\text{Cos}[x]}{4\mu^2} + \frac{\pi\text{Cos}[x]}{2\mu} + \cdots$$

$$=\frac{\pi\text{Cos}[x]}{2\mu}\left(1-\frac{\pi}{2\mu}+\frac{\pi^2}{4\mu^2}-\frac{\pi^3}{8\mu^3}+\frac{\pi^4}{16\mu^4}-\cdots\right)$$

$$=\frac{\pi\ \text{Cos}[x]}{(\pi+2\mu)}$$

Hence, the exact solution $u(x)$ of equation (46) can be obtain by

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \cos(x)$$

Example 4: Consider the following 1-dimensional nonlinear first kind Fredholm integral equation [15]

$$\frac{64}{15}x = \int_{-1}^{1} xtu^4(t)dt \qquad (49)$$

To solve this equation we first set

$$v(x) = u^4(x),\ u(x) = \pm\sqrt[4]{v(x)} \qquad (50)$$

to carry out Eq.(49) into the following 1-dimensional linear first kind Fredholm integral equation

$$\frac{64}{15}x = \int_{-1}^{1} xtv(t)dt \qquad (51)$$

By the regularization method, Eq. (51) can be transformed to the following 1-dimensional linear second kind Fredholm integral equation

$$v_\mu(x) = \frac{64}{15\mu}x - \frac{1}{\mu}\int_{-1}^{1} xtv(x)dt (52)$$

For RqHAM solution, we choose the linear operator $L\left[\theta_\mu(x,q)\right] = \theta_\mu(x,q)$ and initial approximation $v_\mu^{[0]}(x) = \frac{64}{15\mu}x$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,q)-v_\mu^{[0]}(x)\right]=q\,\Psi(x,\gamma)\left[\theta_\mu(x,q)-\frac{64}{15\mu}x+\frac{1}{\mu}\int_{-1}^{1}xt\theta_\mu(t,q)dt\right]$$

Take $\Psi(x,\gamma)=-1$ and the m^{th} order deformation equation is

$$L\left[v_\mu^{[m]}(x)-x_m v_\mu^{[m-1]}(x)\right]=-R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right], \tag{53}$$

where

$$R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right]=v_\mu^{[m-1]}-\left(\frac{64}{15\mu}x\right)\left(1-\frac{1}{\gamma}x_m\right)+\frac{1}{\mu}\int_{-1}^{1}xt\ \left(v_\mu^{[m-1]}(t)\right)dt$$

Now the solution of m^{th} order deformation equation (53) become:

$$v_\mu^{[m]}(x)=x_m v_\mu^{[m-1]}(x)-L^{-1}\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right]\right]$$

Consequently, we obtain the components:

$$v_\mu^{[1]}(x)=-\frac{128x}{45\mu^2}$$

$$v_\mu^{[2]}(x)=\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}$$

$$v_\mu^{[3]}(x)=\gamma\left(\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}\right)-\frac{256x}{135\mu^3}-\frac{128x}{45\mu^2}+\frac{128x\gamma}{45\mu^2}+\frac{256x(-2+3(-1+\gamma)\mu)}{405\mu^4}$$

$$\begin{aligned}v_\mu^{[4]}(x)=&-\gamma\left(\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}\right)+\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}\\&+\frac{256x(2-3(-1+\gamma)\mu)^2}{1215\mu^5}-\frac{256x(-2+3(-1+\gamma)\mu)}{405\mu^4}+\gamma\Big(\gamma\Big(\frac{256x}{135\mu^3}\\&+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}\Big)-\frac{256x}{135\mu^3}-\frac{128x}{45\mu^2}+\frac{128x\gamma}{45\mu^2}\\&+\frac{256x(-2+3(-1+\gamma)\mu)}{405\mu^4}\Big)\end{aligned}$$

⋮

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$v_\mu(x,\gamma,\mu) = v_\mu^{[0]}(x) + \sum_{i=1}^{+\infty} v_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i$$

