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Legendre inversions and Ultra-hypergeometric series identities

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Abstract

In this paper, we extension of hypergeometric series to obtain a new Ultra-hypergeometric series, we establish three terminating Ultra-hypergeometric series identities, by means of Legendre inverse series relations. By using the linear combination between these three identities, we can obtain new terminating Ultra-hypergeometric series identities.

Keywords : Ultra-hypergeometric series; Legendre inversions; convolution formula.

1 Introduction

For complex number x and a natural number n, denote the α-shifted-factorial by(see[5])

$$\langle x|\alpha\rangle_0 = 1, \langle x|\alpha\rangle_n = x(x-\alpha)(x-2\alpha)\cdots(x-n\alpha+\alpha), \tag{1}$$

$$(x|\alpha)_0 = 1, (x|\alpha)_n = x(x+\alpha)(x+2\alpha)\cdots(x+n\alpha-\alpha). \tag{2}$$

Evidently, the classical shifted-factorial is $\alpha = 1$

$$\langle x\rangle_0 = 1, \langle x\rangle_n = x(x-1)(x-2)\cdots(x-n+1),$$

$$(x)_0 = 1, (x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

By formula (1) we give the following definition.

Definition 1 For $(x,y) \in C^2$, let us denote

$$\left(\begin{matrix} x \\ y \end{matrix}\middle|\alpha\right) = \begin{cases} \frac{\langle x|\alpha\rangle_y}{y!}, & x \in C^2, y \in N \\ 0, & x \in C^2, y \notin N \end{cases} \tag{3}$$

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In particular, it is obvious that the binomial coefficients are given by $\alpha = 1, x, y \in N$,

$$\binom{n}{k} = \frac{\langle n\rangle_k}{k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Although we have two kinds of identities by Definition 1

$$\left(\begin{array}{c} x+n\alpha \\ n-k \end{array}\middle|\alpha\right) = \frac{(\alpha+x|\alpha)_n}{(n-k)!(\alpha+x|\alpha)_k}, \tag{4}$$

$$\left(\begin{array}{c} x+n\alpha \\ \varepsilon+n+k \end{array}\middle|\alpha\right) = (-1)^{(\varepsilon+k)}\frac{(\alpha+x|\alpha)_n(-x|\alpha)_{k+\varepsilon}}{(\varepsilon+n+k)!}. \tag{5}$$

Next, we give the definition of α-hypergeometric series.

Definition 2 A Ultra-hypergeometric series is a series Σc_n, such that c_{n+1}/c_n is a rational function of n. On factorizing the polynomials in n, we obtain

$$\frac{c_{n+1}}{c_n} = \frac{(a_1+n\alpha_1)(a_2+n\alpha_2)\cdots(a_p+n\alpha_p)z}{(b_1+n\beta_1)(b_2+n\beta_2)\cdots(b_q+n\beta_q)(n+1)}, \tag{6}$$

from (6), we have

$$\sum_{n=0}^{\infty} c_n = c_0\sum_{n=0}^{\infty}\frac{(a_1|\alpha_1)_n(a_2|\alpha_2)_n\cdots(a_p|\alpha_p)_n}{(b_1|\beta_1)_n(b_2|\beta_2)_n\cdots(b_p|\beta_q)_n}\frac{z^n}{n!} = c_0\ {}_pF_q\left(\begin{array}{c}(a_1|\alpha_1),(a_2|\alpha_2),\cdots,(a_p|\alpha_p)\\(b_1|\beta_1),(b_2|\beta_2),\cdots,(b_q|\beta_q)\end{array};z\right).$$

Here the b_i are not a non-negative integer multiple of β_i or zero, as that would make denominator zero.

For typographical reasons, we shall sometimes denote the sum on the right side of (7) by $\alpha\text{-}{}_pF_q((a_1|\alpha_1),(a_2|\alpha_2),\cdots,(a_p|\alpha_p);(b_1|\beta_1),(b_2|\beta_2),\cdots,(b_q|\beta_q);z)$ or by $\alpha\text{-}{}_pF_q$.

