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Gaussian Generalized Tribonacci Numbers

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Abstract

In this paper, we define Gaussian generalized Tribonacci numbers and as special cases, we investigate Gaussian Tribonacci and Gaussian Tribonacci-Lucas numbers with their properties.

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1. Introduction

In this work, we define Gaussian generalized Tribonacci numbers and give properties of Gaussian Tribonacci and Gaussian Tribonacci-Lucas numbers as special cases. First, we present some background about generalized Tribonacci numbers and Gaussian numbers before defining Gaussian generalized Tribonacci numbers.

Recently, there have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Tribonacci and Tribonacci-Lucas which are special case of generalized Tribonacci numbers. A generalized Tribonacci sequence$\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} \#(1.1)$$

with the initial values$V_0 = c_0, V_1 = c_1, V_2 = c_2$not all being zero.This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example[4], [5], [7],[8],[18], [20], [23], [25], [27], [32], [33].

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1,2,3, \ldots$. Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Tribonacci numbers with positive subscript and negative subscript are given in the following table:

n	0	1	2	3	4	5	$\cdots$
V_n	c_0	c_1	c_2	$c_0+c_1+c_2$	$c_0+2c_1+2c_2$	$2c_0+3c_1+4c_2$	$\cdots$
V_{-n}	c_0	$c_2-c_1-c_0$	$2c_1-c_2$	$2c_0-c_1$	$2c_2-2c_1-3c_0$	$c_0+5c_1-3c_2$	$\cdots$

It is well known that generalized Tribonacci numbers $V_n(V_0, V_1, V_2)$ can be written, for all integers n, in the Binet form

$$V_n = \frac{P\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{Q\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{R\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \quad \#(1.2)$$

where

$$P = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,$$

$$Q = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,$$

$$R = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0,$$

andwhere α, β and $\gamma$are the distinct roots of the cubic equation $x^3 - x^2 - x - 1 = 0$and they are given as

$$\alpha = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\beta = \frac{1 + \omega\sqrt[3]{19+3\sqrt{33}} + \omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19+3\sqrt{33}} + \omega\sqrt[3]{19-3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = exp(2\pi i/3),$$

is a primitive cube root of unity.We consider two special cases ofV_n: $V_n(0,1,1) = T_n$is the sequence of Tribonacci numbers (sequence A000073 in [23]) and $V_n(3,1,3) = K_n$is the sequence of Tribonacci-Lucas numbers (A001644 in [26]).

Recently, there have been so many studies of the sequences of Gaussian numbers in the literature. A Gaussian integer $z$is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib,\ a, b \in \mathbb{Z}$. These numbers were investigated by Gauss in 1832 and the set of them is denoted by $\mathbb{Z}[i]$. With the usual addition and multiplication of complex numbers, $\mathbb{Z}[i]$ forms an integral domain. The norm of a Gaussian integer $a + ib, a, b \in \mathbb{Z}$is its Euclidean norm, that is, $N\,(a+ib) = \sqrt{a^2+b^2} = \sqrt{(a+ib)(a-ib)}$.For more information about this kind of integers, we refer to the work of Fraleigh [10].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tetranacci numbers.

In 1963, Horadam [16] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [22] defined the complex Tribonacci numbers at Gaussian integers, see also [12]. There are other several studies dedicated to these sequences of Gaussian numbers such as the works in [1], [3], [6], [12], [13], [14], [15], [16], [17], [19], [21], [28], [29], [30], among others.

2. Gaussian Generalized Tribonacci Numbers

Gaussian generalized Tribonacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2)\}_{n\geq 0}$are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3}\#(2.1)$$

with the initial conditions

$$GV_0 = c_0 + i(c_2 - c_1 - c_0), GV_1 = c_1 + ic_0, GV_2 = c_2 + ic_1,$$

not all being zero. The sequences $\{GV_n\}_{n\geq 0}$can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} + GV_{-(n-3)}$$

for $n = 1,2,3, \ldots$. Therefore, recurrence (2.1) hold for all integer n. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1} \ \#(2.2)$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}.$$

The first few Gaussian generalized Tribonacci numbers with positive subscript and negative subscript are given in the following table:

n	0	1	2	3
GV_n	$c_0 + i(c_2 - c_1 - c_0)$	$c_1 + ic_0$	$c_2 + ic_1$	$c_0 + c_1 + c_2 + ic_2$
GV_{-n}	$c_0 + i(c_2 - c_1 - c_0)$	$c_2 - c_1 - c_0 + i(2c_1 - c_2)$	$2c_1 - c_2 + i(2c_0 - c_1)$	$2c_0 - c_1 + i(2c_2 - 2c_1 - 3c_0)$