$= \frac{\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}}{\gamma^2} - \frac{128x}{45\gamma\mu^2} + \frac{64x}{15\mu} + \frac{\gamma\left(\frac{256x}{135\mu^3}+\frac{128x}{45\mu^2}-\frac{128x\gamma}{45\mu^2}\right)-\frac{256x}{135\mu^3}-\frac{128x}{45\mu^2}+\frac{128x\gamma}{45\mu^2}+\frac{256x(-2+3(-1+\gamma)\mu)}{405\mu^4}}{\gamma^3} +$
$\frac{1}{\gamma^4}(-\gamma(\frac{256x}{135\mu^3} + \frac{128x}{45\mu^2} - \frac{128x\gamma}{45\mu^2}) + \frac{256x}{135\mu^3} + \frac{128x}{45\mu^2} - \frac{128x\gamma}{45\mu^2} + \frac{256x(2-3(-1+\gamma)\mu)^2}{1215\mu^5} -$
$\frac{256x(-2+3(-1+\gamma)\mu)}{405\mu^4} + \cdots$

When $\gamma = 1$ we have

$$v_\mu(x,\mu) = -\frac{2048x}{3645\mu^6} + \frac{1024x}{1215\mu^5} - \frac{512x}{405\mu^4} + \frac{256x}{135\mu^3} - \frac{128x}{45\mu^2} + \frac{64x}{15\mu} + \cdots$$

$= \frac{64}{15\mu}\left(1 - \frac{2}{3\mu} + \frac{4}{9\mu^2} - \frac{8}{27\mu^3} + \cdots\right)x$

$= \frac{64}{5(2+3\mu)}x$

Hence, the exact solution $v(x)$ of equation (51) can be obtain by

$$v(x) = \lim_{\mu\to 0} v_\mu(x,\mu) = \frac{32}{5}x$$

Using, (50) we obtain the exact solution of equation (49):

$u(x) = \sqrt[4]{\frac{32}{5}x}$.

Example 5: Consider the following 1-dimensional nonlinearfirst kind Fredholm integral equation [15]

$\frac{32}{63}x^2 = \int_{-1}^{1} x^2t^2u^2(t)dt$ (54)

To solve this equation we first set

$$v(x) = u^2(x),\ u(x) = \pm\sqrt{v(x)} \quad (55)$$

to carry out Eq.(54) into the following 1-dimensional linear first kind Fredholm integral equation

$\frac{32}{63}x^2 = \int_{-1}^{1} x^2t^2v(t)dt$ (56)

By the regularization method, Eq. (56) can be transformed to the following 1-dimensional linear second kind Fredholm integral equation

$$v_\mu(x) = \frac{32}{63\mu}x^2 - \frac{1}{\mu}\int_{-1}^{1} x^2 t^2 v(t)dt \quad (57)$$

For RqHAM solution, we choose the linear operator $L[\theta_\mu(x,q)] = \theta_\mu(x,q)$ and initial approximation $v_\mu^{[0]}(x) = \frac{32}{63}x^2$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,q) - v_\mu^{[0]}(x)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,q) - \frac{32}{63\mu}x^2 + \frac{1}{\mu}\int_{-1}^{1} x^2 t^2 \theta_\mu(t,q)dt\right]$$

Take $\Psi(x,\gamma) = -1$ and the m^{th} order deformation equation is

$$L\left[v_\mu^{[m]}(x) - x_m v_\mu^{[m-1]}(x)\right] = -R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right], \quad (58)$$

where

$$R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right] = v_\mu^{[m-1]} - \left(\frac{32}{63}x^2\right)\left(1 - \frac{1}{\gamma}x_m\right) + \frac{1}{\mu}\int_{-1}^{1} x^2 t^2 v_\mu^{[m-1]}(t)dt$$

Now the solution of m^{th} order deformation equation (53) become:

$$v_\mu^{[m]}(x) = x_m v_\mu^{[m-1]}(x) - L^{-1}\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x)\right]\right]$$

Consequently, we obtain the components:

$$v_\mu^{[1]}(x) = -\frac{64x^2}{315\mu^2}$$

$$v_\mu^{[2]}(x)\; v_{\mu_2}(x) = \frac{128x^2}{1575\mu^3} + \frac{64x^2}{315\mu^2} - \frac{64x^2\gamma}{315\mu^2}$$

$$v_\mu^{[3]}(x) = \gamma\left(\frac{128x^2}{1575\mu^3} + \frac{64x^2}{315\mu^2} - \frac{64x^2\gamma}{315\mu^2}\right) - \frac{128x^2}{1575\mu^3} - \frac{64x^2}{315\mu^2} + \frac{64x^2\gamma}{315\mu^2} + \frac{128x^2(-2+5(-1+\gamma)\mu)}{7875\mu^4}$$

