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On the Nature of Solutions of a System of Second Order Nonlinear Difference Equations

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Abstract

In this paper, we investigate the dynamical behaviors of a system of second order non linear difference equations We study local stability of the equilibrium point of the system of the second order rational difference equations and oscillation behaviour of positive solutions of the aforementioned system. Moreover, we establish that the system has unbounded solutions.

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1. Introduction

Recently, there has been great interest in investigating the behavior of solutions of a system of non linear difference equations and discussing the asymptotic stability of their equilibrium points. For example, in [3], El-Owaidy et al. studied stability, boundedness character and oscillation behavior of the difference equation

$$x_{n+1}=\alpha+\frac{x_{n-1}^{p}}{x_{n}^{p}}. \qquad \#(1.1)$$

In [1], Bao investigated the local stability, oscillation and boundedness character of positive solutions of the system of difference equations

$$x_{n+1}=A+\frac{x_{n-1}^{p}}{y_{n}^{p}},\quad y_{n+1}=A+\frac{y_{n-1}^{p}}{x_{n}^{p}},\qquad n=0,1,\ldots,\quad \#(1.2)$$

where$A\in(0,\infty), p\in[1,\infty[$, and initial conditions $x_i, y_i\in(0,\infty), i=-1,0$.

In [6], Gümüş and Soykan considered the dynamical behavior of positive solutions for a system of rational difference equations of the following form

$$u_{n+1}=\frac{\alpha u_{n-1}}{\beta+\gamma v_{n-2}^{p}},\quad v_{n+1}=\frac{\alpha_1 v_{n-1}}{\beta_1+\gamma_1 u_{n-2}^{p}},\quad n=0,1,\ldots,\quad \#(1.3)$$

Where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1,p$ and the initial values u_{-i}, v_{-i} for $i=0,1,2$ are positive real numbers.

In [7], Okumuş and Soykan studied the boundedness, persistence and periodicity of the positive solutions and the global asymptoticst ability of the positive equilibrium points of system of the difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-1}}{y_n}, \quad n = 0,1,\ldots, \quad \#(1.4)$$

where$A \in (0,\infty)$, and initial conditions $x_i, y_i, z_i \in (0,\infty), i = -1,0$.

Also, in [8], the authors investigated the oscillatory, the existence of unbounded solutions and the global behavior of positive solutions for the system of difference equations

$$x_{n+1} = A + \frac{x_{n-m}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-m}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-m}}{y_n}, \quad n = 0,1,\ldots, \quad \#(1.5)$$

where$A$ and the initial values x_i, y_i, z_i, for $i = 0,1,\ldots,m$ are positive real numbers.

Motivated by all above mentioned studies and in the light of the works in [7] and [8], in this paper, we studied the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}^p}{z_n^p}, \quad y_{n+1} = A + \frac{y_{n-1}^p}{z_n^p}, \quad z_{n+1} = A + \frac{z_{n-1}^p}{y_n^p}, \qquad n = 0,1\ldots, \quad \#(1.6)$$

where $A \in (0,\infty), p \in [1,\infty[$, and the initial values $x_i, y_i, z_i \in (0,\infty), i = -1,0$.

For some other papers in which systems of difference equations have been studied, see [1-14].

2. Preliminaries

We recall some basic definitions that we afterwards need in the paper.

Let us introduce the discrete dynamical system:

$$\begin{aligned} x_{n+1} &= f_1(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}, z_n, z_{n-1}, \ldots, z_{n-k}), \\ y_{n+1} &= f_2(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}, z_n, z_{n-1}, \ldots, z_{n-k}), \quad \#(2.1) \\ z_{n+1} &= f_3(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}, z_n, z_{n-1}, \ldots, z_{n-k}), \end{aligned}$$

$n \in \mathbb{N}$ where $f_1: I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_1$,$f_2: I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_2$ and $f_3: I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_3$ are continuously differentiable functions and I_1, I_2, I_3, are some intervals of real numbers. Also, a solution $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ of system (2.1) is uniquely determined by initial values $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$for $i \in \{0,1,\ldots,k\}$.