We consider two special cases of GV_n: $GV_n(0,1,1 + i) = GT_n$ is the sequence of Gaussian Tribonacci numbers and $GV_n(3 - i, 1 + 3i, 3 + i) = GK_n$is the sequence of Gaussian Tribonacci-Lucas numbers. We formally define them as follows:

Gaussian Tribonacci numbers are defined by

$$GT_n = GT_{n-1} + GT_{n-2} + GT_{n-3}\#(2.3)$$

with the initial conditions

$$GT_0 = 0, GT_1 = 1, GT_2 = 1 + i$$

and Gaussian Tribonacci-Lucas numbers are defined by

$$GK_n = GK_{n-1} + GK_{n-2} + GK_{n-3}\#(2.4)$$

with the initial conditions

$$GK_0 = 3 - i, GK_1 = 1 + 3i, GK_2 = 3 + i.$$

Note that for $n \geq 0$

$$GT_{-n} = T_{-n} + iT_{-n-1}$$

and

$$GK_{-n} = K_{-n} + iK_{-n-1}.$$

The first few values of Gaussian Tribonacci numbers with positive and negative subscript are given in the following table.

n	0	1	2	3	4	5	6	7	8	9
GT_n	0	1	$1+i$	$2+i$	$4+2i$	$7+4i$	$13+7i$	$24+13i$	$44+24i$	$81+44i$
GT_{-n}	0	i	$1-i$	-1	$2i$	$2-3i$	$-3+i$	$1+4i$	$4-8i$	$-8+5i$

The first few values of Gaussian Tribonacci-Lucas numbers with positive and negative subscript are given in the following table.

n	0	1	2	3	4	5	6	7	8
GK_n	$3-i$	$1+3i$	$3+i$	$7+3i$	$11+7i$	$21+11i$	$39+21i$	$71+39i$	$131+71i$
GK_{-n}	$3-i$	$-1-i$	$-1+5i$	$5-5i$	$-5-i$	$-1+11i$	$11-15i$	$-15+3i$	$3+23i$

The following Theorem presents the generating function of Gaussian generalized Tribonacci numbers.

Theorem 2.1 The generating function of Gaussian generalized Tribonacci numbers is given as

$$f_{GV_n}(x)=\sum_{n=0}^{\infty} GV_n x^n=\frac{GV_0+(GV_1-GV_0)x+(GV_2-GV_1-GV_0)x^2}{1-x-x^2-x^3}. \quad \#(2.5)$$

Proof. Let

$$f_{GV_n}(x)=\sum_{n=0}^{\infty} GV_n x^n$$

be generating function of Gaussian generalized Tribonacci numbers. Then using the definition of Gaussian Tribonacci numbers, and substracting$xf(x)$, $x^2f(x)$ and $x^3f(x)$from $f(x)$we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-x-x^2-x^3)f_{GV_n}(x) &= \sum_{n=0}^{\infty} GV_n x^n - x\sum_{n=0}^{\infty} GV_n x^n - x^2\sum_{n=0}^{\infty} GV_n x^n - x^3\sum_{n=0}^{\infty} GV_n x^n \\
&= \sum_{n=0}^{\infty} GV_n x^n - \sum_{n=0}^{\infty} GV_n x^{n+1} - \sum_{n=0}^{\infty} GV_n x^{n+2} - \sum_{n=0}^{\infty} GV_n x^{n+3} \\
&= \sum_{n=0}^{\infty} GV_n x^n - \sum_{n=1}^{\infty} GV_{n-1} x^{n} - \sum_{n=2}^{\infty} GV_{n-2} x^{n} - \sum_{n=0}^{\infty} GV_{n-3} x^{n} \\
&= (GV_0+GV_1x+GV_2x^2)-(GV_0x+GV_1x^2)-GV_0x^2 \\
&\quad +\sum_{n=3}^{\infty}(GV_n-GV_{n-1}-GV_{n-2}-GV_{n-3})x^n \\
&= GV_0+GV_1x+GV_2x^2-GV_0x-GV_1x^2-GV_0x^2 \\
&= GV_0+(GV_1-GV_0)x+(GV_2-GV_1-GV_0)x^2.
\end{aligned}$$

