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Hypersurfaces Of Constant Curvature In Hyperbolic Space

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Abstract. We continue the work done in [2],[3] which investigates the problem of finding Weingarten hypersurfaces of constant curvature satisfying (1), (2) below in hyperbolic space $\mathbb{H}^{n+1}$ with a prescribed asymptotic boundary at infinity. In [2], the focus is on the case of finding complete hypersurfaces with positive hyperbolic principal curvatures everywhere; in [3], the focus is on finding graphs over a domain with nonnegative mean curvature. In [2] and [3], some restriction is imposed on σ to assure us of the existence. The main aim of this article is to remove these restrictions. The results stated in the manuscript, as well as more general ones have been proved in [4] and [5] with a less elementary approach.

In this paper, we continue the work done in [2],[3] which investigates Weingarten hypersurfaces of constant curvature in hyperbolic space $\mathbb{H}^{n+1}$ with a prescribed asymptotic boundary at infinity. More precisely, given a disjoint collection $\Gamma = \{\Gamma_1, \cdots, \Gamma_n\}$ of closed embedded $(n-1)$-dimensional submanifolds of $\partial_\infty \mathbb{H}^{n+1}$ at infinity, the ideal boundary of $\mathbb{H}^{n+1}$ at infinity, and a smooth function f of n variables, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1}$$

where $\kappa[\Sigma] = \{\kappa_1, \cdots, \kappa_n\}$ denotes the hyperbolic principal curvature of Σ and σ is a constant, with the asymptotic boundary

$$\partial\Sigma = \Gamma. \tag{2}$$

Letting $K \subset \mathbb{R}^n$ is an open symmetric convex cone such that

$$K_+^n := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subset K,$$

the function f is a concave function in K which satisfies the fundamental structural conditions:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0, \quad 1 \le i \le n, \quad f > 0 \text{ in } K, \quad f = 0 \text{ on } \partial K. \tag{3}$$

In addition, we shall assume that f is homogeneous of degree one, normalized $f(1, \cdots, 1) = 1$ and $\lim_{r\to\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \le 1 + \varepsilon_0$, uniformly in $B_{\delta_0}(\mathbf{1})$ for some fixed $\varepsilon > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$. All these assumptions are satisfied by $f = H_k^{1/k}$

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and $(H_k/H_\ell)^{1/(k-\ell)}$, $0 \le \ell < k \le n$, defined in K_k, where H_k is the normalized k-th elemenrary symmetric polynomial ($H_0 = 1$) and $K_k = \{\lambda \in \mathbb{R}^n : H_j(\lambda) > 0, \forall 1 \le j \le k\}$.

We will use the half-space model $\mathbb{H}^{n+1} = \{(x_1, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ equipped with the hyperbolic metric $ds^2 = \frac{\Sigma_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}$. Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (2) may be understood in the Euclidean case. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

In [2], the focus is on the case of finding complete hypersurfaces satisfying (1)-(2) with positive hyperbolic principal curvatures everywhere; for convenience we call such hypersurfaces (hyperbolically) locally strictly convex. In [3], the focus is on graphs over a domain with nonnegative mean curvature. In [2] and [3], some restriction is imposed on σ to assure us of the relevant existence. The main aim of this article is to remove this restriction. The results stated in the manuscript, as well as more general ones have been proved in [4] and [5] with a less elementary approach.

Part I. Strictly convex hypersurfaces.

According to Theorem 1.1 in [2], a complete locally strictly convex C^2 hypersurface Σ in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity must be the (vertical) graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, for some domain $\Omega \subset \mathbb{R}^n$: $\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1}_+ : x \in \Omega\}$ such that

$$\{\delta_{ij} + u_i u_j + u u_{ij}\} > 0 \quad \text{in } \Omega. \tag{4}$$

That is, the function $u^2 + |x|^2$ is strictly convex.

Therefore, problem (1)-(2) for complete locally strictly convex hypersurfaces reduces to the Dirichlet problem for a nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \frac{\sigma}{u}, \quad u > 0 \ \text{ in } \Omega \subset \mathbb{R}^n \tag{5}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{6}$$

In particular, the asymptotic boundary Γ must be the boundary of some bounded domain Ω in $\mathbb{R}^n$. The exact form of G is given as (2.9) of [2].

