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Solving 2D Time-Fractional Diffusion Equations by Preconditioned Fractional EDG Method

Abdulkafi Mohammed Saeed [1]
[1]Department of Mathematics, College of Science, Qassim University, Saudi Arabia,
Email: abdulkafi.ahmed@qu.edu.sa

Abstract

Fractional differential equations play a significant role in science and technology given that several scientific problems in mathematics, physics, engineering and chemistry can be resolved using fractional partial differential equations in terms of space and/or time fractional derivative. Because of new developments in the analysis and understanding of many complex systems in engineering and sciences, it has been observed that several phenomena are more realistically and accurately described by differential equations of fractional order. Fast computational methods for solving fractional partial differential equations using finite difference schemes derived from skewed (rotated) difference operators have been extensively investigated over the years. The main aim of this paper is to examine a new fractional group iterative method which is called Preconditioned Fractional Explicit Decoupled Group (PFEDG) method in solving 2D time-fractional diffusion equations. Numerical experiments and comparison with other existing methods are given to confirm the superiority of our proposed method.

Keywords: Preconditioned Fractional Explicit Decoupled Group Method; Time-Fractional Diffusion Equation.

1. Introduction

Fractional differential equations have been the focus on many studies due to their frequent appearance in various fields such as physics, chemistry and engineering. The differential equations with a fractional derivative serve as superior models in subjects as diverse as astrophysics, chaotic dynamics, fractal network, signal processing, continuum mechanics, turbulent flow and wave propagation ([1], [2], [3]). Fractional partial differential equations (FPDE's) started to play an important role, in particular, during the last few decades, in modeling of the so-called anomalous phenomena and in the theory of the complex or fractal systems [4]. As it was in the classical PDE's there is no general method that can be used in solving FPDE's. Numerical solution of FPDE's has received great progress in the recent years ([5], [6]).

We consider the following time fractional diffusion equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \tag{1.1}$$

where α is the order of the time fractional derivative in Caputo sense which is defined as [7]

$$\frac{\partial^{\alpha} u(x, y, t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, y, \xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^{\alpha}}, \quad 0 < \alpha < 1. \tag{1.2}$$

In the last few years there has been a growing interest in the development of numerical techniques appropriate for the approximation of FPDE's ([8], [9]). The construction of a specific splitting-type preconditioner in block formulation applied to a class of group relaxation iterative methods derived from the rotated (skewed) finite difference approximations have been shown to improve the convergence rates of these methods ([10], [11], [12]).

In this article, we aim to construct an efficient method which is called Preconditioned Fractional Explicit Decoupled

Group (PFEDG) method in solving equation (1.1). The paper is organized in five sections: Section 2 describes the formulation of the Fractional Explicit Group (FEG) Method. In Section 3, the derivation of Fractional Explicit Decoupled Group (FEDG) Method will be presented. The accelerated version of Fractional Explicit Decoupled Group (PFEDG) Method will be introduced in Section 4. In Section 5, the numerical results are presented in order to show the efficiency of the new proposed method. Finally, the conclusion is given in Section 6.

2. Formulation of Fractional Explicit Group Method

In this section, it is assumed that the domains are constant for both x and y, while the grid dimensions in relation to space and time for the positive integers n and l are respectively represented by $h = \frac{1}{n}$ and $\tau = \frac{T}{l}$. The grid points in the space in the space interval $[0,1]$ are denoted $x_i = ih,\ x_j = jh,\ \{i, j = 0,1,\ldots\}$ and the grid points for time are designated $t_k = k\tau,\ k = 0,1,\ldots,l.$ Discretization with regard to time fractional with utilization of Crank-Nicolson finite difference approximations at $(x_i, y_j, t_{n+1/2})$ is realized through the formula displayed below [13]