Rearranging above equation, we get

$$f_{GV_n}(x)=\frac{GV_0+(GV_1-GV_0)x+(GV_2-GV_1-GV_0)x^2}{1-x-x^2-x^3}.$$

The previous Theorem gives the following results as particular examples:

$$f_{GT_n}(x)=\frac{x+ix^2}{1-x-x^2-x^3}\#(2.6)$$

and

$$f_{GK_n}(x)=\frac{-(1+i)x^2-(2-4i)x+3-i}{1-x-x^2-x^3}. \quad \#(2.7)$$

The result (2.6) is already known, see [12].

We now present the Binet formula for the Gaussian generalized Tribonacci numbers.

Theorem 2.2 TheBinet formula for the Gaussian generalized Tribonacci numbers is

$$GV_n = \left(\frac{P\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{Q\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{R\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right)$$
$$+i\left(\frac{P\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{Q\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{R\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

whereP, Q and R are as in (1.2).

Proof. The proof follows from $(1.2)$and (2.2).

The previous Theorem gives the following results as particular examples: the Binet formula for the Gaussian Tribonaccinumbers is

$$GT_n = \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$
$$+i\left(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

and the Binet formula for the Gaussian Tribonacci-Lucas numbers is

$$GK_n = (\alpha^n + \beta^n + \gamma^n) + i(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1}).$$

The following Theorem present some formulas of Gaussian generalized Tribonacci numbers.

Theorem 2.3 For$n \geq 1$ we have the following formulas:

(a) (Sum of the Gaussian Generalized Tribonacci numbers)

$$\sum_{k=1}^{n} GV_k = \frac{1}{2}(GV_{n+3} - GV_{n+1} - GV_2 - GV_0)$$

(b): $\sum_{k=1}^{n} GV_{2k+1} = \frac{1}{2}(GV_{2n+2} + GV_{2n+1} + GV_1 - GV_2)$

(c): $\sum_{k=1}^{n} GV_{2k} = \frac{1}{2}(GV_{2n+4} - 2GV_{2n+2} - GV_{2n+1} - GV_0 - 3GV_1).$

Proof.

(a) Using the recurrence relation

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3}$$

i.e.

$$GV_{n-3} = GV_n - GV_{n-1} - GV_{n-2}$$

we obtain

$$GV_0 = GV_3 - GV_2 - GV_1$$

$$GV_1 = GV_4 - GV_3 - GV_2$$

$$GV_2 = GV_5 - GV_4 - GV_3$$

$$GV_3 = GV_6 - GV_5 - GV_4$$

$$GV_4 = GV_7 - GV_6 - GV_5$$

$$\vdots$$

$$GV_{n-3} = GV_n - GV_{n-1} - GV_{n-2}$$

$$GV_{n-2} = GV_{n+1} - GV_n - GV_{n-1}$$

$$GV_{n-1} = GV_{n+2} - GV_{n+1} - GV_n$$

$$GV_n = GV_{n+3} - GV_{n+2} - GV_{n+1}.$$

If we add the equations by side by, we get

$$\sum_{k=1}^{n} GV_k = \frac{1}{2}(GV_{n+3} - GV_{n+1} - GV_2 - GV_0).$$

(b) When we use (2.1), we obtain the following equalities:

$$GV_k = GV_{k-1} + GV_{k-2} + GV_{k-3}$$

$$GV_4 = GV_3 + GV_2 + GV_1$$

$$GV_6 = GV_5 + GV_4 + GV_3$$

$$GV_8 = GV_7 + GV_6 + GV_5$$

$$GV_{10} = GV_9 + GV_8 + GV_7$$

$$\vdots$$

$$GV_{2n+2} = GV_{2n+1} + GV_{2n} + GV_{2n-1}.$$

If we rearrange the above equalities, we obtain

$$GV_3 = GV_4 - GV_2 - GV_1$$

$$GV_5 = GV_6 - GV_4 - GV_3$$

$$GV_7 = GV_8 - GV_6 - GV_5$$

$$GV_9 = GV_{10} - GV_8 - GV_7$$

$$\vdots$$

$$GV_{2n+1} = GV_{2n+2} - GV_{2n} - GV_{2n-1}.$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}\sum_{k=1}^{n} GV_{2k+1} &= GV_{2n+2} - GV_2 - \sum_{k=1}^{n} GV_{2k-1} \\ &= GV_{2n+2} - GV_2 - \left(\sum_{k=1}^{n} GV_{2k+1} - GV_{2n+1} - GV_1\right)\end{aligned}$$