Following the literature we define the class of *admissible* functions

$$\mathcal{A} = \{u \in C^2(\Omega : \kappa[u] \in K\}.$$

Thus in [1] we call solutions of (5) satisfying (4) *admissible* with $K = K_n^+$. By [1] condition (3) implies that (5) is elliptic for admissible solutions.

Our goal in Part I is to show that the Dirichlet problem (5)-(6) admits smooth solutions for all $0 < \sigma < 1$, removing the restriction $\sigma^2 > \frac{1}{8}$ imposed on [2]. Namely, we shall establish the following:

Theorem 1. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+2}$, where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that $\sigma \in (0, 1)$ and $K = K_n^+$. Under conditions (3), there exists a complete locally strictly convex hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1)-(2) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma]| \le C \quad on\ \Sigma.$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem (5)-(6). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\sqrt{1+|Du|^2} \le \frac{1}{\sigma}, \quad u|D^2 u| \le C \quad \text{in } \Omega,$$

$$\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega.$$

To remove the restriction imposed on σ in [2], we examine closely the proof of Theorem 1.1 in [2] with this restriction on σ. First recall that (5) is singular where $u = 0$, and therefore in [2] we approximate the boundary condition (6) by

$$u = \varepsilon > 0 \quad \text{on } \partial\Omega. \tag{7}$$

It is shown in [2] that for any $\varepsilon > 0$ sufficiently small, there exists an admissible solution $u^\varepsilon \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (5)-(7). Because the linearized operator of G at u is not necessarily invertible, this existence result is not proved in [2] by the continuity method directly. Instead an iterative procedure is carried out in Section 6 of [2]. Namely, we construct a monotone sequence $\{u_k\}$ of admissible functions satisfying (2) in Ω, starting from $u_0 \equiv \varepsilon$. To show that $\{u_k\}$ converges to a solution of (5), we need second derivative estimates which is independent of k. These estimates are obtained in Section 6 of [2] by means of Theorem 5.1 of [2], without any restriction on σ.

To finish the proof of Theorem 1, we need to establish for $\sigma \in (0, 1)$, an estimate for $\sup_\Omega \kappa_{\max}$ which is independent of ε as ε tends to zero. It is here that in [2] we impose the restriction on σ to be $\sigma^2 > \frac{1}{8}$, which we shall remove below. Namely, we consider

$$M_0 = \max_{x\in\overline{\Omega}} \frac{\kappa_{\max}(x)}{\eta - a},$$

where $\eta = \mathbf{e}\cdot\nu$, ν is the upward (Euclidean) unit normal to Σ, and a is a constant such that $\inf \eta > a$. If M_0 is achieved on $\partial\Omega$, then a uniform bound is obtained from Theorem 4.2 of [2]. Otherwise, M_0 is attained at an interior point $x_0 \in \Omega$ and we let $X_0 = (x_0, u(x_0))$. After a horizontal translation of the origin in $\mathbb{R}^{n+1}$, we may write Σ locally near X_0 as a radial graph

$$X = e^v \mathbf{z}, \quad \mathbf{z} \in \mathbb{S}^{n+1}_+ \subset \mathbb{R}^{n+1},$$

with $X_0 = e^{v(\mathbf{z}_0)}\mathbf{z}_0$, $\mathbf{z}_0 \in \mathbb{S}^n_+$, such that $\nu(X_0) = \mathbf{z}_0$. Let the hyperbolic principal curvatures $\kappa_1, \cdots, \kappa_n$ be the eigenvalues of the matrix $\{a_{ij}[v]\}$. We choose an orthonormal local frame $\tau_1, \cdots, \tau_n$ around $\mathbf{z}_0$ on $\mathbb{S}^n_+$ such that $v_{ij}(\mathbf{z}_0)$ is diagonal. Then, letting $y = \mathbf{e}\cdot\mathbf{z}$ for $\mathbf{z} \in \mathbb{S}^n_+$, we have $\nabla v(\mathbf{z}_0) = 0$ and at $\mathbf{z}_0$

$$a_{ij} = y v_{ij} = \kappa_i \delta_{ij}.$$

We assume

$$\kappa_1 = \kappa_{\max}(X_0).$$

The function $\frac{a_{11}}{\phi}$, $\phi := \eta - a$, which is defined locally near $\mathbf{z}_0$, then achieves its maximum at $\mathbf{z}_0$ at which therefore

$$\left(\frac{a_{11}}{\phi}\right)_i = 0, \quad 1 \le i \le n$$

and

$$\left(\frac{a_{11}}{\phi}\right)_{ii} = \frac{1}{\phi} F^{ii} a_{11,ii} - \frac{\kappa_1}{\phi^2} F^{ii} \phi_{ii} \le 0, \quad 1 \le i \le n.$$