Definition 2.1 An *equilibrium point* of system (2.1) is a point $(\overline{x}, \overline{y}, \overline{z})$ that satisfies

$$\begin{aligned} \overline{x} &= f_1(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}, \overline{z}, \overline{z}, \ldots, \overline{z}), \\ \overline{y} &= f_2(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}, \overline{z}, \overline{z}, \ldots, \overline{z}), \\ \overline{z} &= f_3(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}, \overline{z}, \overline{z}, \ldots, \overline{z}). \end{aligned}$$

Together with system (2.1), if we consider the associated vector map

$$F = (f_1, x_n, x_{n-1}, \ldots, x_{n-k}, f_2, y_n, y_{n-1}, \ldots, y_{n-k}, f_3, z_n, z_{n-1}, \ldots, z_{n-k}),$$

then the point $(\overline{x}, \overline{y}, \overline{z})$ is also called a fixed point of the vector map F.

Definition 2.2 Let $(\overline{x}, \overline{y}, \overline{z})$ be an equilibrium point of system (2.1).

(a) An equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ is called *stable* if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that for every initial value $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$, with

$$\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta, \sum_{i=-k}^{0} |y_i - \overline{y}| < \delta, \sum_{i=-k}^{0} |z_i - \overline{z}| < \delta$$

implying $|x_n - \overline{x}| < \varepsilon, |y_n - \overline{y}| < \varepsilon, |z_n - \overline{z}| < \varepsilon$ for $n \in \mathbb{N}$.

(b) If an equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ of system (2.1) is called *unstable* if it is not stable.

(c) An equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ of system (2.1) is called *locally asymptotically stable* if, it is stable, and if in addition there exists $\gamma > 0$ such that

$$\sum_{i=-k}^{0}|x_i-\bar{x}|<\gamma,\sum_{i=-k}^{0}|y_i-\bar{y}|<\gamma,\sum_{i=-k}^{0}|z_i-\bar{z}|<\gamma$$

and$(x_n,y_n,z_n)\to(\bar{x},\bar{y},\bar{z})$ as $n\to\infty$.

(d) An equilibrium point $(\bar{x},\bar{y},\bar{z})$of system (2.1) is called a *global attractor* if, $(x_n,y_n,z_n)\to(\bar{x},\bar{y},\bar{z})$ as $n\to\infty$.

(e) An equilibrium point $(\bar{x},\bar{y},\bar{z})$ of system (2.1) is called *globally asymptotically stable* if it is stable, and a global attractor.

Definition 2.3 Let $(\bar{x},\bar{y},\bar{z})$ be an equilibrium point of the map F where f_1, $f_2$and $f_3$are continuously differentiable functions at $(\bar{x},\bar{y},\bar{z})$. The linearized system of system (2.1) about the equilibrium point $(\bar{x},\bar{y},\bar{z})$ is

$$X_{n+1}=F(X_n)=BX_n,$$

where

$$X_n=\begin{pmatrix} x_n \\ \vdots \\ x_{n-k} \\ y_n \\ \vdots \\ y_{n-k} \\ z_n \\ \vdots \\ z_{n-k} \end{pmatrix}$$

and B is a Jacobian matrix of system (2.1) about the equilibrium point $(\bar{x},\bar{y},\bar{z})$.

Definition 2.4 Assume that

$$X_{n+1}=F(X_n),\ \ n=0,1,\ldots,$$

be a system of difference equations such that $\bar{X}$ is a fixed point of F. If no eigenvalues of the Jacobian matrix B about $\bar{X}$ have absolute value equal to one, then $\bar{X}$ is called hyperbolic. If there exists an eigenvalue of the Jacobian matrix B about $\bar{X}$ with absolute value equal to one, then $\bar{X}$ is called nonhyperbolic.

Theorem 2.5 (The Linearized Stability Theorem)

Assume that

$$X_{n+1}=F(X_n),\ \ n=0,1,\ldots,$$

be a system of difference equations such that $\bar{X}$ is a fixed point of F.

(a) If all eigenvalues of the Jacobian matrix B about $\bar{X}$ lie inside the open unit disk $|\lambda|<1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.

(b) If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.

Definition 2.6 Let $(\bar{x},\bar{y},\bar{z})$ be an equilibrium point of system (2.1), and assume that $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ is a solution of the system (2.1).