In particular, in formula (7) when $\alpha_i = \beta_i = 1$ is classical hypergeometric series (see[1,2,3])

$$\sum_{n=0}^{\infty} c_n = c_0\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!} = c_0\ {}_pF_q\left(\begin{array}{c}a_1,a_2,\cdots,a_p\\b_1,b_2,\cdots,b_q\end{array};z\right).$$

Lemma 1[4] We can introduction Legendre two pairs inversions, let p be a fixed real number the follow inverse series relations hold:

$$f(n) = \sum_{n=0}^{\infty}(-1)^k\binom{p+2n}{n-k}\frac{p+2k}{p+2n}g(k), \tag{8}$$

$$g(n) = \sum_{n=0}^{\infty}(-1)^k\binom{p+n+k-1}{n-k}f(k). \tag{9}$$

We shall also use another pair of inversions

$$F(n) = \sum_{n=0}^{\infty}(-1)^k\binom{p+2n}{n-k}G(k), \tag{10}$$

$$G(n) = \sum_{n=0}^{\infty}(-1)^k\binom{p+n+k}{n-k}\frac{p+2n}{p+n+k}F(k), \tag{11}$$

this follows directly from the replacements

$$F(k) = (p+2k)f(k),\ \ and\ G(k) = (p+2k)g(k).$$

Lemma 2[5] For generalized Vandermonde convolution formula

$$\left(\begin{array}{c} x+y \\ m \end{array}\middle|\alpha\right) = \sum_{k=0}^{m}\left(\begin{array}{c} x \\ k \end{array}\middle|\alpha\right)\left(\begin{array}{c} y \\ m-k \end{array}\middle|\alpha\right). \tag{12}$$

Especially $\alpha = 1$ is the Chu-vandermonde convolution formula(see[6])

$$\binom{x+y}{m} = \sum_{k=0}^{m} \binom{x}{k}\binom{y}{m-k}.$$

2 Terminating Ultra-hypergeometric series identity

This section mainly uses the Legendre inversion relationship to give three useful Ultra-hypergeometric series identities.

Theorem 1 For $\delta = 0, 1$terminating Ultra-hypergeometric series identity

$$\begin{aligned}&{}_4F_3\left(\begin{matrix} -n, n+1+\delta, (\frac{a}{2}|\alpha), (\frac{a+\alpha}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha)\end{matrix} ;1\right) \\ &= \frac{n!}{(a-2c)(\alpha-a|\alpha)_{\delta+1}(\delta+1)_n}\left\{\frac{(-c|\alpha)_{n+1+\delta}}{(\alpha+a-c|\alpha)_n} - \frac{(c-a|\alpha)_{n+1+\delta}}{(\alpha+c|\alpha)_n}\right\}.\end{aligned}$$

Proof Applying lemma 2 it is not difficult to show that

$$\frac{a-2c}{a+2n\alpha}\left(\begin{matrix} a+2n\alpha \\ \delta+1+2n\end{matrix}\middle|\alpha\right) = \sum_{k=0}^{\delta+1+2n} \frac{\delta+1+2n-2k}{\delta+1+2n}\left(\begin{matrix} c+n\alpha \\ k\end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+1+2n-k\end{matrix}\middle|\alpha\right).$$

In the above formula, then for $\delta = 0, 1$, splitting the last sum into two parts and then performing replacements $k \longrightarrow n-k$, $k \longrightarrow \delta+1+n+k$, respectively, for the first and the second sum, we can manipulate the sum as follow:

$$\begin{aligned}\frac{a-2c}{a+2n\alpha}\left(\begin{matrix} a+2n\alpha \\ \delta+1+2n\end{matrix}\middle|\alpha\right) &= \left\{\sum_{k=0}^{n} + \sum_{k=\delta+1+n}^{\delta+1+2n}\right\}\frac{\delta+1+2n-2k}{\delta+1+2n}\left(\begin{matrix} c+n\alpha \\ k\end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+1+2n-k\end{matrix}\middle|\alpha\right) \\ &= \sum_{k=0}^{n}\frac{\delta+1+2k}{\delta+1+2n}\left\{\left(\begin{matrix} c+n\alpha \\ n-k\end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+1+n+k\end{matrix}\middle|\alpha\right) - \left(\begin{matrix} c+n\alpha \\ n-k\end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+1+n+k\end{matrix}\middle|\alpha\right)\right\}.\end{aligned}$$