⋮

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$v_\mu(x,\gamma,\mu) = v_\mu^{[0]}(x) + \sum_{i=1}^{+\infty} v_\mu^{[i]}(x)\left(\frac{1}{\gamma}\right)^i$$

$$= \frac{\frac{128x^2}{1575\mu^3}+\frac{64x^2}{315\mu^2}-\frac{64x^2\gamma}{315\mu^2}}{\gamma^2} - \frac{64x^2}{315\gamma\mu^2} + \frac{32x^2}{63\mu} + \frac{\gamma\left(\frac{128x^2}{1575\mu^3}+\frac{64x^2}{315\mu^2}-\frac{64x^2\gamma}{315\mu^2}\right)-\frac{128x^2}{1575\mu^3}-\frac{64x^2}{315\mu^2}+\frac{64x^2\gamma}{315\mu^2}+\frac{128x^2(-2+5(-1+\gamma)\mu)}{7875\mu^4}}{\gamma^3} + \cdots$$

When $\gamma = 1$ we have

$$v_\mu(x,\mu) = \frac{512x^2}{39375\mu^5} - \frac{256x^2}{7875\mu^4} + \frac{128x^2}{1575\mu^3} - \frac{64x^2}{315\mu^2} + \frac{32x^2}{63\mu} + \cdots$$

$$= \frac{32}{63\mu}\left(1 - \frac{2}{5\mu} + \frac{4}{25\mu^2} - \frac{8}{125\mu^3} + \cdots\right)x^2$$

$$= \frac{160}{5(2+3\mu)}x^2$$

Hence, the exact solution $v(x)$ of equation (56) can be obtain by

$$v(x) = \lim_{\mu\to 0} v_\mu(x,\mu) = \frac{80}{63}x^2$$

Using, (55) we obtain the exact solution of equation (54):

$$u(x) = \sqrt{\frac{80}{63}}x\ .$$

Example 6: Consider the following 2-dimensional linear first kind Fredholm integral equation [17]:

$$\frac{7}{12}(x+t) = \int_0^1\int_0^1 (x+t)yu(y,z)dy\,dz \quad (59)$$

By the regularization method, Eq. (59) can be transformed to the following 2-dimensional linear second kind Fredholm integral equation

$$u_\mu(x,t) = \frac{7}{12\mu}(x+t) - \frac{1}{\mu}\int_0^1\int_0^1 (x+t)yu_\mu(y,z)dy\,dz \quad (60)$$

For RqHAM solution, we choose the linear operator $L[\theta_\mu(x,t,q)] = \theta_\mu(x,t,q)$ and initial approximation $u_\mu^{[0]}(x,t) = \frac{7}{12\mu}(x+t)$

We construct the 0^{th} order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,t,q)-u_\mu^{[0]}(x,t)\right]$$
$$=q\,\Psi(x,\gamma)\left[\theta_\mu(x,t,q)-\frac{7}{12\mu}(x+t)\right.$$
$$\left.+\frac{1}{\mu}\int_0^1\int_0^1(x+t)y\,\theta_\mu(y,z,q)dy\,dz\right]$$

Take $\Psi(x,\gamma)=-1$ and the m^{th}order deformation equation is

$$L\left[u_\mu^{[m]}(x,t)-x_m u_\mu^{[m-1]}(x,t)\right]=-\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x,t)\right]\right],(61)$$

where

$$\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x,t)\right]\right]=u_\mu^{[m-1]}-\left(\frac{7}{12\mu}(x+t)\right)\left(1-\frac{1}{\gamma}x_m\right)$$
$$+\frac{1}{\mu}\int_0^1\int_0^1(x+t)y\,u_\mu^{[m-1]}(y,z)dy\,dz$$

Now the solution of m^{th}order deformation equation (61) become:

$$u_\mu^{[m]}(x,t)=x_m u_\mu^{[m-1]}(x,t)-L^{-1}\left[R_\mu^{[m]}\left[u_\mu^{[m-1]}(x,t)\right]\right]$$