$$= GV_{2n+2} + GV_{2n+1} + GV_1 - GV_2 - \sum_{k=1}^{n} GV_{2k+1}$$

and so

$$\sum_{k=1}^{n} GV_{2k+1} = \frac{GV_{2n+2} + GV_{2n+1} + GV_1 - GV_2}{2}.$$

(c) Since

$$\sum_{k=1}^{n} GV_{2k+1} + \sum_{k=1}^{n} GV_{2k} = \sum_{k=1}^{2n+1} GV_k - GV_1$$

we have

$$\begin{aligned}\sum_{k=1}^{n} GV_{2k} &= \sum_{k=1}^{2n+1} GV_k - \sum_{k=1}^{n} GV_{2k+1} - GV_1 \\ &= \frac{1}{2}\left(GV_{(2n+1)+3} - GV_{(2n+1)+1} - GV_2 - GV_0\right) - \frac{GV_{2n+2} + GV_{2n+1} + GV_1 - GV_2}{2} - GV_1 \\ &= \frac{1}{2}(GV_{2n+4} - GV_{2n+2} - GV_2 - GV_0) - \frac{GV_{2n+2} + GV_{2n+1} + GV_1 - GV_2}{2} - GV_1 \\ &= \frac{1}{2}(GV_{2n+4} - 2GV_{2n+2} - GV_{2n+1} - GV_0 - 3GV_1)\end{aligned}$$

This completes the proof.

As special cases of above Theorem, we have the following two Corollary. First one present some formulas of Gaussian Tribonacci numbers.

COROLLARY 2.4 For $n \geq 1$we have the following formulas:

(a):(Sum of the Gaussian Tribonacci numbers)

$$\sum_{k=1}^{n} GT_k = \frac{1}{2}(GT_{n+3} - GT_{n+1} + (1+i))$$

(b):$\sum_{k=1}^{n} GT_{2k+1} = \frac{1}{2}(GT_{2n+2} + GT_{2n+1} - 1)$

(c):$\sum_{k=1}^{n} GT_{2k} = \frac{1}{2}(GT_{2n+4} - 2GT_{2n+2} - GT_{2n+1} - 3)$.

Second Corollary gives some formulas of Gaussian Tribonacci-Lucas numbers.

COROLLARY 2.5 For $n \geq 1$we have the following formulas:

(a): (Sum of the Gaussian Tribonacci-Lucas numbers)

$$\sum_{k=1}^{n} GK_k = \frac{1}{2}(GK_{n+3} - GK_{n+1} + 2i)$$

(b): $\sum_{k=1}^{n} GK_{2k+1} = \frac{1}{2}(GK_{2n+2} + GK_{2n+1} - 2 + 2i)$

(c): $\sum_{k=1}^{n} GK_{2k} = \frac{1}{2}(GK_{2n+4} - 2GK_{2n+2} - GK_{2n+1} - 10i)$.

3. Some Identities Connecting Gaussian Tribonacci and Gaussian Tribonacci-Lucas Numbers

In this section, we obtain some identities of Gaussian Tribonacci numbers and Gaussian Tribonacci-Lucas numbers.

First, we can give a few basic relations between $\{GT_n\}$ and $\{GK_n\}$ as

$$GK_n = -GT_{n+2} + 4GT_{n+1} - GT_n \quad \#(3.1)$$
$$GK_n = 3GT_{n+1} - 2GT_n - GT_{n-1} \quad \#(3.2)$$

and also

$$GK_n = GT_n + 2GT_{n-1} + 3GT_{n-2}. \quad \#(3.3)$$

Note that the last three identities hold for all integers n. For example, to show (3.1), writing

$$GK_n = -GT_{n+2} + 4GT_{n+1} - GT_n$$

and solving the system of equations

$$GK_0 = aGT_2 + bGT_1 + cGT_0$$

$$GK_1 = aGT_3 + bGT_2 + cGT_1$$

$$GK_2 = aGT_4 + bGT_3 + cGT_2$$

we find that $a = -1, b = 4, c = -1$. Or using the relations $GT_n = T_n + iT_{n-1}$, $GK_n = K_n + iK_{n-1}$ and identity $K_n = 4T_{n+1} - T_n - T_{n+2}$ we obtain the identity (3.1). The others can be found similarly.