$$\frac{\partial^\alpha u(x_i, y_j, t_{k+1})}{\partial t^\alpha} = \{w_1 u^k + \sum_{s=1}^{k-1}[w_{k-s+1} - w_{k-s}]u_{i,j}^s - w_k u_{i,j}^0 + \sigma \frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}}\} + 0(\tau^{2-\alpha}), \quad (2.1)$$

where $\sigma = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}, \quad w_s = \sigma\{(\frac{s+1}{2})^{1-\alpha} - (\frac{s-1}{2})^{1-\alpha}\}$

utilization of the standard second order Crank-Nicolson difference scheme with the TFD formula (2.1) for finite difference discretization of (1.1) will result in the standard Crank-Nicolson formula portrayed below

$$\begin{aligned} u_{i,j}^{k+1} = \frac{1}{1+2r}\{&\frac{r}{2}[(u_{i+1,j}^{k+1} + u_{i-1,j}^{k+1} + u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1}) + (u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k)] + (1 - 2^{1-\alpha} w_1^* - 2r)u_{i,j}^k \\ &+ 2^{1-\alpha}\sum_{s=1}^{k-1}[(w_{k-s}^* - w_{k-s+1}^*)u_{i,j}^s + 2^{1-\alpha} w_k^* u_{i,j}^0 + m_0 f_{i,j}^{\frac{k+1}{2}}\}, \end{aligned} \quad (2.2)$$

where $m_0 = \tau^\alpha \Gamma(2-\alpha) * 2^{1-\alpha}, \quad r = \frac{m_0}{h^2}, \ w_s^* = [(\frac{s+1}{2})^{1-\alpha} - (\frac{s-1}{2})^{1-\alpha}].$

The application of (2.2) to whichever group of four points on a discretized solution domain will lead to a 4×4 system displayed as follows

$$\begin{pmatrix} a_1 & -a_2 & a_3 & -a_2 \\ -a_2 & a_1 & -a_2 & a_3 \\ a_3 & -a_2 & a_1 & -a_2 \\ -a_2 & a_3 & -a_2 & a_1 \end{pmatrix} \begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix}, \quad (2.3)$$

where

$$a_1 = 1 + 2r, \quad a_2 = \frac{r}{2}, \quad a_3 = 0,$$

$$\begin{aligned} rhs_{i,j} = {} & a_1[u_{i-1,j}^{k+1} + u_{i,j+1}^{k+1}] + a_2[u_{i-1,j}^k + u_{i+1,j}^k + u_{i,j-1}^k + u_{i,j+1}^k] + (1 - 2^{1-\alpha} w_1^* - 2r)u_{i,j}^k \\ & + 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^* - w_{k-s+1}^*]u_{i,j}^s + 2^{1-\alpha} w_k^* u_{i,j}^0 + m_0 f_{i,j}^{\frac{k+1}{2}} \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j} = {} & a_1[u_{i+2,j}^{k+1} + u_{i+1,j-1}^{k+1}] + a_2[u_{i,j}^k + u_{i+2,j}^k + u_{i+1,j-1}^k + u_{i+1,j+1}^k] + (1 - 2^{1-\alpha} w_1^* - 2r)u_{i+1,j}^k \\ & + 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^* - w_{k-s+1}^*]u_{i+1,j}^s + 2^{1-\alpha} w_k^* u_{i+1,j}^0 + m_0 f_{i+1,j}^{\frac{k+1}{2}}, \end{aligned}$$

$$rhs_{i+1,j+1} = a_1[u^{k+1}_{i+2,j+1} + u^{k+1}_{i+1,j+2}] + a_2[u^{k}_{i,j+1} + u^{k}_{i+2,j+1} + u^{k}_{i+1,j} + u^{k}_{i+1,j+2}] + (1 - 2^{1-\alpha} w^*_1 - 2r)u^{k}_{i+1,j+1}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w^*_{k-s} - w^*_{k-s+1}]u^{s}_{i+1,j+1} + 2^{1-\alpha} w^*_k u^{0}_{i+1,j+1} + m_0\, f^{\frac{k+1}{2}}_{i+1,j+1},$$

