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EQUIVALENCE OF FERMAT'S LAST THEOREM AND BEAL'S CONJECTURE

James E. Joseph and Bhamini M. P. Nayar

ABSTRACT. It is proved in this paper that (1) **Fermat's Last Theorem:** If π is an odd prime, there are no relatively prime solutions x, y, z to the equation $z^\pi = x^\pi + y^\pi$, and (2) **Beal's Conjecture :** The equation $z^\xi = x^\mu + y^\nu$ has no solution in relatively prime positive integers x, y, z with μ, ξ, ν odd primes at least 3. It is proved that these two statements are equivalent.

(1) (Fermat's Last Theorem) If π is an odd prime, there are no relatively prime solutions x, y, z to the equation $z^\pi = x^\pi + y^\pi$,
(2) (Beal's Conjecture) The equation $z^\xi = x^\mu + y^\nu$ has no solution in relatively prime positive integers x, y, z with μ, ξ, ν odd primes at least 3.

See [1], [2] and [3] for history of these problems.

First, the **Fermat's last Theorem** will be proved and then it will be shown that the Beal's Conjecture is equivalent to the Fermat's Theorem.

Proof of Fermat's last Theorem. It will be shown that if x, y, z are relativity prime positive integers, π is an odd prime and if $z^\pi = x^\pi + y^\pi$, then we arrive at a contradiction. Edwards [1] has proved that $z^4 \neq x^4 + y^4$ for relatively prime positive integers x, y and z.

It is clear that if $z^\pi = x^\pi + y^\pi$, then either x or y or z is divisible by 2. Suppose z is divisible by 2. Then x and y are odd. Since $z^\pi = x^\pi + y^\pi$, z^π is $2^{m\pi}$ times an odd integer, where m is an integer, and $x^\pi + y^\pi = (x+y)(\sum_{k=0}^{\pi-1} x^k y^{\pi-1-k})$, by prime factorization and since $x + y$ is even. Hence,

$$x + y = 2^{m\pi}. \qquad (1)$$

Also,

$$x + y - z \equiv 0 \ (mod\ 2). \qquad (2)$$

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Also,

$$x + y - z \equiv 0 \ (mod\ 2). \qquad (2)$$

So,

$$(x + y - z)^{\pi} \equiv 0 \ (mod\ 2^{\pi});$$

and

$$(x + y)^{\pi} - z^{\pi} \equiv 0 \ (mod\ 2^{\pi}), \qquad (3)$$

since, by expanding $(x + y - z)^{\pi}$ using binomial expansion,

$$(x + y - z)^{\pi} - ((x + y)^{\pi} - z^{\pi}) = \sum_{k=1}^{\pi-1} C(\pi, k)(x + y)^{\pi-k}(-z)^{k}.$$

Hence, in view of equation (2) and (3),

$$z^{\pi} - x^{\pi} - y^{\pi} = (x + y)^{\pi} - x^{\pi} - y^{\pi}$$

$$= \sum_{k=1}^{\pi-1} C(\pi, k)x^{\pi-k}y^{k} \equiv 0 \ (mod\ 2^{\pi}). \qquad (4)$$

So, $y \equiv 0 \ (mod\ 2)$ and $x \equiv 0 \ (mod\ 2)$. That is, if z is even, x and y are even.

Now assume that x is even and we have $x^{\pi} = z^{\pi} - y^{\pi}$

Since x is even, z and y are odd; $z - y = 2^{nx}$ for some integer n and hence

$$z - y - x \equiv 0 \ (mod\ 2). \qquad (5)$$

So,

$$(z - y - x)^{\pi} \equiv 0 \ (mod\ 2^{\pi}). \qquad (6)$$

Also

$$(z-y-x)^{\pi} - ((z-y)^{\pi} - x^{\pi}) = \sum_{k=1}^{\pi-1} C(\pi, k)(z-y)^{\pi-k}(-x)^{k} \equiv 0 \ (mod\ 2^{\pi}). \quad (7)$$

So,

$$(z - y)^{\pi} - x^{\pi} \equiv 0 \ (mod\ 2^{\pi}). \qquad (8)$$

Hence,

$$x^{\pi} - z^{\pi} + y^{\pi} = (z - y)^{\pi} - z^{\pi} + y^{\pi}$$

$$= \sum_{k=1}^{\pi-1} C(\pi, k)z^{\pi-k}(-y)^{k} \equiv 0 \ (mod\ 2^{\pi})$$

So, $z \equiv 0 \ (mod\ 2)$; and $y \equiv 0 \ (mod\ 2)$.

The case when y is even is similar to the case when x is even. So, if either x or y or z is even then, all are even which leads to a contradiction of the equation. Hence Fermat's last Theorem.

Now, consider **Beal's conjecture.** Assume Fermat's Last Theorem and let $\xi, \mu, \nu, \geq 3$. Then,

$$(z^{\xi})^{\pi} \neq (x^{\mu})^{\pi} + (y^{\nu})^{\pi}$$

Suppose that $z^{\xi} = x^{\mu} + y^{\nu}$, for any x, y and z.

Then $(z^{\xi})^{\xi} = (x^{\xi})^{\mu} + (y^{\xi})^{\nu}$, replacing x, y and z with x^{ξ}, y^{ξ} and z^{ξ}. Hence $(z^{\xi})^{\xi} = (x^{\mu})^{\xi} + (v^{\nu})^{\xi}$. As in the proof of Fermat's Last Theorem, it can be shown that each x^{μ}, y^{ν} and z^{ξ} is divisible by 2. Therefore, each x, y and z is divisible by 2, which implies that x, y and z are not relatively prime. Thus Fermat's Last Theorem implies Beal's conjecture.

For the converse, take, $\xi = \mu = \nu = \pi$, an odd prime. Thus the proof of the equivalence is complete.

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Department of Mathematics, Howard University, Washington, DC 20059, USA
E-mail address: jjoseph@Howard.edu
Current address: 35 E Street NW #709, Washington, DC 20001, USA
E-mail address: j122437@yahoo.com

Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA
E-mail address: Bhamini.Nayar@morgan.edu