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ON THE APPROXIMATION OF THE WALLIS RATIO

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Abstract. In this review article, we study the recent investigations and developments on the Wallis ratio. Some best constants for the approximation of the Wallis ratio are introduced. Some double inequalities for bounding the Wallis ratio are also introduced.

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1. Recent developments on the Wallis ratio

Let $\mathbb{N}$ be the set of all positive integers. For $n \in \mathbb{N}$, the double factorial $n!!$ is defined by

$$n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n-2i), \tag{1}$$

where in (1) the floor function $\lfloor t \rfloor$ denotes the largest integer less than or equal to t. For our own convenience, in what follows, we denote the ratio of two neighbouring double factorials by

$$W_n = \frac{(2n-1)!!}{(2n)!!}, \ n \in \mathbb{N}, \tag{2}$$

which is called Wallis ratio in literature.

Consider the Wallis ratio W_n, defined by (2). This quantity is important in the probability theory – for example, the three events, (a) a return to the origin takes place at time $2n$, (b) no return occurs up to and including time $2n$, and (c) the path is non -negative between 0 and $2n$, have the common probability W_n. And the two events, (a) the first return to the origin takes place at time $2n$, (b) the first passage through -1 occurs at time $2n-1$, have the common probability $W_n/(2n-1)$. For details of these interesting results one may see [1, Chapter III].

The Wallis ratio W_n can be represented as follows (see [2, p. 258]):

$$W_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)}, \tag{3}$$

where in (3) $\Gamma(x)$ is the classical Euler's gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t. \tag{4}$$

Also note that [2, p. 258]

$$W_n = \left[(2n+1)\int_0^{\pi/2} \sin^{2n+1} x dx\right]^{-1} \tag{5}$$

$$= \left[(2n+1)\int_0^{\pi/2} \cos^{2n+1} x dx\right]^{-1} \tag{6}$$

and the Wallis sine(cosine) formula

$$W_n = \frac{2}{\pi}\int_0^{\pi/2} \sin^{2n} x dx \tag{7}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos^{2n} x dx. \tag{8}$$

The Wallis ratio W_n is closely related to the Wallis formula:

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} \tag{9}$$

In fact, since [3, pp. 181-184]

$$\prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} = \lim_{n\to\infty} (2n+1) W_n^2, \tag{10}$$

another important form of the Wallis formula is

$$\lim_{n\to\infty} (2n+1) W_n^2 = \frac{2}{\pi}. \tag{11}$$

For the approximation of the Wallis ratio W_n, in [4] the authors proved

Theorem 1. *For all* $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le W_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{12}$$

In [5], the following result was established:

Theorem 2. *For all* $n \in \mathbb{N}$,

$$W_n > \left[n\pi\left(1+\frac{1}{4n-\frac{1}{2}}\right)\right]^{-1/2}. \tag{13}$$

The authors [5] also proved that

Theorem 3. *For* $\varepsilon \in (0, \frac{1}{2})$,

$$W_n < \left[n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \varepsilon} \right) \right]^{-1/2}, \quad n > n^*, \tag{14}$$

where n^* *is the maximal root of the following equation on* n*:*

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

In [6] the author showed that

Theorem 4. *For all* $n \in \mathbb{N}$,

$$\left[n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n + \frac{15}{4n}}} \right) \right]^{-1/2} < W_n < \left[n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n}} \right) \right]^{-1/2}. \tag{15}$$

In [7] the authors proved the result below.

Theorem 5. *For all* $n \in \mathbb{N}$,

$$\frac{\sqrt{\pi}}{2\sqrt{n + \frac{9\pi}{16} - 1}} \leq \frac{(2n)!!}{(2n+1)!!} < \frac{\sqrt{\pi}}{2\sqrt{n + \frac{3}{4}}}, \tag{16}$$

which is equivalent to the following:

$$\frac{2\sqrt{n + \frac{3}{4}}}{(2n+1)\sqrt{\pi}} < W_n \leq \frac{2\sqrt{n + \frac{9\pi}{16} - 1}}{(2n+1)\sqrt{\pi}}. \tag{17}$$

The following several results for the approximation of the Wallis ratio are in the form:

$$C_1 P_n < W_n < C_2 P_n, \tag{18}$$

where in (18) the constants $C_1 > 0$ and $C_2 > 0$ are best possible - which mean that the constant C_1 in (18) can not be replaced by a number which is greater than C_1 and the constant C_2 in (18) can not be replaced by a number which is less than C_2.

In [8] the authors showed

Theorem 6. *For all* $n \in \mathbb{N}$,

$$\frac{e}{\sqrt{\pi}} \frac{(n - 1/2)^n}{(n + 1/2)^{n+1/2}} < W_n \leq \left(\frac{3}{2} \right)^{3/2} \frac{(n - 1/2)^n}{(n + 1/2)^{n+1/2}}. \tag{19}$$

The constants $\frac{e}{\sqrt{\pi}}$ *and* $\left(\frac{3}{2}\right)^{3/2}$ *in (19) are best possible.*

In [9] the authors proved the following result:

Theorem 7. *For all $n \geq 2$,*

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}. \tag{20}$$

The constants $\sqrt{\frac{e}{\pi}}$ and $\frac{4}{3}$ in (20) are best possible.

In [10] the authors established

Theorem 8. *For all $n \geq 2$,*

$$\left(\frac{2}{3}\right)^{3/2}\left(1-\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}} \leq W_n < \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}}. \tag{21}$$

The constants $\left(\frac{2}{3}\right)^{3/2}$ and $\sqrt{\frac{e}{\pi}}$ in (21) are best possible.

2. Conclusion

In this review article, we introduced the recent investigations and developments on the Wallis ratio. Some best constants for the approximation of the Wallis ratio are introduced. Some double inequalities for bounding the Wallis ratio are also introduced.

Competing interests

The author declares that there are no competing interests.

References

[1] W. Feller, An introduction to probability theory and its applications, Vol 1, Wiley, New York, 1966.

[2] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, Applied Mathematics Series **55**, Washington, 1970.

[3] D. S. Mitrinović, Analytic Inequalities, Springer-verlag, Berlin, 1970.

[4] C. Chen and F. Qi, The best bounds in wallis' inequality, Proc. Amer. Math. Soc., **133**(2005), 397-401.

[5] Y. Zhao and Q. Wu, Wallis inequality with a parameter. J. Inequal. Pure Appl. Math. **7**(2006), Article ID 56.

[6] C. Mortici, Completely monotone functions and the Wallis ratio. Appl. Math. Lett. **25**(2012), 717–722.

[7] B. Guo and F. Qi, A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications. J. Korean Math. Soc. **48**(2011), 655–667.

[8] S. Guo and C. Song, A new result for the approximation of the quantity involved in the Wallis' formula, Proceedings of the 2012 International Conference on Computer Science, Mathematics, Physical Education, and Management, Wuhan, China, September, 2012, IEEE Press, pp. 259–262.

[9] S. Guo, J. Xu and F. Qi, Some exact constants for the approximation of the quantity in the Wallis' formula, J. Inequal. Appl. **2013**(2013), Article ID 67.

[10] S. Guo, Q. Feng, Y. Bi and Q. Luo, A sharp two-sided inequality for bounding the Wallis ratio, J. Inequal. Appl. , **2015**(2015), Article ID 43.

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