Proposition 5.3 and Lemma 5.4 in [2] give

$$\sigma(y-a)\kappa_1^2 + (a - 2(1-y^2)(y-a))\kappa_1\Sigma f_i \le 4\sigma\kappa_1, \tag{8}$$

which is (6.5) in [2]. We attempt to drop the second term on the left hand side of (8) so as to avoid dealing with the somewhat unfathomable function $\kappa_1\Sigma f_i$. For this, we proceed to find conditions under which the coefficient

$$\gamma(y) := a - 2(1-y^2)(y-a) = 2y^2(y-a) + 3a - 2y \tag{9}$$

is nonnegative. In [2], the condition $\sigma^2 > \frac{1}{8}$ is imposed to make sure that

$$\gamma(y) > 0 \quad \text{for all } y \in [a,1].$$

To improve this, we notice that, by Lemma 3.5 in [2], for a sufficiently small $\varepsilon_1 > 0$, we have $y - \sigma > -C\varepsilon_1$ if $0 < \varepsilon < \varepsilon_1$, where C is a uniform constant. Hence, in particular, $\inf y > -\infty$.

First recall from Lemma 3.1 in [2] that near the boundary $\partial\Omega$, we have $|y - \sigma| < C\sigma$, where C is the uniform constant as above. Suppose the interior maximum point X_0 of M_0 is so close to the boundary that at X_0 we have $y = y_0 < \sigma + C\sigma$. In this case, we may fix ε_1 with $\varepsilon_1 < \frac{\sigma}{8C}$ and choose a close to σ such that

$$\sigma - \frac{3}{2}C\varepsilon_1 > a > \sigma - 2C\varepsilon_1. \tag{10}$$

Then

$$\gamma(y_0) = 2y_0^2(y_0 - a) + 3a - 2y_0 > 3a - 2y_0 > \sigma - 8C\varepsilon_1 > 0.$$

We are now allowed in this case to throw away the second term on the left hand side of (8) and then obtain from (8) that $\kappa_1 \le \frac{4}{y_0 - a}$. Now that $y_0 - a > \frac{1}{2}C\varepsilon_1$, we have

$$\kappa_1 \le \frac{8}{C\varepsilon_1}. \tag{11}$$

And then

$$\max_{\overline{\Omega}} \kappa_{\max} \le \kappa_1(\mathbf{z}_0)\frac{\max y - a}{\min y - a} \le \frac{2\kappa_1(\mathbf{z}_0)}{C\varepsilon_1} \le \frac{16}{(C\varepsilon_1)^2}. \tag{12}$$

which is independent of ε.

Next let X_ε be the points at which the function y for u^ε attains its maximum value 1. We have $\gamma(1) = a > 0$ for each $a \in (0,1)$. Moreover, for some sufficiently small number δ, $\gamma_\varepsilon(y)$ is still positive if $1 - \delta \le y \le 1$ and $\varepsilon < \varepsilon_1$. If the interior maximum point X_0 of M_0 is sufficiently close to X_ε, then at X_0 we have $y = y_0 > 1 - \delta$ and $\gamma_\varepsilon(y)$ is still positive for $\varepsilon < \varepsilon_1$. By choosing a to satisfy (10), we still obtain (11) and (12).

It remains to consider the case where the interior maximum of M_0 is attained at a point where $y = y_0$, $1 - \delta > y_0 > \sigma + C\varepsilon_1$. To treat this, we may let $L_\varepsilon(\widetilde{y})$ be the curve on which $y = \widetilde{y}$. Also denote by $E_\varepsilon(\widetilde{y})$ the region in the graph of u^ε which contains X_ε and is enclosed by $L_\varepsilon(\widetilde{y})$. Thus $y \ge \widetilde{y}$ in $E_\varepsilon(\widetilde{y})$. We take a_1 with $\frac{3}{4}y_0 > a_1 > \frac{2}{3}y_0$ and let $\gamma_1(y) = \gamma(y)$ with $a = a_1$. Then $\gamma_1(y_0) > 0$ and hence $\gamma_1(y_0 \pm \delta) > 0$ for sufficiently small δ, say $\delta < \hat{\delta}_1$. Take $\delta_1 = \min\{\hat{\delta}_1, \frac{1}{8}y_0\}$. and consider the function

$$M_1 = \max_{x \in \overline{E}_1} \frac{\kappa_{\max}(x)}{\eta - a_1},$$

defined in the region $E_1 := E_\varepsilon(y_0-\delta_1)\setminus E_\varepsilon(y_0+\delta_1)$; here we may notice that in the region E_1 we have $y \ge a_1$. As above, we consider two cases separately.