We will present some other identities between Gaussian Tribonacci and Gaussian Tribonacci-Lucas numbers with the help of generating functions. Firstly, we give the ordinary generating function of the sequence V_n.

LEMMA 3.1 Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} a_n x^n$ is the ordinary generating function of the sequence V_n. Then $f_{V_n}(x)$ is given by

$$f_{V_n}(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3}. \quad \#(3.4)$$

Proof. Using (1.1) and some calculation, we obtain

$$f_{V_n}(x) - xf_{V_n}(x) - x^2 f_{V_n}(x) - x^3 f_{V_n}(x) = V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2$$

which gives (3.4).

The previous Lemma gives the following results as particular examples:

$$f_{T_n}(x) = \frac{x}{1 - x - x^2 - x^3}$$

and

$$f_{K_n}(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}.$$

Both results are very well known.

The following lemma will help us to derive the generating functions of even and odd-indexed Gaussian Tribonacci and Gaussian Tribonacci-Lucas sequences.

LEMMA 3.2 ([11]) Suppose that $f(x)=\sum_{n=0}^{\infty}a_n x^n$ is the generating function of the sequence $\{a_n\}_{n\geq 0}$. Then the generating functions of the sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$ are given as

$$f_{a_{2n}}(x)=\sum_{n=0}^{\infty}a_{2n}x^n=\frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x)=\sum_{n=0}^{\infty}a_{2n+1}x^n=\frac{f(\sqrt{x})-f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The next Theorem presents the generating functions of even and odd-indexed generalized Tribonacci sequences.

THEOREM 3.3 ([11]) The generating functions of the sequences V_{2n} and V_{2n+1} are given by

$$f_{V_{2n}}(x)=\frac{V_0+(V_2-3V_0)x+(2V_1-V_2)x^2}{1-3x-x^2-x^3}$$

and

$$f_{V_{2n+1}}(x)=\frac{V_1+(V_2-2V_1+V_0)x+(V_2-V_1-V_0)x^2}{1-3x-x^2-x^3}$$

respectively.

From the previous Theorem we get the following results as particular examples:

$$f_{T_{2n}}(x)=\frac{x+x^2}{1-3x-x^2-x^3}\text{ and }f_{T_{2n+1}}(x)=\frac{1-x}{1-3x-x^2-x^3}$$

and

$$f_{K_{2n}}(x)=\frac{3-6x-x^2}{1-3x-x^2-x^3}\text{ and }f_{K_{2n+1}}(x)=\frac{1+4x-x^2}{1-3x-x^2-x^3}.$$

The next Theorem presents the generating functions of even and odd-indexed Gaussian generalized Tribonacci sequences.

THEOREM 3.4 The generating functions of the sequences GV_{2n} and GV_{2n+1} are given by

$$f_{GV_{2n}}=\frac{GV_0+(GV_2-3GV_0)x+(2GV_1-GV_2)x^2}{1-3x-x^2-x^3}$$

and

$$f_{GV_{2n+1}}=\frac{GV_1+(GV_2-2GV_1+GV_0)x+(GV_2-GV_1-GV_2)x^2}{1-3x-x^2-x^3}$$

respectively.

Proof. Both statements are consequences of Lemma 3.2 applied to (2.5) and some lengthy algebraic calculations.

The previous theorem gives the following two corollaries as particular examples. Firstly, the next one presents the generating functions of even and odd-indexed Gaussian Tribonacci sequences.

COROLLARY 3.5 The generating functions of the sequences GT_{2n} and GT_{2n+1} are given by

$$f_{GT_{2n}} = \frac{(1+i)x + (1-i)x^2}{1-3x-x^2-x^3} \#(3.5)$$

and

$$f_{GT_{2n+1}} = \frac{1+(i-1)x + ix^2}{1-3x-x^2-x^3} \#(3.6)$$

respectively.

The following Corollary gives the generating functions of even and odd-indexed Gaussian Tribonacci-Lucas sequences.

COROLLARY 3.6 The generating functions of the sequences GK_{2n} and GK_{2n+1} are given by

$$f_{GK_{2n}} = \frac{(3-i)+(-6+4i)x+(-1+5i)x^2}{1-3x-x^2-x^3} \#(3.7)$$

and

$$f_{GK_{2n+1}} = \frac{(1+3i)+(4-6i)x+(-1-i)x^2}{1-3x-x^2-x^3} \#(3.8)$$

respectively.