A "string" of consecutive terms $\{x_s,\ldots,x_m\}$ (resp. $\{y_s,\ldots,y_m\}$, $\{z_s,\ldots,z_m\}$), $s\geq-1,m\leq\infty$is said to be a positive semicycle if $x_i\geq\bar{x}$(resp. $y_i\geq\bar{y},z_i\geq\bar{z}$),$i\in\{s,\ldots,m\}$, $x_{s-1}<\bar{x}$ (resp. $y_{s-1}<\bar{y},z_{s-1}<\bar{z}$), and $x_{m+1}<\bar{x}$ (resp. $y_{m+1}<\bar{y},z_{m+1}<\bar{z}$).

A "string" of consecutive terms $\{x_s,\ldots,x_m\}$ (resp. $\{y_s,\ldots,y_m\}$, $\{z_s,\ldots,z_m\}$), $s\geq-1,m\leq\infty$ is said to be a negative semicycle if $x_i<\bar{x}$(resp. $y_i<\bar{y},z_i<\bar{z}$),$i\in\{s,\ldots,m\}$, $x_{s-1}\geq\bar{x}$ (resp. $y_{s-1}\geq\bar{y},z_{s-1}\geq\bar{z}$), and $x_{m+1}\geq\bar{x}$ (resp. $y_{m+1}\geq\bar{y},z_{m+1}\geq\bar{z}$).

A "string" of consecutive terms $\{(x_s,y_s,z_s),\ldots,(x_m,y_m,z_m)\}$, is said to be a positive (resp. negative) semicycle if $\{x_s,\ldots,x_m\}$, $\{y_s,\ldots,y_m\}$, $\{z_s,\ldots,z_m\}$ are positive (resp. negative) semicycles.

Definition 2.7A solution $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ of system (2.1) is called non oscillatory about $(\bar{x},\bar{y},\bar{z})$, or simply nonoscillatory, if there exists $N\geq-k$ such that either

$$x_n\geq\bar{x},y_n\geq\bar{y},z_n\geq\bar{z},for\ all\ n\geq N$$

or

$$x_n < \overline{x},\ \ y_n < \overline{y},\ \ z_n < \overline{z}, for\ all\ n \geq N.$$

Otherwise, the solution $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is called oscillatory about $(\overline{x}, \overline{y}, \overline{z})$, or simply oscillatory.

3. Main Results

In this section, we prove our main results.

Theorem 3.1 The following statements are true:

(i) The system (1.6) has a positive equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$.

(ii) If $A > 2p - 1$, then the equilibrium point of system (1.6) is locally asymptotically stable.

(iii) If $A < 2p - 1$, then the equilibrium point of system (1.6) is unstable.

(iv) Also, when $A = 2p - 1$and $p = 1$, the results has been investigated in [7].

Proof.

(i) It is easily seen from the definition of equilibrium point.

(ii) We consider the following transformation to build the corresponding linearized form of system (1.6):

$$(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}) \to (f, f_1, g, g_1, h, h_1)$$

where

$$f = A + \frac{x_{n-1}^p}{z_n^p},$$

$$f_1 = x_n,$$

$$g = A + \frac{y_{n-1}^p}{z_n^p},$$

$$g_1 = y_n,$$

$$h = A + \frac{z_{n-1}^p}{y_n^p},$$

$$h_1 = z_n.$$

The Jacobian matrix about the equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ under the above transformation is given by

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \frac{p\overline{x}^{p-1}}{\overline{z}^p} & 0 & 0 & -\frac{p\overline{x}^p}{\overline{z}^{p+1}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p\overline{y}^{p-1}}{\overline{z}^p} & -\frac{p\overline{y}^p}{\overline{z}^{p+1}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p\overline{z}^p}{\overline{y}^{p+1}} & 0 & 0 & \frac{p\overline{z}^{p-1}}{\overline{y}^p} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \#(3.1)$$

Hence, the linearized system of system (1.6) about the equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$ is

$$X_{n+1} = B(\overline{x}, \overline{y}, \overline{z})X_n,$$

where

$$X_n = (x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1})^T$$

and

$$B(\overline{x},\overline{y},\overline{z}) = \begin{pmatrix} 0 & \frac{p}{A+1} & 0 & 0 & -\frac{p}{A+1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{A+1} & -\frac{p}{A+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p}{A+1} & 0 & 0 & \frac{p}{A+1} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic equation of $B(\overline{x},\overline{y},\overline{z})$ about $(\overline{x},\overline{y},\overline{z}) = (A+1, A+1, A+1)$ is

$$\lambda^6 - \left(\frac{p^2}{(A+1)^2} + 3\frac{p}{A+1}\right)\lambda^4 + \left(\frac{p^3}{(A+1)^3} + 3\frac{p^2}{(A+1)^2}\right)\lambda^2 - \frac{p^3}{(A+1)^3} = 0. \quad \#(3.2)$$