Case a. Suppose the maximum of M_1 is attained at an interior point of E_1 Then the previous discussion on M_0 can be applied to M_1 on E_1 to derive that at the interior maximum point of M_1 there holds

$$\kappa_1 \le \frac{4}{y_0-\delta_1-a_1} \le \frac{4}{(1/8)y_0} \le \frac{32}{\sigma - C\varepsilon_1}. \tag{20}$$

And then

$$\max_{\overline{E}_1}\kappa_{\max} \le \kappa_1(\mathbf{z}_0)\frac{\max_{E_1} y-a_1}{\min_{E_1} y - a_1} \le \frac{8\kappa_1(\mathbf{z}_0)}{\sigma - C\varepsilon_1} \le \frac{256}{(\sigma-C\varepsilon_1)^2}.$$

Moreover, since the interior maximum of M_0 is attained at a point in E_1, we have

$$\max_{\overline{\Omega}}\kappa_{\max} \le \max_{\overline{E}_1}\kappa_{\max}\frac{\max_\Omega y-a}{\min_\Omega y-a} \le \frac{2\max_{\overline{E}_1}\kappa_{\max}}{C\varepsilon_1} \le \frac{512}{(\sigma-C\varepsilon_1)^2 C\varepsilon_1}, \tag{21}$$

which is independent of ε; here a is taken to satisfy (17). We notice that to begin the discussion we may have to make a horizontal translation of the origin in $\mathbb{R}^{n+1}$; however, the value y is invariant under such a horizontal translation.

Case b. Suppose the maximum of M_1 is taken at a boundary point of E_1. We observe:

Lemma 1. *If the maximum of M_1 is attained at a boundary point of E_1, then at this boundary point $y = y_0 - \delta$.*

Proof of Lemma 1. Suppose that the maximum of M_1 is attained at a boundary point of E_1 where $y = y_1^* = y_0 + \delta$. Denote by κ_1^0 and κ_1^1 the value of $\kappa_{\max}$ taken respectively at y_0 and the maximum point of M_1. Then, since the maximum of M_0 is attained at y_0, we have

$$\kappa_1^1 \le \kappa_1^0\frac{y_1^*-a}{y_0-a}.$$

On the other hand, since the maximum of M_1 is assumed to be attained at a point with $y = y_1^*$, we have

$$\kappa_1^1 \ge \kappa_1^0\frac{y_1^*-a_1}{y_0-a_1}.$$

However, since $a_1 > a$ and $y_1^* > y_0$, we have

$$\frac{y_1^*-a_1}{y_0-a_1} \ge \frac{y_1^*-a}{y_0-a},$$

a contradiction which proves Lemma 1.

If **Case b** occurs, then we set $y_1 = y_0 - \delta$ and then take a_2 with $\frac{3}{4}y_1 > a_2 > \frac{2}{3}y_1$. Let $\gamma_2(y) = \gamma(y)$ with $a = a_2$. Then $\gamma_2(y_1) > 0$ and hence $\gamma_2(y_1 \pm \delta) > 0$ for sufficiently small δ and $\varepsilon < \varepsilon_1$, say $\delta < \hat{\delta}_2$. Take $\delta_2 = \min\{\hat{\delta}_2, \frac{1}{8}y_1\}$. Consider the function

$$M_2 = \max_{x\in\overline{E}_2}\frac{\kappa_{\max}(x)}{\eta - a_2},$$

defined in the region $E_2 := E_\varepsilon(y_1-\delta_2)\setminus E_\varepsilon(y_1+\delta_2)$. Then there holds $y \ge a_2$ in E_2. We consider two cases separately as above:

Case $\mathbf{a}^2$. Suppose the maximum of M_2 is attained at an interior point of E_2. Then, we can obtain the estimates

$$\kappa_1 \leq \frac{256}{\sigma - C\varepsilon_1} \quad \text{and} \quad \max_{\overline{\Omega}} \kappa_{\max} \leq \frac{512}{(\sigma - C\varepsilon_1)^2 C\varepsilon_1} \tag{15}$$

as in (13) and (14).