The next Corollary present identities between GausssianTribonacci and Gaussian Tribonacci-Lucas sequences.

Corollary 3.7 We have the following identities:

$$(3-i)GT_{2n} + (-6+4i)GT_{2n-2} + (-1+5i)GT_{2n-4} = (1+i)GK_{2n-2} + (1-i)GK_{2n-4},$$

$$(1+3i)GT_{2n} + (4-6i)GT_{2n-2} + (-1-i)GT_{2n-4} = (1+i)GK_{2n-1} + (1-i)GK_{2n-3},$$

$$(3-i)GT_{2n+1} + (-6+4i)GT_{2n-1} + (-1+5i)GT_{2n-3} = GK_{2n} + (i-1)GK_{2n-2} + iGK_{2n-4},$$

$$(1+3i)GT_{2n+1} + (4-6i)GT_{2n-1} + (-1-i)GT_{2n-3} = GK_{2n+1} + (i-1)GK_{2n-1} + iGK_{2n-3}.$$

Proof. From (3.5) and (3.7) we obtain

$$\left((3-i)+(-6+4i)x+(-1+5i)x^2\right)f_{GT_{2n}} = \left((1+i)x+(1-i)x^2\right)f_{GK_{2n}}.$$

The LHS (left hand side) is equal to

$$LHS = \left((3-i)+(-6+4i)x+(-1+5i)x^2\right)\sum_{n=0}^{\infty} GT_{2n}x^n$$

$$= (3-i)(1+i)x + \sum_{n=2}^{\infty}((3-i)GT_{2n} + (-6+4i)GT_{2n-2} + (-1+5i)GT_{2n-4})x^n$$

whereas the RHS is

$$RHS = \left((1+i)x+(1-i)x^2\right)\sum_{n=0}^{\infty} GK_{2n}x^n$$

$$= (1+i)(3-i)x + \sum_{n=2}^{\infty}((1+i)GK_{2n-2} + (1-i)GK_{2n-4})x^n.$$

Compare the coefficients and the proof of the first identity is done. The other identities can be proved similarly by using (3.5)- (3.8).

We present an identity related with Gaussian general Tribonacci numbers and Tribonacci numbers.

Theorem 3.8 For $n \geq 0$ and $m \geq 0$ the following identity holds:

$$GV_{m+n} = T_{m-1}GV_{n+2} + (T_{m-2} + T_{m-3})GV_{n+1} + T_{m-2}GV_n. \ \#(3.9)$$

Proof. We prove the identity by strong induction on m. If $m = 0$ then

$$GV_n = T_{-1}GV_{n+2} + (T_{-2} + T_{-3})GV_{n+1} + T_{-2}GV_n,$$

which is true because$T_{-1} = 0, T_{-2} = 1, T_{-3} = -1$. Assume that the equality holds for $m \leq k$. For $m = k+1$, we have

$$\begin{aligned}
GV_{(k+1)+n} &= GV_{n+k} + GV_{n+k-1} + GV_{n+k-2} \\
&= (T_{k-1}GV_{n+2} + (T_{k-2} + T_{k-3})GV_{n+1} + T_{k-2}GV_n) \\
&\quad + (T_{k-2}GV_{n+2} + (T_{k-3} + T_{k-4})GV_{n+1} + T_{k-3}GV_n) \\
&\quad + (T_{k-3}GV_{n+2} + (T_{k-4} + T_{k-5})GV_{n+1} + T_{k-4}GV_n) \\
&= (T_{k-1} + T_{k-2} + T_{k-3})GV_{n+2} + ((T_{k-2} + T_{k-3} + T_{k-4}) \\
&\quad + (T_{k-3} + T_{k-4} + T_{k-5}))GV_{n+1} + (T_{k-2} + T_{k-3} + T_{k-4})GV_n \\
&= T_kGV_{n+2} + (T_{k-1} + T_{k-2})GV_{n+1} + T_{k-1}GV_n \\
&= T_{(k+1)-1}GV_{n+2} + \big(T_{(k+1)-2} + T_{(k+1)-3}\big)GV_{n+1} + T_{(k+1)-2}GV_n.
\end{aligned}$$

By strong induction on m, this proves (3.9).