Applying (4), (5) it is not difficult to binomial coefficient identity

$$\frac{(2c-a)(a|\alpha)_{2n}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n} = \sum_{k=0}^{n}(-1)^k\binom{\delta+1+2n}{n-k}\frac{\delta+1+2k}{\delta+1+2n}\left\{\frac{(c-a|\alpha)_{k+\delta+1}}{(\alpha+c|\alpha)_k} - \frac{(-c|\alpha)_{k+\delta+1}}{(\alpha+a-c|\alpha)_k}\right\},$$

the last identity matches (8) $p = \delta+1$ and

$$f(n) = \frac{(2c-a)(a|\alpha)_{2n}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n},$$

$$g(k) = \frac{(c-a|\alpha)_{k+\delta+1}}{(\alpha+c|\alpha)_k} - \frac{(-c|\alpha)_{k+\delta+1}}{(\alpha+a-c|\alpha)_k}.$$

The dual relation corresponding to (9) gives us the following identity

$$\frac{(c-a|\alpha)_{n+\delta+1}}{(\alpha+c|\alpha)_n} - \frac{(-c|\alpha)_{n+\delta+1}}{(\alpha+a-c|\alpha)_n} = \sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{(2c-a)(a|\alpha)_{2k}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_k(\alpha+a-c|\alpha)_k}.$$

Rewriting the binomial coefficient in terms of factions(see[7])

$$\binom{\delta+n+k}{n-k} = (-1)^k\frac{(\delta+1)_n}{n!}\frac{(-n)_k(\delta+1+n)_k}{(\delta+1)_{2k}},$$

So,we easily find the following terminating Ultra-hypergeometric series identity:

$$
{}_4F_3\left(\begin{matrix} -n, n+1+\delta, (\frac{a}{2}|\alpha), (\frac{a+\alpha}{2}|\alpha) \\ \frac{1}{2}+\delta, \alpha+c|\alpha, \alpha+a-c|\alpha \end{matrix}; 1\right)
= \frac{n!}{(a-2c)(\alpha-a|\alpha)_\delta(\delta+1)_n}\left\{\frac{(-c|\alpha)_{n+1+\delta}}{(\alpha+a-c|\alpha)_n} - \frac{(c-a|\alpha)_{n+1+\delta}}{(\alpha+c|\alpha)_n}\right\}.
$$

Corollary 2 In Theorem 1, when $\alpha = 1$, for $\delta = 0, 1$, we have terminating balanced hypergeometric series identity:

$$
{}_4F_3\left(\begin{matrix} -n, n+1+\delta, \frac{a}{2}, \frac{a+1}{2} \\ \delta+\frac{1}{2}, c, 1+a-c \end{matrix}; 1\right) = \frac{n!}{(\delta+1)_n(a-2c)(1-a)_\delta}\left\{\frac{(-c)_{n+1+\delta}}{(1+a-c)_n} - \frac{(c-a)_{n+1+\delta}}{(1+c)_n}\right\}.
$$

this is the corollary of Theorem 1 in the Reference [7].