$$rhs_{i,j+1} = a_1[u^{k+1}_{i-1,j+1} + u^{k+1}_{i,j+2}] + a_2[u^{k}_{i-1,j+1} + u^{k}_{i,j} + u^{k}_{i+1,j+1} + u^{k}_{i,j+2}] + (1 - 2^{1-\alpha} w^*_1 - 2r)u^{k}_{i,j+1}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w^*_{k-s} - w^*_{k-s+1}]u^{s}_{i,j+1} + 2^{1-\alpha} w^*_k u^{0}_{i,j+1} + m_0\, f^{\frac{k+1}{2}}_{i,j+1},$$

A four point FEG equation can be generated through a reversal of the matrix above as the following

$$\begin{pmatrix} u^{k+1}_{i,j} \\ u^{k+1}_{i+1,j} \\ u^{k+1}_{i+1,j+1} \\ u^{k+1}_{i,j+1} \end{pmatrix} = \frac{1}{2(1+r)(1+2r)(1+3r)} \begin{pmatrix} b_1 & b_2 & b_3 & b_2 \\ b_2 & b_1 & b_2 & b_3 \\ b_3 & b_2 & b_1 & b_2 \\ b_2 & b_3 & b_2 & b_1 \end{pmatrix} \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix}, \tag{2.4}$$

$$b_1 = 7r^2 + 8r + 2,\quad b_2 = r(1+2r),\quad b_3 = r^2.$$

3. Formulation of Fractional Explicit Decoupled Group Method

It can be seen that for the rotated five-point finite difference approximation the following transformations take place

$$i, j \pm 1 \to i \pm 1, j \pm 1$$
$$i \pm 1, j \to i \pm 1, j \mp 1$$
$$h \to \sqrt{2}\, h.$$

Therefore, the rotated finite difference approximation (achieved through 45^0 degree clockwise rotation of the x-y axis) for equation (1.1) can be utilized to unveil the following:

$$u^{k+1}_{i,j} = \frac{1}{1+r}\{\frac{r}{4}[(u^{k+1}_{i+1,j+1} + u^{k+1}_{i-1,j-1} + u^{k+1}_{i-1,j+1} + u^{k+1}_{i+1,j-1}) + (u^{k}_{i+1,j+1} + u^{k}_{i-1,j-1} + u^{k}_{i-1,j+1} + u^{k}_{i+1,j-1})]$$
$$+ (1 - 2^{1-\alpha} w^*_1 - r)u^{k}_{i,j} + 2^{1-\alpha}\sum_{s=1}^{k-1}[(w^*_{k-s} - w^*_{k-s+1})u^{s}_{i,j} + 2^{1-\alpha} w^*_k u^{0}_{i,j} + m_0\, f^{\frac{k+1}{2}}_{i,j}, \tag{3.1}$$

Wher m_0, r and w^*_s as mentioned before, the application of (2.3) to whichever four-point group on a discretized solution domain will lead to a 4×4 system of equations. This is illustrated as follows:

$$\begin{pmatrix} a_1 & -a_2 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & -a_2 \\ 0 & 0 & -a_2 & a_1 \end{pmatrix} \begin{pmatrix} u^{k+1}_{i,j} \\ u^{k+1}_{i+1,j} \\ u^{k+1}_{i+1,j+1} \\ u^{k+1}_{i,j+1} \end{pmatrix} = \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j+1} \\ rhs_{i+1,j} \\ rhs_{i,j+1} \end{pmatrix}, \tag{3.2}$$