The previous Theorem gives the following results as particular examples: For $n \geq 0$and $m \geq 0$, we have (taking $GV_n = GT_n$)

$$GT_{m+n} = T_{m-1}GT_{n+2} + (T_{m-2} + T_{m-3})GT_{n+1} + T_{m-2}GT_n$$

and (taking $GV_n = GK_n$)

$$GK_{m+n} = T_{m-1}GK_{n+2} + (T_{m-2} + T_{m-3})GK_{n+1} + T_{m-2}GK_n.$$

4. Matrix Formulation of V_n

Consider the sequence$\{U_n\}$ which is defined by the third-order recurrence relation

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}, \qquad U_0 = U_1 = 0, U_2 = 1.$$

Note that some authors call $\{U_n\}$ as a Tribonacci sequence instead of $\{T_n\}$. The numbers U_n can be expressed using Binet's formula

$$U_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

and the negative numbers $U_{-n}(n=1,2,3,\dots)$ satisfies the recurrence relation

$$U_{-n}=\begin{vmatrix}U_{n+1} & U_{n+2}\\ U_n & U_{n+1}\end{vmatrix}=U_{n+1}^2-U_{n+2}U_n.$$

The matrix method is very useful method in order to obtain some identities for special sequences. We define the square matrix M of order 3 as:

$$M=\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

such that $detM=1$. Note that

$$M^n=\begin{pmatrix}U_{n+2} & U_{n+1}+U_n & U_{n+1}\\ U_{n+1} & U_n+U_{n-1} & U_n\\ U_n & U_{n-1}+U_{n-2} & U_{n-1}\end{pmatrix}.\#(4.1)$$

For a proof of(4.1), see [2]. Matrix formulation of T_n and K_n can be given as

$$\begin{pmatrix}T_{n+2}\\ T_{n+1}\\ T_n\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^n\begin{pmatrix}T_2\\ T_1\\ T_0\end{pmatrix}\#(4.2)$$

and

$$\begin{pmatrix}K_{n+2}\\ K_{n+1}\\ K_n\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^n\begin{pmatrix}K_2\\ K_1\\ K_0\end{pmatrix}.\#(4.3)$$

The matrix M was defined and used in [24]. For matrix formulations (4.2)and (4.3), see [31] and [34]. Note that

$$GT_n=iU_n+U_{n+1}$$

and

$$GK_n=(3-i)U_{n+2}-(2-4i)U_{n+1}-(1+i)U_n.$$

Consider the matrices N_T,E_T defined by as follows:

$$N_T=\begin{pmatrix}1+i & 1 & 1\\ 1 & 0 & i\\ 0 & i & 1-i\end{pmatrix},$$

$$E_T=\begin{pmatrix}GT_{n+2} & GT_{n+1} & GT_n\\ GT_{n+1} & GT_n & GT_{n-1}\\ GT_n & GT_{n-1} & GT_{n-2}\end{pmatrix}.$$

Next Theorem presents the relations between M^n, $N_T$and E_T.

Theorem 4.1 ([12]) For $n\geq 2$, we have

$$M^nN_T=E_T.$$

Proof. Using the relation

$$GT_n=iU_n+U_{n+1}$$

we get

$$M^nN_T=\begin{pmatrix}U_{n+2} & U_{n+1}+U_n & U_{n+1}\\ U_{n+1} & U_n+U_{n-1} & U_n\\ U_n & U_{n-1}+U_{n-2} & U_{n-1}\end{pmatrix}\begin{pmatrix}1+i & 1 & 1\\ 1 & 0 & i\\ 0 & i & 1-i\end{pmatrix}$$

$$=\begin{pmatrix}U_n+U_{n+1}+(1+i)U_{n+2} & iU_{n+1}+U_{n+2} & iU_n+U_{n+1}\\ U_n+U_{n-1}+(1+i)U_{n+1} & iU_n+U_{n+1} & U_n+iU_{n-1}\\ (1+i)U_n+U_{n-1}+U_{n-2} & U_n+iU_{n-1} & U_{n-1}+iU_{n-2}\end{pmatrix}$$

$$=\begin{pmatrix}GT_{n+2} & GT_{n+1} & GT_n\\ GT_{n+1} & GT_n & GT_{n-1}\\ GT_n & GT_{n-1} & GT_{n-2}\end{pmatrix}.$$

Above Theorem can be proved by mathematical induction as well.