Corollary 3 In Theorem 1,when $\alpha = 0$ and $\delta = 0$ we can established generating functions identity as follows:

$$
\Psi(1,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\delta+n)_k(\frac{a}{2})^{2k}}{(\delta+\frac{1}{2})_k[c(a-c)]^k}\frac{1}{k!}\frac{t^n}{n!} = \frac{c}{2c-a}\exp(\frac{c}{c-a}t) - \frac{c-a}{a-2c}\exp(\frac{c-a}{c}t).
$$

Theorem 4 For $\delta = 0, 1$ terminating Ultra-hypergeometric series identity

$$
{}_4F_3\left(\begin{matrix} -n, n+\delta, (\frac{a}{2}|\alpha), (\frac{a+\alpha}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{matrix}; 1\right)
= \begin{cases} \frac{a-2c+2n\alpha}{2(a-2c)}\frac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n} + \frac{a-2c-2n\alpha}{2(a-2c)}\frac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta = 0 \\ \frac{1}{(1+2n)(\alpha-a)}\left\{\frac{[a-2c+(1+2n)\alpha](-c|\alpha)_{n+1}}{(a-2c)(\alpha+a-c|\alpha)_n} + \frac{[a-2c-(1+2n)\alpha](c-a|\alpha)_{n+1}}{(a-2c)(\alpha+c|\alpha)_n}\right\}, & \delta = 1 \end{cases}
$$

Proof Recall again applying lemma 2 it is not difficult to show that

$$
\left(\begin{matrix} a+2n-\alpha \\ \delta+2n \end{matrix}\middle|\alpha\right) = \sum_{k=0}^{\delta+2n}\frac{\delta+a-2c+2n-2k}{a-2c}\left(\begin{matrix} c+n\alpha \\ \delta+2n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ k \end{matrix}\middle|\alpha\right).
$$

In the above formula then for $\delta = 1, 2$, splitting the last sum into two parts and then performing replacements $k \longrightarrow n-k$, $k \longrightarrow \delta+n+k$, respectively, for the first and the second sum, we can manipulate the sum as follow:

$$
\begin{aligned}
&\left(\begin{matrix} a+2n\alpha-\alpha \\ \delta+2n \end{matrix}\middle|\alpha\right) + (1-\delta)\left(\begin{matrix} c+n\alpha \\ n \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ n \end{matrix}\middle|\alpha\right) \\
&= \left\{\sum_{k=0}^{n} + \sum_{k=\delta+n}^{\delta+2n}\right\}\frac{a-2c+(2n-2k+\delta)\alpha}{a-2c}\left(\begin{matrix} c+n\alpha \\ \delta+2n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ k \end{matrix}\middle|\alpha\right) \\
&= \sum_{k=0}^{n}\left\{\frac{a-2c+(2k+\delta)\alpha}{a-2c}\left(\begin{matrix} c+n\alpha \\ \delta+n+k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ n-k \end{matrix}\middle|\alpha\right)\right. \\
&\qquad \left. + \frac{a-2c-(2k+\delta)\alpha}{a-2c}\left(\begin{matrix} c+n\alpha \\ n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+n+k \end{matrix}\middle|\alpha\right)\right\}.
\end{aligned}
$$

Applying (4), (5) it is not difficult to binomial coefficient identity

$$\frac{(a|\alpha)_{2n}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n}+(1-\delta)\binom{\delta+2n}{n}$$

$$=\sum_{k=0}^{n}(-1)^k\binom{\delta+2n}{n-k}\left\{\frac{a-2c+(\delta+2k)\alpha}{a-2c}\frac{(-c|\alpha)_{k+\delta}}{(\alpha+a-c|\alpha)_k}+\frac{a-2c-(\delta+2k)\alpha}{a-2c}\frac{(c-a|\alpha)_{k+\delta}}{(\alpha+c|\alpha)_k}\right\},$$

the last identity matches (10) $p=\delta$ and

$$F(n)=\frac{(a|\alpha)_{2n}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n}+(1-\delta)\binom{\delta+2n}{n},$$

$$G(k)=\left\{\frac{a-2c+(\delta+2k)\alpha}{a-2c}\frac{(-c|\alpha)_{k+\delta}}{(\alpha+a-c|\alpha)_k}+\frac{a-2c-(\delta+2k)\alpha}{a-2c}\frac{(c-a|\alpha)_{k+\delta}}{(\alpha+c|\alpha)_k}\right\}.$$