$$a_1 = 1 + r,\quad a_2 = \frac{r}{4},$$

$$rhs_{i,j} = a_1[u^{k+1}_{i-1,j-1} + u^{k+1}_{i-1,j+1} + u^{k+1}_{i+1,j-1}] + a_2[u^{k}_{i+1,j+1} + u^{k}_{i-1,j-1} + u^{k}_{i-1,j+1} + u^{k}_{i+1,j-1}] + (1 - 2^{1-\alpha} w^*_1 - r)u^{k}_{i,j}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w^*_{k-s} - w^*_{k-s+1}]u^{s}_{i,j} + 2^{1-\alpha} w^*_k u^{0}_{i,j} + m_0\, f^{k+1}_{i,j}$$

$$rhs_{i+1,j+1} = a_1[u_{i+2,j}^{k+1} + u_{i+2,j+2}^{k+1} + u_{i,j+2}^{k+1}] + a_2[u_{i,j}^{k} + u_{i+2,j}^{k} + u_{i+2,j+2}^{k} + u_{i,j+2}^{k}] + (1 - 2^{1-\alpha} w_1^* - r)u_{i+1,j+1}^{k}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^* - w_{k-s+1}^*]u_{i+1,j+1}^{s} + 2^{1-\alpha} w_k^* u_{i+1,j+1}^{0} + m_0 f_{i+1,j+1}^{\frac{k+1}{2}},$$

$$rhs_{i+1,j} = a_1[u_{i,j-1}^{k+1} + u_{i+2,j+1}^{k+1} + u_{i+2,j-1}^{k+1}] + a_2[u_{i,j+1}^{k} + u_{i,j-1}^{k} + u_{i+2,j+1}^{k} + u_{i+2,j-1}^{k}] + (1 - 2^{1-\alpha} w_1^* - r)u_{i+1,j}^{k}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^* - w_{k-s+1}^*]u_{i+1,j}^{s} + 2^{1-\alpha} w_k^* u_{i+1,j}^{0} + m_0 f_{i+1,j}^{\frac{k+1}{2}},$$

$$rhs_{i,j+1} = a_1[u_{i-1,j}^{k+1} + u_{i-1,j+2}^{k+1} + u_{i+1,j+2}^{k+1}] + a_2[u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i-1,j}^{k} + u_{i+1,j+2}^{k}] + (1 - 2^{1-\alpha} w_1^* - r)u_{i,j+1}^{k}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^* - w_{k-s+1}^*]u_{i,j+1}^{s} + 2^{1-\alpha} w_k^* u_{i,j+1}^{0} + m_0 f_{i,j+1}^{\frac{k+1}{2}},$$

The 4×4 system can be effortlessly reversed into the de-coupled system of 2×2 equations

$$\begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \end{pmatrix} = \frac{16}{q}\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j+1} \end{pmatrix} \tag{3.3}$$

and

$$\begin{pmatrix} u_{i+1,j}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \frac{16}{q}\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} rhs_{i+1,j} \\ rhs_{i,j+1} \end{pmatrix} \tag{3.4}$$

where $q = 15r^2 + 32r + 16$.

The FEDG scheme involves group comprising two varieties of points located in the x-y plane of the solution domain. The iteration of one variety was achieved through the utilization of equation (3.3) until convergence was realized. Subsequent to the attainment of convergence, equation (2.3) was applied for a direct assessment of the solutions at the residual points [8].

4. Preconditioned Fractional EDG Method

It is well known that preconditioners play a vital role in accelerating the convergence rates of iterative methods, several preconditioned strategies have been used for improving the convergence rate of the iterative methods derived from the standard and skewed (rotated) finite difference operators ([14], [15], [16]). Dramatic improvements are possible, but the difficulty is to construct the suitable preconditioner. In general, a good preconditioner should satisfy the following prosperities: the first one is that, the preconditioned system should be easy to solve and the second one is that the preconditioner should be cheap to construct and apply.

By multiplication the following preconditioner matrix P_U^* for both sides of equation (3.2) such that:

$$P_U^* = \begin{pmatrix} a_1 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & -a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_1 = 1 + r, \quad a_2 = \frac{r}{4},$$

The resulted preconditioned system PFEDG has the same solution of system (3.2), but that has more favourable spectral properties. The effectiveness of this preconditioned PFEDG method will be shown in the next section.