From this, the roots of characteristic equation (3.2) are

$$\lambda_1 = \sqrt{\frac{p}{A+1}},$$

$$\lambda_2 = -\sqrt{\frac{p}{A+1}},$$

$$\lambda_3 = -\frac{1}{2(A+1)}\left(p + \sqrt{p^2 + 4Ap + 4p}\right),$$

$$\lambda_4 = \frac{1}{2(A+1)}\left(-p + \sqrt{p^2 + 4Ap + 4p}\right),$$

$$\lambda_5 = \frac{1}{2(A+1)}\left(p + \sqrt{p^2 + 4Ap + 4p}\right),$$

$$\lambda_6 = \frac{1}{2(A+1)}\left(p - \sqrt{p^2 + 4Ap + 4p}\right).$$

From the Linearized Stability Theorem, since $A > 2p - 1$, all roots of the characterictic equation lie inside the open unit disk $|\lambda| < 1$. Therefore, the positive equilibrium point of system (1.6) is locally asymptotically stable.

(iii) From the proof of (ii), it is true.

Theorem 3.2 Let $0 < A < 1$ and $\{(x_n, y_n, z_n)\}$ be an arbitrary positive solution of system (1.6). Then, the following statements are true.

(i) If

$$0 < x_{-1} < 1, \ 0 < y_{-1} < 1, \ 0 < z_{-1} < 1,$$

$$x_0 \geq \frac{1}{(1-A)^{1/p}}, \ y_0 \geq \frac{1}{(1-A)^{1/p}}, \ z_0 \geq \frac{1}{(1-A)^{1/p}}, \quad \#(3.3)$$

then

$$\lim_{n\to\infty} x_{2n+1} = A, \ \lim_{n\to\infty} y_{2n+1} = A, \ \lim_{n\to\infty} z_{2n+1} = A,$$

$$\lim_{n\to\infty} x_{2n} = \infty, \lim_{n\to\infty} y_{2n} = \infty, \lim_{n\to\infty} z_{2n} = \infty.$$

(ii) If

$$0 < x_0 < 1, \ 0 < y_0 < 1, \ 0 < z_0 < 1,$$

$$x_{-1} \geq \frac{1}{(1-A)^{1/p}}, \ y_{-1} \geq \frac{1}{(1-A)^{1/p}}, \ z_{-1} \geq \frac{1}{(1-A)^{1/p}}, \quad \#(3.4)$$

then

$$\lim_{n\to\infty} x_{2n+1} = \infty, \lim_{n\to\infty} y_{2n+1} = \infty, \lim_{n\to\infty} z_{2n+1} = \infty,$$

$$\lim_{n\to\infty} x_{2n} = A, \quad \lim_{n\to\infty} y_{2n} = A, \quad \lim_{n\to\infty} z_{2n} = A.$$

Proof.

(i) From system (1.6) and (3.3), we have

$$x_1 = A + \frac{x_{-1}^p}{z_0^p} \le A + \frac{1}{z_0^p} \le A + (1 - A) = 1,$$

$$y_1 = A + \frac{y_{-1}^p}{z_0^p} \le A + \frac{1}{z_0^p} \le A + (1 - A) = 1,$$

$$z_1 = A + \frac{z_{-1}^p}{y_0^p} \le A + \frac{1}{y_0^p} \le A + (1 - A) = 1,$$

$$x_1 = A + \frac{x_{-1}^p}{z_0^p} > A,$$

$$y_1 = A + \frac{y_{-1}^p}{z_0^p} > A,$$

$$z_1 = A + \frac{z_{-1}^p}{y_0^p} > A.$$

Hence,

$$x_1 \in]A, 1], \quad y_1 \in]A, 1], \quad z_1 \in]A, 1].$$

Also,

$$x_2 = A + \frac{x_0^p}{z_1^p} \ge A + x_0^p,$$

$$y_2 = A + \frac{y_0^p}{z_1^p} \ge A + y_0^p,$$

$$z_2 = A + \frac{z_0^p}{y_1^p} \ge A + z_0^p.$$

Similarly, we get

$$x_3 = A + \frac{x_1^p}{z_2^p} \le A + \frac{1}{\left(A + z_0^p\right)^p} \le A + \frac{1}{\left(A + z_0^p\right)} \le A + \frac{1}{z_0^p} \le A + (1 - A) = 1,$$