Consequently, we obtain the components:

$$u_\mu^{[1]}(x,t)=-\frac{49(t+x)}{144\mu^2}$$

$$u_\mu^{[2]}(x,t)=\frac{343(t+x)}{1728\mu^3}+\frac{49(t+x)}{144\mu^2}-\frac{49(t+x)\gamma}{144\mu^2}$$

$$u_\mu^{[3]}(x,t)=\gamma\left(\frac{343(t+x)}{1728\mu^3}+\frac{49(t+x)}{144\mu^2}-\frac{49(t+x)\gamma}{144\mu^2}\right)-\frac{343(t+x)}{1728\mu^3}-\frac{49(t+x)}{144\mu^2}$$
$$+\frac{49(t+x)\gamma}{144\mu^2}+\frac{343(t+x)(-7+12(-1+\gamma)\mu)}{20736\mu^4}$$

$$\vdots$$

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$u_\mu(x,\gamma,\mu) = u_\mu^{[0]}(x,t) + \sum_{i=1}^{+\infty} u_\mu^{[i]}(x,t)\left(\frac{1}{\gamma}\right)^i$$

$$= \frac{\frac{343(t+x)}{1728\mu^3}+\frac{49(t+x)}{144\mu^2}-\frac{49(t+x)\gamma}{144\mu^2}}{\gamma^2} - \frac{49(t+x)}{144\gamma\mu^2} + \frac{7(t+x)}{12\mu} + \frac{\gamma\left(\frac{343(t+x)}{1728\mu^3}+\frac{49(t+x)}{144\mu^2}-\frac{49(t+x)\gamma}{144\mu^2}\right)-\frac{343(t+x)}{1728\mu^3}-\frac{49(t+x)}{144\mu^2}+\frac{49(t+x)\gamma}{144\mu^2}+\frac{343(t+x)(-7+12(-1+\gamma)\mu)}{20736\mu^4}}{\gamma^3} + \ldots$$

When $\gamma = 1$ we have

$$u_\mu(x,t) = \frac{16807(t+x)}{248832\mu^5} - \frac{2401(t+x)}{20736\mu^4} + \frac{343(t+x)}{1728\mu^3} - \frac{49(t+x)}{144\mu^2} + \frac{7(t+x)}{12\mu} + \cdots$$

$$= \frac{7}{12\mu}(x+t)\left(1 - \frac{7}{12\mu} + \frac{49}{144\mu^2} - \frac{434}{1728\mu^3} + \frac{2401}{20736\mu^4} + \cdots\right)$$

$$= \frac{7}{(12\mu+7)}(x+t)$$

Hence, the exact solution $u(x,t)$ of equation (59) can be obtain by

$$u(x,t) = \lim_{\mu\to 0} u_\mu(x,t) = (x+t)$$

Example 7: Consider the following 2-dimensional nonlinear first kind Fredholm integral equation [17]:

$$2(x+t) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(x+t)\sin(u(y,z))dy\,dz \quad (62)$$

To solve this equation we first set

$$v(x,t) = \sin(u(x,t)) \quad (63)$$

to carry out Eq.(62) into the following 2-dimensional linear first kind Fredholm integral equation

$$2(x+t) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(x+t)v(y,z)dy\,dz \quad (64)$$

By the regularization method, Eq. (64) can be transformed to the following 2-dimensional linear second kind Fredholm integral equation

$$v_\mu(x,t) = \frac{2}{\mu}(x+t) - \frac{1}{\mu}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(x+t)v_\mu(y,z)dy\,dz \quad (65)$$

For RqHAM solution, we choose the linear operator $L[\theta_\mu(x,t,q)] = \theta_\mu(x,t,q)$ and initial approximation $v_\mu^{[0]}(x,t) = \frac{2}{\mu}(x+t)$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,t,q) - v_\mu^{[0]}(x,t)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,t,q) - \frac{2}{\mu}(x+t) + \frac{1}{\mu}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(x+t)\theta_\mu(y,z,q)dy\,dz\right]$$

Take $\Psi(x,\gamma) = -1$ and the m^{th}order deformation equation is

$$L\left[v_\mu^{[m]}(x,t) - x_m v_\mu^{[m-1]}(x,t)\right] = -\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right], (66)$$

where

$$\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right] = v_\mu^{[m-1]} - \left(\frac{2}{\mu}(x+t)\right)\left(1-\frac{1}{\gamma}x_m\right) + \frac{1}{\mu}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(x+t)$$
$$v_\mu^{[m-1]}(y,z)dy\,dz$$

Now the solution of m^{th}order deformation equation (66) become:

$$v_\mu^{[m]}(x,t) = x_m v_\mu^{[m-1]}(x,t) - L^{-1}\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right]$$

Consequently, we obtain the components:

$v_\mu^{[1]}(x,t) = -\frac{\pi^3(t+x)}{4\mu^2}$

$$v_\mu^{[2]}(x,t) = \frac{\pi^6(t+x)}{32\mu^3} + \frac{\pi^3(t+x)}{4\mu^2} - \frac{\pi^3(t+x)\gamma}{4\mu^2}$$

$$v_\mu^{[3]}(x,t) = \gamma\left(\frac{\pi^6(t+x)}{32\mu^3} + \frac{\pi^3(t+x)}{4\mu^2} - \frac{\pi^3(t+x)\gamma}{4\mu^2}\right) - \frac{\pi^6(t+x)}{32\mu^3} - \frac{\pi^3(t+x)}{4\mu^2} + \frac{\pi^3(t+x)\gamma}{4\mu^2} - \frac{\pi^6(t+x)(\pi^3 - 8(-1+\gamma)\mu)}{256\mu^4}$$

⋮

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$v_\mu(x,t,\gamma,\mu) = v_\mu^{[0]}(x,t) + \sum_{i=1}^{+\infty} v_\mu^{[i]}(x,t)\left(\frac{1}{\gamma}\right)^i$$

$$= \frac{\frac{\pi^6(t+x)}{32\mu^3}+\frac{\pi^3(t+x)}{4\mu^2}-\frac{\pi^3(t+x)\gamma}{4\mu^2}}{\gamma^2} - \frac{\pi^3(t+x)}{4\gamma\mu^2} + \frac{2(t+x)}{\mu} + \frac{\gamma\left(\frac{\pi^6(t+x)}{32\mu^3}+\frac{\pi^3(t+x)}{4\mu^2}-\frac{\pi^3(t+x)\gamma}{4\mu^2}\right)-\frac{\pi^6(t+x)}{32\mu^3}-\frac{\pi^3(t+x)}{4\mu^2}+\frac{\pi^3(t+x)\gamma}{4\mu^2}-\frac{\pi^6(t+x)(\pi^3-8(-1+\gamma)\mu)}{256\mu^4}}{\gamma^3} + \cdots$$

When $\gamma = 1$ we have

$$v_\mu(x,t) = \frac{\pi^{12}(t+x)}{2048\mu^5} - \frac{\pi^9(t+x)}{256\mu^4} + \frac{\pi^6(t+x)}{32\mu^3} - \frac{\pi^3(t+x)}{4\mu^2} + \frac{2(t+x)}{\mu} + \cdots$$

$$= \frac{2}{\mu}(x+t)\left(1 - \frac{\pi^3}{8\mu} + \frac{\pi^6}{64\mu^2} - \frac{\pi^9}{512\mu^3} + \frac{\pi^{12}}{4096\mu^4} + \cdots\right)$$

$$= \frac{2}{\mu}(x+t)\frac{8\mu}{8\mu+\pi^3}$$

$$= \frac{16}{8\mu+\pi^3}(x+t)$$

Hence, the exact solution $v(x,t)$ of equation (64) can be obtain by

$$v(x,t) = \lim_{\mu\to 0} v_\mu(x,t) = \frac{16}{\pi^3}(x+t)$$

Using, (63) we obtain the exact solution of equation (62):

$$u(x,t) = \sin^{-1}\left(\frac{16}{\pi^3}(x+t)\right).$$

Example 8: Consider the following 2-dimensional nonlinear first kind Fredholm integral equation [1]

$$\frac{x}{6(1+t)} = \int_0^1\int_0^1 \frac{x}{1+t}\,(1+\mathrm{y}+\mathrm{z})u^2(y,z)dy\,dz \qquad (67)$$

To solve this equation we first set

$$v(y,z) = u^2(y,z) \qquad (68)$$

to carry out Eq.(67) into the following 2-dimensional linear first kind Fredholm integral equation

$$\frac{x}{6(1+t)} = \int_0^1\int_0^1 \frac{x}{1+t}\,(1+\mathrm{y}+\mathrm{z})v(y,z)dy\,dz \qquad (69)$$

By the regularization method, Eq. (69) can be transformed to the following 2-dimensional linear second kind Fredholm integral equation

$$v_\mu(x,t) = \frac{x}{6\mu(1+t)} - \frac{1}{\mu}\int_0^1\int_0^1 \frac{x}{1+t} v_\mu(y,z)\,(1+\mathrm{y}+\mathrm{z})dy\,dz \qquad (70)$$