Case $\mathbf{b}^2$. Suppose the maximum of M_2 is attained at a boundary point of E_2. Then again at such a boundary point we have $y = y_2 := y_1 - \delta_2$.

If **Case $\mathbf{b}^2$** occurs, we can adapt the discussion made in **Case b** and subsequently consider separately the corresponding **Case $\mathbf{a}^3$** and **Case $\mathbf{b}^3$**, which can be formulated in an obvious manner analogously to **Case $\mathbf{a}^2$** and **Case $\mathbf{b}^2$**. In this manner we make the discussion iteratively and then, putting inductively, there hold alternatively the following two:

(i) for some m, Case $\mathbf{a}^{m-1}$ occurs and we obtain the estimate (15);

(ii) Case $\mathbf{b}^{m-1}$ occurs with $y = y_{m-1}$, and we take a_m with $\frac{3}{4}y_{m-1} > a_m > \frac{2}{3}y_{m-1}$ and let $\gamma_m(y) = \gamma(y)$ with $a = a_m$. Then $\gamma_m(y_{m-1}) > 0$ and hence $\gamma_m(y_{m-1} \pm \delta) > 0$ for sufficiently small δ and $\varepsilon < \varepsilon_1$, say $\delta < \hat{\delta}_m$. Take $\delta_m = \min\{\hat{\delta}_m, \frac{1}{8}y_{m-1}\}$. Consider the function

$$M_m = \max_{x \in \overline{\Omega}} \frac{\kappa_{\max}(x)}{\eta - a_m},$$

defined in the region $E_m := E_\varepsilon(y_{m-1} - \delta_m) \setminus E_\varepsilon(y_{m-1} + \delta_m)$. We subsequently consider two cases separately.

Case $\mathbf{a}^m$. Suppose the maximum of M_m is attained at an interior point of E_m. Then, we can obtain the estimate (15).

Case $\mathbf{b}^m$. Suppose the maximum of M_m is attained at a boundary point y_m of E_m. Again $y = y_m = y_{m-1} - \delta_m$ and we subsequently consider separately **Case $\mathbf{a}^{m+1}$**, **Case $\mathbf{b}^{m+1}$**.

The iteration process is terminated if either **Case $\mathbf{a}^m$** occurs or if $y_m < \sigma + C\varepsilon_1$ for some m. In the latter case, we however have the estimates (11) and (12) as indicated above.

In conclusion, we obtain

$$\max_{\overline{\Omega}} \kappa_{\max} \leq \max\{\frac{16}{(C\varepsilon_1)^2}, \frac{512}{(\sigma - C\varepsilon_1)^2 C\varepsilon_1}\}, \tag{16}$$

which is independent of ε. This completes the proof of Theorem 1.

Part II. Graphs over a domain with nonnegative mean curvature.

In [3] the focus is on finding graphs over a domain with nonnegative mean curvature and problem (1.1)-(1.2) reduces to the Dirichlet problem for a fully nonlinear second order equation which we write in the form

$$\widetilde{G}(D^2u, Du, u) = \sigma, \quad u > 0 \ \text{ in } \Omega \subset \mathbb{R}^n, \quad \text{where } \widetilde{G} = uG. \tag{17}$$

Following the literature we define the class of *admissible* functions

$$\mathcal{A} = \{u \in C^2(\Omega : \kappa[u] \in K\}.$$

Our goal in Part II is to show that the Dirichlet problem (17)-(6) admits smooth solutions for all $0 < \sigma < 1$, removing the restriction imposed on σ in [3]. Namely, we shall establish the following.

Theorem 2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$, where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that the Euclidean mean curvature $\mathcal{H}_{\partial\Omega}$ is nonnegative and $\sigma \in (0,1)$. Under conditions (4)-(10), there exists a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1)-(2) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma]| \leq C \quad \text{on } \Sigma.$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem (25)-(13). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\sqrt{1+|Du|^2} \leq \frac{1}{\sigma}, \quad u|D^2u| \leq C \quad \text{in } \Omega,$$

$$\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega.$$

In [3], Theorem 2 is proved under the restriction that $\sigma \in (0,1)$ satisfies $\sigma > \sigma_0$, where σ_0 is the unique zero in $(0,1)$ of

$$\phi(a) := \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2+3)^{3/2};$$

(numerical calculations show $0.3703 < \sigma_0 < 0.3704$.) To remove this restriction, recall that again in [3] we approximate the boundary condition (6) by (7), and it is shown in [3] that for any $\varepsilon > 0$ sufficiently small, there exists an admissible solution $u^\varepsilon \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (17)-(7).