Consider the matrices N_K, E_K defined by as follows:

$$N_K=\begin{pmatrix}3+i & 1+3i & 3-i\\ 1+3i & 3-i & -1-i\\ 3-i & -1-i & -1+5i\end{pmatrix},$$

$$E_K=\begin{pmatrix}GK_{n+2} & GK_{n+1} & GK_n\\ GK_{n+1} & GK_n & GK_{n-1}\\ GK_n & GK_{n-1} & GK_{n-2}\end{pmatrix}.$$

The following Theorem presents the relations between M^n, N_K and E_K.

Theorem 4.2 For $n\geq 2$, we have

$$M^nN_K=E_K.$$

Proof. Using the relation

$$GK_n=(3-i)U_{n+2}-(2-4i)U_{n+1}-(1+i)U_n$$

we get

$$M^nN_K=\begin{pmatrix}U_{n+2} & U_{n+1}+U_n & U_{n+1}\\ U_{n+1} & U_n+U_{n-1} & U_n\\ U_n & U_{n-1}+U_{n-2} & U_{n-1}\end{pmatrix}\begin{pmatrix}3+i & 1+3i & 3-i\\ 1+3i & 3-i & -1-i\\ 3-i & -1-i & -1+5i\end{pmatrix}$$

$$=\begin{pmatrix}GK_{n+2} & GK_{n+1} & GK_n\\ GK_{n+1} & GK_n & GK_{n-1}\\ GK_n & GK_{n-1} & GK_{n-2}\end{pmatrix}.$$

The previous Theorem, also, can be proved by mathematical induction.

Similarly, matrix formulation of V_n can be given as

$$\begin{pmatrix}V_{n+2}\\ V_{n+1}\\ V_n\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^n\begin{pmatrix}V_2\\ V_1\\ V_0\end{pmatrix}.$$

Consider the matrices N_V, E_V defined by as follows:

$$N_V=\begin{pmatrix}c_2+ic_1 & c_1+ic_0 & c_0+i(-c_0-c_1+c_2)\\ c_1+ic_0 & c_0+i(-c_0-c_1+c_2) & -c_0-c_1+c_2+i(2c_1-c_2)\\ c_0+i(-c_0-c_1+c_2) & -c_0-c_1+c_2+i(2c_1-c_2) & 2c_1-c_2+i(2c_0-c_1)\end{pmatrix},$$

$$E_V = \begin{pmatrix} GV_{n+2} & GV_{n+1} & GV_n \\ GV_{n+1} & GV_n & GV_{n-1} \\ GV_n & GV_{n-1} & GV_{n-2} \end{pmatrix}.$$

We show that for $n \geq 2$, $M^n N_V = E_V$. Note that

$$GV_n = (c_1 + ic_0)U_{n-2} + (c_0 + ic_2 + (1-i)c_1)U_{n-1} + (c_2 + ic_1)U_n$$

$$= c_0 U_{n-1} + ic_0 U_{n-2} + (1-i)c_1 U_{n-1} + c_1 U_{n-2} + ic_2 U_{n-1} + ic_1 U_n + c_2 U_n$$

$$= (c_1 U_{n-2} + c_0 U_{n-1} + c_1 U_{n-1} + c_2 U_n) + i(c_0 U_{n-2} - c_1 U_{n-1} + c_2 U_{n-1} + c_1 U_n).$$

We now present our final Theorem.

Theorem 4.3 For $n \geq 2$, we have

$$M^n N_V = E_V.$$

Proof.

$$M^n N_V = \begin{pmatrix} U_{n+2} & U_{n+1} + U_n & U_{n+1} \\ U_{n+1} & U_n + U_{n-1} & U_n \\ U_n & U_{n-1} + U_{n-2} & U_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} c_2 + ic_1 & c_1 + ic_0 & c_0 + i(-c_0 - c_1 + c_2) \\ c_1 + ic_0 & c_0 + i(-c_0 - c_1 + c_2) & -c_0 - c_1 + c_2 + i(2c_1 - c_2) \\ c_0 + i(-c_0 - c_1 + c_2) & -c_0 - c_1 + c_2 + i(2c_1 - c_2) & 2c_1 - c_2 + i(2c_0 - c_1) \end{pmatrix}$$

$$= \begin{pmatrix} GV_{n+2} & GV_{n+1} & GV_n \\ GV_{n+1} & GV_n & GV_{n-1} \\ GV_n & GV_{n-1} & GV_{n-2} \end{pmatrix}.$$

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