For RqHAM solution, we choose the linear operator $L[\theta_\mu(x,t,q)] = \theta_\mu(x,t,q)$ and initial approximation $v_\mu^{[0]}(x,t) = \frac{x}{6\mu(1+t)}$

We construct the 0th order deformation equation

$$(1-\gamma q)L\left[\theta_\mu(x,t,q) - v_\mu^{[0]}(x,t)\right] = q\,\Psi(x,\gamma)\left[\theta_\mu(x,t,q) - \frac{x}{6\mu(1+t)} + \frac{1}{\mu}\int_0^1\int_0^1 \frac{x}{1+t}\,(1+\mathrm{y}+\mathrm{z})\theta_\mu(y,z,q)dy\,dz\right]$$

Take $\Psi(x,\gamma) = -1$ and the m^{th} order deformation equation is

$$L\left[v_\mu^{[m]}(x,t) - x_m v_\mu^{[m-1]}(x,t)\right] = -\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right], \qquad (71)$$

where

$$\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right] = v_\mu^{[m-1]} - \left(\frac{x}{6\mu(1+t)}\right)\left(1 - \frac{1}{\gamma}x_m\right) + \frac{1}{\mu}\int_0^1\int_0^1 \frac{x}{1+t}(1+\mathrm{y}+\mathrm{z})v_\mu^{[m-1]}(y,z)dy\,dz$$

Now the solution of m^{th} order deformation equation (71) become:

$$v_\mu^{[m]}(x,t) = x_m v_\mu^{[m-1]}(x,t) - L^{-1}\left[R_\mu^{[m]}\left[v_\mu^{[m-1]}(x,t)\right]\right]$$

Consequently, we obtain the components:

$$v_\mu^{[1]}(x,t) = -\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2}$$

$$v_\mu^{[2]}(x,t) = \frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3}$$

$$v_\mu^{[3]}(x,t) = -\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3} + \frac{x(-3+6(-1+\gamma)\mu-\mathrm{Log}[4])(3+\mathrm{Log}[4])^2}{1296(1+t)\mu^4} + \gamma(\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3})$$

⋮

Then the approximate solution obtaining by RqHAM can be written as the following series:

$$v_\mu(x,t,\gamma,\mu) = v_\mu^{[0]}(x,t) + \sum_{i=1}^{+\infty} v_\mu^{[i]}(x,t)\left(\frac{1}{\gamma}\right)^i$$

$$= \frac{x}{6(1+t)\mu} - \frac{x(3+\mathrm{Log}[4])}{36(1+t)\gamma\mu^2} + \frac{\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3}}{\gamma^2} + \frac{1}{\gamma^3}(-\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3} + \frac{x(-3+6(-1+\gamma)\mu-\mathrm{Log}[4])(3+\mathrm{Log}[4])^2}{1296(1+t)\mu^4} + \gamma(\frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} - \frac{x\gamma(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3}))+\ldots$$

When $\gamma = 1$ we have

$$v_\mu(x,t) = \frac{x}{6(1+t)\mu} - \frac{x(3+\mathrm{Log}[4])}{36(1+t)\mu^2} + \frac{x(3+\mathrm{Log}[4])^2}{216(1+t)\mu^3} + \frac{x(-3-\mathrm{Log}[4])(3+\mathrm{Log}[4])^2}{1296(1+t)\mu^4} + \cdots$$

$$= \frac{x}{6(1+t)\mu}(\frac{6\mu}{6\mu+3+\log[4]})$$

Hence, the exact solution $v(x,t)$ of equation (69) can be obtain by

$$v(x,t) = \lim_{\mu\to 0} v_\mu(x,t) = \frac{x}{(1+t)(3+\log[4])}$$

Using, (68) we obtain the exact solution of equation (67):

$$u(x,t) = \pm\sqrt{\frac{x}{(1+t)(3+\log[4])}}$$

5. Conclusion

In this article, new powerful modification of q-homotopy analysis method namely regularization q-homotopy analysis method(RqHAM) was proposed to solve (1 and 2)-

dimensional linear and nonlinear first kind Fredholm integral equations. The regularization q-homotopy analysis method is a combination of the regularization method and q-homotopy analysis method. Illustrative examples showed, this method is simple and it often gives the exact solution. Depending on the results of this work, we can say that the regularization q-homotopy analysis method is very effective to solve first kind Fredholm integral equations and it may be used to solve multiple first kind Fredholm integral equations.

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