The dual relation corresponding to (11) gives us the following identity

$$\frac{a-2c+(\delta+2n)\alpha}{a-2c}\frac{(-c|\alpha)_{n+\delta}}{(\alpha+a-c|\alpha)_n}-\frac{a-2c-(\delta+2n)\alpha}{a-2c}\frac{(c-a|\alpha)_{n+\delta}}{(\alpha+c|\alpha)_n}$$

$$=\sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{\delta+2n}{\delta+n+k}\left\{\frac{(a|\alpha)_{2k}(\alpha-a|\alpha)_\delta}{(\alpha+c|\alpha)_k(\alpha+a-c|\alpha)_k}+(1-\delta)\binom{2k}{k}\right\}.$$

The following expression for $\delta=0,1$ results is zero[see7]

$$\sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{\delta+2n}{\delta+n+k}(1-\delta)\binom{2k}{k}=0.$$

Hence we have the Ultra-hypergeometric series identity in Theorem 4.

Corollary 5 In Theorem 4, when $\alpha=1$, for $\delta=0,1$ terminating 2-balanced hypergeometric series identity:

$$ {}_4F_3\left(\begin{matrix}-n,\ n+\delta,\ \frac{a}{2},\ \frac{a+1}{2}\\ \delta+\frac{1}{2},1+c,1+a-c\end{matrix};1\right)$$

$$=\begin{cases}\dfrac{a-2c+2n}{a-2c}\dfrac{(-c)_n}{(1+a-c)_n}+\dfrac{a-2c-2n}{a-2c}\dfrac{(c-a)_n}{(1+c)_n}, & \delta=0\\ \dfrac{1}{(2n+1)(1-a)}\left\{\dfrac{a-2c+1+2n}{a-2c}\dfrac{(-c)_{n+1}}{(1+a-c)_n}+\dfrac{a-2c-(2n+1)}{a-2c}\dfrac{(c-a)_{n+1}}{(1+c)_n}\right\}, & \delta=1\end{cases}$$

this is the corollary of Theorem 1 in the Reference [7].

Corollary 6 In Theorem 4 when $\alpha=0$ and $\delta=0$ we can established generating functions identity as follows:

$$\Psi(1,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-n)_k(n)_k(\frac{a}{2})^{2k}}{(\frac{1}{2})_k[c(a-c)]^k}\frac{1}{k!}\frac{t^n}{n!}=\frac{1}{2}\exp\left(\frac{c}{c-a}t\right)+\frac{1}{2}\exp\left(\frac{c-a}{c}t\right).$$

Theorem 7 For $\delta=0,1$ terminating Ultra-hypergeometric series identity

$$ {}_4F_3\left(\begin{matrix}-n,\delta+n,(\frac{a+\alpha}{2}|\alpha),(\alpha+\frac{a}{2}|\alpha)\\ \delta+\frac{1}{2},(\alpha+c|\alpha),(\alpha+a-c|\alpha)\end{matrix};1\right)$$

$$=\begin{cases}\dfrac{1}{2}\dfrac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n}+\dfrac{1}{2}\dfrac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta=0\\ \dfrac{1}{a(1+2n)}\dfrac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n}+\dfrac{1}{a(1+2n)}\dfrac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}, & \delta=1\end{cases}$$

Proof Recall again applying lemma 2 it is not difficult to show that

$$\left(\begin{matrix} a+2n\alpha \\ \delta+2n \end{matrix}\middle|\alpha\right)=\sum_{k=0}^{\delta+2n}\left(\begin{matrix} c+n\alpha \\ \delta+2n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ k \end{matrix}\middle|\alpha\right).$$

In the above formula then for $\delta=1,2$, splitting the last sum into two parts and then performing replacements $k\longrightarrow n-k$, $k\longrightarrow \delta+n+k$, respectively, for the first and the second sum, we can manipulate the sum as follow:

$$\left(\begin{matrix} a+2n\alpha \\ \delta+2n \end{matrix}\middle|\alpha\right)+(1-\delta)\left(\begin{matrix} c+n\alpha \\ n \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ n \end{matrix}\middle|\alpha\right)$$
$$=\left\{\sum_{k=0}^{n}+\sum_{k=\delta+n}^{\delta+2n}\right\}\left(\begin{matrix} c+n\alpha \\ \delta+2n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ k \end{matrix}\middle|\alpha\right)$$
$$=\sum_{k=0}^{n}\left\{\left(\begin{matrix} c+n\alpha \\ \delta+n+k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ n-k \end{matrix}\middle|\alpha\right)+\left(\begin{matrix} c+n\alpha \\ n-k \end{matrix}\middle|\alpha\right)\left(\begin{matrix} a-c+n\alpha \\ \delta+n+k \end{matrix}\middle|\alpha\right)\right\}.$$