$$y_3 = A + \frac{y_1^p}{z_2^p} \le A + \frac{1}{\left(A + z_0^p\right)^p} \le A + \frac{1}{\left(A + z_0^p\right)} \le A + \frac{1}{z_0^p} \le A + (1 - A) = 1,$$

$$z_3 = A + \frac{z_1^p}{y_2^p} \le A + \frac{1}{\left(A + y_0^p\right)^p} \le A + \frac{1}{\left(A + y_0^p\right)} \le A + \frac{1}{y_0^p} \le A + (1 - A) = 1.$$

Thus,

$$x_3 \in]A, 1], \quad y_3 \in]A, 1], \quad z_3 \in]A, 1].$$

Also,

$$x_4 = A + \frac{x_2^p}{z_3^p} \ge A + x_2^p \ge A + \left(A + x_0^p\right)^p \ge A + \left(A + x_0^p\right) = 2A + x_0^p,$$

$$y_4 = A + \frac{y_2^p}{z_3^p} \ge A + y_2^p \ge A + \left(A + y_0^p\right)^p \ge A + \left(A + y_0^p\right) = 2A + y_0^p,$$

$$z_4 = A + \frac{z_2^p}{y_3^p} \ge A + z_2^p \ge A + \left(A + z_0^p\right)^p \ge A + \left(A + z_0^p\right) = 2A + z_0^p.$$

By induction for $n = 1,2,\ldots$, we obtain

$$A < x_{2n-1} < 1,\ \ A < y_{2n-1} < 1,\ \ A < z_{2n-1} < 1,$$
$$x_{2n} \geq nA + x_0^p,\ \ y_{2n} \geq nA + y_0^p,\ \ z_{2n} \geq nA + z_0^p. \quad \#(3.5)$$

From system (1.6) and (3.5), it follows that

$$\lim_{n\to\infty} x_{2n} = \infty, \lim_{n\to\infty} y_{2n} = \infty, \lim_{n\to\infty} z_{2n} = \infty,$$

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}\left(A + \frac{x_{2n-1}^p}{z_{2n}^p}\right) = A,$$

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty}\left(A + \frac{y_{2n-1}^p}{z_{2n}^p}\right) = A,$$

$$\lim_{n\to\infty} z_{2n+1} = \lim_{n\to\infty}\left(A + \frac{z_{2n-1}^p}{y_{2n}^p}\right) = A.$$

(ii) The proof is similar to the proof of (i), so we omit it. The proof is completed.

Theorem 3.3 Let $\{(x_n, y_n, z_n)\}$be a positive solution of system (1.6) which consists of at least two semicycles. Then $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ is oscillatory.

Proof. Since $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$has at least two semicycles, there exists $N \geq 0$ such that either

$$x_{N-1} < A + 1 \leq x_N,$$
$$y_{N-1} < A + 1 \leq y_N, \quad \#(3.6)$$
$$z_{N-1} < A + 1 \leq z_N,$$

or

$$x_N < A + 1 \leq x_{N-1},$$
$$y_N < A + 1 \leq y_{N-1}, \quad \#(3.7)$$
$$z_N < A + 1 \leq z_{N-1}.$$

First, we suppose the case (3.6). Then

$$x_{N+1} = A + \frac{x_{N-1}^p}{z_N^p} < A + 1,$$

$$y_{N+1} = A + \frac{y_{N-1}^p}{z_N^p} < A + 1,$$

$$z_{N+1} = A + \frac{z_{N-1}^p}{y_N^p} < A + 1,$$

$$x_{N+2} = A + \frac{x_N^p}{z_{N+1}^p} > A + 1,$$

$$y_{N+2} = A + \frac{y_N^p}{z_{N+1}^p} > A + 1,$$

$$z_{N+2} = A + \frac{z_N^p}{y_{N+1}^p} > A + 1.$$

So, we have

$$x_{N+1} < A + 1 < x_{N+2},$$
$$y_{N+1} < A + 1 < y_{N+2},$$
$$z_{N+1} < A + 1 < z_{N+2}.$$

Last, we suppose the case (3.7). The case is similar to the first case, so we leave it to readers.

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