5. Numerical Results and Discussion

Numerical experiments are conducted to examine the effectiveness of the proposed PFEDG method by using the following problem [17],

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [\Gamma(2+\alpha)t - 2t^{1+\alpha}]e^{x+y}$$

where the solution domain is $\Omega = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$, with Dirichlet boundary requisites which comply with the exact solution $u(x, y, t) = e^{x+y} t^{1+\alpha}$. Through the numerical experiments, the FEG, FEDG and PFEDG iterative methods applied on a variety of grid dimensions (4, 8,16, 20 and 24) with varying time steps $(\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{20}, \frac{1}{24})$ for $0 < t < 1$. Preconditioned methods were deemed efficient through investigations which revealed their superiority in the context of execution time (measured in seconds), number of iterations (Ite) and maximum absolute error (Max) with tolerance $\varepsilon = 10^{-6}$. From the following tables (1) and (2), we can observe that the proposed preconditioned system (PFEDG) is the most superior among the FEG and FEDG methods in term of the number of iterations and execution times with $\alpha = 0.25$ and $\alpha = 0.75$ respectively which yield very encouraging results. Furthermore, we can observe that the results reveal the significant improvement in number of iterations and execution timings of the proposed PFEDG iterative method compared to the results obtained in([18], [19]).

Table 1. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.25$

Δt	n	**Method**	**Time**	**iterations**	**Max Error**
$\frac{1}{4}$	4	FEG	0.0297	9	8.24E-5
		FEDG	0.0124	8	8.13E-4
		PFEDG	0.0063	4	8.09E-5
$\frac{1}{8}$	8	FEG	0.8704	26	4.62E-4
		FEDG	0.4922	19	4.62E-4
		PFEDG	0.3379	11	4.21E-5
$\frac{1}{16}$	16	FEG	6.9461	43	9.86E-4
		FEDG	4.2013	37	9.75E-4
		PFEDG	3.6812	25	9.06E-4
$\frac{1}{20}$	20	FEG	218.035	67	8.52E-5
		FEDG	198.016	56	7.66E-5
		PFEDG	164.486	32	7.26E-5
$\frac{1}{24}$	24	FEG	224.492	121	4.52E-6
		FEDG	202.012	93	2.98E-5
		PFEDG	172.324	64	2.44E-6

Table 2. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.75$

Δt	n	**Method**	**Time**	**iterations**	**Max Error**
$\frac{1}{4}$	4	FEG	0.1463	8	3.36E-3
		FEDG	0.0104	7	3.33E-3
		PFEDG	0.0063	4	3.32E-3
$\frac{1}{8}$	8	FEG	0.7553	21	1.41E-3
		FEDG	0.4241	17	1.48E-3
		PFEDG	0.3182	10	1.47E-3
$\frac{1}{16}$	16	FEG	6.4563	41	5.22E-4
		FEDG	4.0014	35	4.71E-4
		PFEDG	3.3231	23	4.68E-4
$\frac{1}{20}$	20	FEG	218.035	52	3.76E-4
		FEDG	146.235	48	2.61E-4
		PFEDG	144.221	27	2.57E-4
$\frac{1}{24}$	24	FEG	186.673	96	3.15E-3
		FEDG	172.545	82	2.12E-3
		PFEDG	134.981	54	2.08E-3

6. Conclusion

In this paper, we have developed a new fractional explicit decoupled group method for solving two-dimensional time-fractional diffusion equation. The proposed PFEDG is derived from the rotated fractional formula with $\sqrt{2}\,h$ spacing. It has been demonstrated that the PFEDG method requires less iterations and less computing times to solve the equation, making it a more effective method in solving fractional diffusion equation problems comparing with the FEG and FEDG methods.

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