Recall again applying (4), (5) it is not difficult to binomial coefficient identity

$$\frac{(\alpha+a|\alpha)_{2n}(a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n}+(1-\delta)\binom{\delta+2n}{n}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{\delta+2n}{n-k}\left\{\frac{(-c|\alpha)_{k+\delta}(-1)^\delta}{(\alpha+a-c|\alpha)_k}-\frac{(c-a|\alpha)_{k+\delta}(-1)^\delta}{(\alpha+c|\alpha)_k}\right\}.$$

the last identity matches (10) $p=\delta$ and

$$F(n)=\frac{(\alpha+a|\alpha)_{2n}(a|\alpha)_\delta}{(\alpha+c|\alpha)_n(\alpha+a-c|\alpha)_n}+(1-\delta)\binom{\delta+2n}{n},$$
$$G(k)=\left\{\frac{(-c|\alpha)_{k+\delta}(-1)^\delta}{(\alpha+a-c|\alpha)_k}+\frac{(c-a|\alpha)_{k+\delta}(-1)^\delta}{(\alpha+c|\alpha)_k}\right\}.$$

The dual relation corresponding to (11) given us the following identity

$$\frac{(-c|\alpha)_{n+\delta}(-1)^\delta}{(\alpha+a-c|\alpha)_n}+\frac{(c-a|\alpha)_{n+\delta}(-1)^\delta}{(\alpha+c|\alpha)_n}$$
$$=\sum_{k=0}^{n}(-1)^k\frac{\delta+2n}{\delta+n+k}\binom{\delta+n+k}{n-k}\left\{\frac{(\alpha+a|\alpha)_{2k}(a|\alpha)_\delta}{(\alpha+c|\alpha)_k(\alpha+a-c|\alpha)_k}+(1-\delta)\binom{\delta+2k}{k}\right\}.$$

The following expression for $\delta=0,1$ results is zero[see7]

$$\sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{\delta+2n}{\delta+n+k}(1-\delta)\binom{2k}{k}=0.$$

Hence we have the Ultra-hypergeometric series identity in Theorem 7.

Corollary 8 In Theorem 7, when $\alpha=1$, for $\delta=0,1$, we have terminating balanced hypergeometric series identity:

$${}_4F_3\left(\begin{matrix} -n,\delta+n,\frac{a+1}{2},1+\frac{a}{2} \\ \delta+\frac{1}{2},1+c,1+a-c \end{matrix};1\right)=\begin{cases}\frac{1}{2}\left\{\frac{(-c)_n}{(1+a-c)_n}+\frac{(c-a)_n}{(1+c)_n}\right\}, & \delta=0\\ \frac{1}{a(1+2n)}\frac{(-c)_{n+1}}{(1+a-c)_n}+\frac{1}{a(1+2n)}\frac{(c-a)_{n+1}}{(1+c)_n}, & \delta=1\end{cases}$$

3 Linear combinations and related Ultra-hypergeometric formulae

This section mainly uses the linear combination between the three theorems in the previous section to obtain new terminating Ultra-hypergeometric series identities.