In this case, the linearized operator of G at u is invertible for all $\varepsilon \in (0,1)$. A sharp gradient estimate is obtained under the assumption $\mathcal{H}_{\partial\Omega} \geq 0$. Using this, the C^2 estimate is obtained in Section 5 of [3]. To finish the proof of Theorem 2, we again need to show that for $\sigma \in (0,1)$, an estimate can be obtained for $\sup_\Omega \kappa_{\max}$ which is independent of ε as ε tends to zero. For this, we again consider

$$M_0 = \max_{x\in\overline{\Omega}} \frac{\kappa_{\max}(x)}{\eta - a},$$

with the notation used in Part I. Using Proposition 5.3 and Lemma 5.4 of [2], we obtain after some manipulation, the inequality (6.18) in [3]; i.e., fixing $\theta \in (0,1)$ which is chosen later and letting $\alpha = a\kappa_1/(\kappa_1 - (y-a))$ and

$$J = \{i : \kappa_i > \alpha > a,\ f_1 \leq \theta f_i\},$$

we have

$$\sigma(y-a)\kappa_1^2 + \phi_\theta(y)\kappa_1 \sum_{i\in J} f_i \leq 2\sigma\kappa_1, \tag{18}$$

where the coefficient of $\kappa_1 \sum_{i\in J} f_i$ is

$$\phi_\theta(y) = \gamma(y) - \frac{a-\gamma(y)}{4(1-\theta)} + a^3.$$

Again we desire to throw away the second term on the left hand side of (18) by finding conditions under which its coefficient is nonnegative. It is here in [3] we make the restriction on σ which we shall remove below. Namely, setting $\theta = 0$, we obtain from (9),

$$\begin{aligned}
\phi_0(y) =& \frac{5}{4}\gamma(y) - \frac{1}{4}a + a^3 \\
=& \frac{5}{4}(3a-2y) - \frac{1}{4}a + \frac{5}{8}y^2(y-a) \\
>& \frac{7}{2}a - \frac{9}{4}y.
\end{aligned}$$

And again, having prescribed σ, we know that for a sufficiently small $\varepsilon_1 > 0$, if $0 < \varepsilon < \varepsilon_1$ then we have $y > \sigma - C\varepsilon_1$, where C is a uniform constant.

For points near the boundary, we obtain from Lemma 3.2 in [3] that $|y - \sigma| < C\varepsilon_1$. If the interior maximum of M_0 is attained at a point so close to the boundary that $|y - \sigma| < C\varepsilon_1$, we may as above fix ε_1 with $\varepsilon_1 < \frac{\sigma}{8C}$ and again choose a close to σ such that (10) is satisfied. Then

$$\phi_0(y) > \frac{7}{2}a - \frac{9}{4}y > \frac{5}{4}\sigma - \frac{19}{4}C\varepsilon_1 > 0.$$

Hence we have $\phi_\theta(y) > 0$ if $\theta > 0$ is chosen small. We are now allowed to throw away the second term on the left hand side of (18) in this case and then obtain from (18) that $\kappa_1 \leq \frac{4}{y-a}$. Now that $y - a > \frac{1}{2}C\varepsilon_1$, we again obtain in this case the estimates (11) and (12), which is independent of ε.

Also we have $\phi_0(1) > 0$. Therefore if the interior maximum X_0 of M_0 is close to the point where $p = 1$ and at X_0 the function u^ε takes $p > 1 - \delta$, for some sufficiently small δ, we still obtain the estimates (11) and (12), by taking θ small enough.

On the other hand, suppose at the interior maximum point of M_0 we have $\sigma - C\varepsilon_1 < p < 1 - \delta$. If we choose $\frac{3}{4}y > a > \frac{3}{2}y$, then as above we have $\phi_0(y) > 0$. Thus the iteration process used in the end of Part 1 can be adapted to obtain estimates (15), by taking θ small enough. We then again obtain the estimates (16) and completes the proof of Theorem 2.

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