Theorem 9 For $\delta = 0, 1$ terminating Ultra-hypergeometric series identity

$$ {}_5F_4\left(\begin{array}{c} -n, n+\delta, (n+\delta\alpha|\alpha), (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (n+\delta|\alpha), (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) $$

$$ = \begin{cases} \left\{\frac{(1-\alpha)(a-2c+2n\alpha)}{2(a-2c)} + \frac{(-c+n\alpha)\alpha}{(a-2c)(\alpha-a)}\right\}\frac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n} + \left\{\frac{(1-\alpha)(a-2c-2n\alpha)}{2(a-2c)} - \frac{(c-a+n\alpha)\alpha}{(a-2c)(\alpha-a)}\right\}\frac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta=0 \\ \left\{\frac{(1-\alpha)(a-2c+2n\alpha+\alpha)}{(2n+1)(a-2c)(\alpha-a)} + \frac{(-c+n\alpha+\alpha)\alpha}{(a-2c)(n+1)(\alpha-a|\alpha)_2}\right\}\frac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n} + \left\{\frac{(1-\alpha)(a-2c-2n\alpha-\alpha)}{(2n+1)(a-2c)(\alpha-a)} - \right. & \\ \left. \frac{(c-a+n\alpha+\alpha)\alpha}{(a-2c)(n+1)(\alpha-a|\alpha)_2}\right\}\frac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}, & \delta=1 \end{cases} $$

Proof Because

$$ {}_5F_4\left(\begin{array}{c} -n, n+\delta, (\delta+n+\alpha|\alpha), (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\delta+n|\alpha)(\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) $$

$$ = (1-\alpha){}_4F_3\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) + \alpha{}_4F_3\left(\begin{array}{c} -n, n+1+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right). $$

We then use Theorem 1 and Theorem 4 to immediately get the conclusion of Theorem 9.

Theorem 10 For $\delta = 0, 1$ terminating Ultra-hypergeometric series identity

$$ {}_5F_4\left(\begin{array}{c} -n, n, 2n+1+\delta, (\frac{a}{2}|\alpha), (\frac{a+\alpha}{2}|\alpha) \\ \frac{1}{2}, 2n+\delta, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) $$

$$ = \begin{cases} \left\{\frac{a-2c+2n\alpha}{4(a-2c)} + \frac{-c+n\alpha}{2(a-2c)(\alpha-a)}\right\}\frac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n} + \left\{\frac{a-2c-2n\alpha}{4(a-2c)} - \frac{c-a+n\alpha}{2(a-2c)(\alpha-a)}\right\}\frac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta=0 \\ \left\{\frac{a-2c+(2n+1)\alpha}{2(1+2n)(a-2c)(\alpha-a)} + \frac{-c+n\alpha+\alpha}{2(n+1)(a-2c)(\alpha-a|\alpha)_2}\right\}\frac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n} + \left\{\frac{a-2c-(2n+1)\alpha}{2(2n+1)(a-2c)(\alpha-a)}\right. & \\ \left. -\frac{c-a+n\alpha+\alpha}{2(n+1)(a-2c)(\alpha-a|\alpha)_2}\right\}\frac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}, & \delta=1 \end{cases} $$

Proof Because

$$ {}_5F_4\left(\begin{array}{c} -n, n+\delta, (n+\delta\alpha|\alpha), (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (n|\alpha)(\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) $$

$$ = \frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right) + \frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, n+1+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array}; 1\right). $$

We then use Theorem 1 and Theorem 4 to immediately get the conclusion of Theorem 10.

Theorem 11 For $\delta = 0, 1$ terminating Ultra-hypergeometric series identity

$$ {}_5F_4\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (2(n+\delta)a+[a+2(n+\delta)\alpha]|a+2(n+\delta)\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), (2(n+\delta)a|a+2(n+\delta)\alpha) \end{array}; 1\right) $$

$$ = \begin{cases} \left\{\frac{-c+n\alpha}{2(a-2c)(\alpha-a)} + \frac{1}{4}\right\}\frac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n} + \left\{\frac{1}{4} - \frac{c-a+n\alpha}{2(a-2c)(\alpha-a)}\right\}\frac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta=0 \\ \left\{\frac{-c+n\alpha+\alpha}{2(n+1)(a-2c)(\alpha-a|\alpha)_2} + \frac{1}{2a(2n+1)}\right\}\frac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n} + \left\{\frac{1}{2a(2n+1)} - \frac{c-a+n\alpha+\alpha}{2(n+1)(a-2c)(\alpha-a|\alpha)_2}\right\}\frac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}, & \delta=1 \end{cases} $$

Proof Because

$$
{}_5F_4\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (2(n+\delta)a+[a+2(n+\delta)\alpha]|a+2(n+\delta)\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), ((n+\delta)a|a+2(n+\delta)\alpha) \end{array};1\right)
$$

$$
=\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, n+1+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right)+\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\alpha+\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right).
$$

We then use Theorem 1 and Theorem 7 to immediately get the conclusion of Theorem 11.

Theorem 12 For $\delta=0,1$ terminating Ultra-hypergeometric series identity

$$
{}_5F_4\left(\begin{array}{c} -n, n+\delta, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (a+\alpha|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), (a|\alpha) \end{array};1\right)
$$

$$
=\begin{cases} \left\{\frac{a-2c+2n\alpha}{4(a-2c)}+\frac{1}{4}\right\}\frac{(-c|\alpha)_n}{(\alpha+a-c|\alpha)_n}+\left\{\frac{a-2c-2n\alpha}{4(a-2c)}+\frac{1}{4}\right\}\frac{(c-a|\alpha)_n}{(\alpha+c|\alpha)_n}, & \delta=0 \\ \left\{\frac{a-2c+(2n+)\alpha}{2(2n+1)(a-2c)(\alpha-a)}+\frac{1}{2a(2n+1)}\right\}\frac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n}+\left\{\frac{a-2c-(2n+1)\alpha}{2(2n+1)(a-2c)(\alpha-a)}+\frac{1}{2a(2n+1)}\right\}\frac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}, & \delta=1 \end{cases}
$$

Proof Because

$$
{}_5F_4\left(\begin{array}{c} -n, \delta+n, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (a+\alpha|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), (a|\alpha) \end{array};1\right)
$$

$$
=\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, \delta+n, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right)+\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, \delta+n, (\frac{a+\alpha}{2}|\alpha), (\alpha+\frac{a}{2}|\alpha) \\ \delta+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right).
$$

We then use Theorem 4 and Theorem 7 to immediately get the conclusion of Theorem 12.

Corollary 13 In Theorem12,when $a=c=\alpha$ and $\delta=0$ we have

$$
\sum_{k=0}^{\infty}\binom{-n}{k}\binom{n}{k}=0.
$$

Theorem 14 Terminating Ultra-hypergeometric series identity

$$
{}_5F_4\left(\begin{array}{c} -n, n+1, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (2a+\alpha|a+\alpha) \\ 1+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), (a|a+\alpha) \end{array};1\right)
$$

$$
=\left\{\frac{1}{2(a-2c)(\alpha-a)}+\frac{1}{2a(2n+1)}\right\}\frac{(-c|\alpha)_{n+1}}{(\alpha+a-c|\alpha)_n}+\left\{\frac{1}{2(a-2c)(\alpha-a)}+\frac{1}{a(2n+1)}\right\}\frac{(c-a|\alpha)_{n+1}}{(\alpha+c|\alpha)_n}.
$$

Proof Because

$$
{}_5F_4\left(\begin{array}{c} -n, 1+n, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha), (a+\frac{\alpha}{2}|\frac{\alpha}{2}) \\ \frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha), (a|\frac{\alpha}{2}) \end{array};1\right)
$$

$$
=\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, 1+n, (\frac{a+\alpha}{2}|\alpha), (\frac{a}{2}|\alpha) \\ \frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right)+\frac{1}{2}{}_4F_3\left(\begin{array}{c} -n, 1+n, (\frac{a+\alpha}{2}|\alpha), (\alpha+\frac{a}{2}|\alpha) \\ 1+\frac{1}{2}, (\alpha+c|\alpha), (\alpha+a-c|\alpha) \end{array};1\right).
$$

In Theorem 1 when $\delta=0$ and Theorem 7 when $\delta=1$ to immediately get the conclusion of Theorem 14.

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