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Some Numerical Techniques for Solve Nonlinear Fredholm-Volterra Integral Equation.

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Abstract. In this paper, the existence and uniqueness of the solution of nonlinear Fredholm – Volterra integral equation is consider(NF-VIE) with continuous kernel , then we use a numerical method to reduce this type of equations to a system of Fredholm integral equation . Trapeziodal rule, Simpson rule, and Romberg integral method are used to solve the Fredholm integral equation of the second kind with continuous kernel. The error in each case is calculated.

Keywords: Nonlinear Fredholm-Volterra integral equation; system of nonlinear Fredholm integral equation; Trapezoidal rule; Simpson's rule; Romberg integration.

1 Introduction:

System of integral equations are used as mathematical models for many physical situations, and also occur as reformulations of other mathematical problems [1].There are many references that dealt with the fundamental concepts of integral equations [2],[3]. Also proved a lot of numerical methods to solve the kinds of integral equations, such as the Trapezoidal Rule, Nystrom method, the Galerkin method; Simpson's Rule and Romberg Integration (see [4], [5],[6]). The discussions of F-VIE with is applications in contact problems in the theory of elasticity started by Abdou [7]. Abdou in [7] discussed the solution of F-VIE of the first kind in one, two and three dimensions, using separation of variable method. Also, the same author, in [8-10] used some different methods to obtain the solution of F-VIE of the first kind and of the second kind. Hendi and Albugami, in [11], obtained numerically, the solution of F-VIE of the second kind, using collocation and Galerkin methods. In [12,13], the authors used two numerical methods to obtain the solution of F-VIE of the second kind when the kernel takes a logarithmic form and Hilbert kernel, respectively. In the references[14-17] the authors considered many different methods to solve the mixed integral equation numerically.

$$\mu\phi(x,t)=f(x,t)+\lambda\int_a^b K(x,y)\gamma(y,t,\phi(y,t))dy+\lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))d\tau \qquad (1)$$

The equation (1) is called nonlinearFredholm- Volterra integral type. This formula is measured in the space $L_2[a,b]\times C[0,T],T<\infty,$ where the Fredholm integral term is mrasured with respect to position and its kernel $K(x,y)$ is positive and continuous for all

$x,y\in[a,b]$. While the Volterra integral term is consider in time and its kernel $F(t,\tau)$ is positive and continuous for all $t,\tau\in[0,T],T<\infty$, and $f(x,t)$ is known function which is name the free term. The numerical coefficient λ is called the parameter of the integral equation, may be complex,and has physical meaning, while the constant parameter μ defines the kind of the integral equation (1).

In order to guarantee the existence of a unique solution of equation (2.1), we assume the following conditions:

(i) The kernel of the Fredholm term $K(x,y)\in C([a,b]\times[a,b])$ satisfies :

$$|K(x,y)|\le A_1 \qquad ,(\forall x,y\in[a,b],A_1 \text{ is a constant})$$

(ii) The kernel of the Volterra term $F(t,\tau)\in C([0,\mathrm{T}]\times[0,\mathrm{T}]), 0\le\tau\le t\le T\le\infty$, satisfies:

$$|F(t,\tau)|\le A_2 \quad ,(A_2 \text{ is a constant})$$

(iii) The given function $f(x,t)$ with its derivatives with respect to x and t are continuous in $\mathrm{L}_2[a,b]\times\mathrm{C}[0,\mathrm{T}]$ where:

$$\|f(x,t)\|=\max_{0\le t\le T}\int_0^t[\int_a^b|f(x,\tau)|^2dx]^{\frac{1}{2}}d\tau=A_3 \quad ,(A_3 \text{ is a constant})$$

(iv) The unknown function $\gamma(x,t,\phi(x,t))$ satisfies for the constant $B>B_1,B>p$, the following conditions:

$$a)\int_0^t\int_a^b(|\gamma(x,t,\phi(x,t))|^2dxdt)^{\frac{1}{2}}\le B_1\|\phi(x,t)\|_{L_2[a,b]\times C[0,T]}$$

$$b)\|\gamma(x,t,\phi_1(x,t))-\gamma(x,t,\phi_2(x,t))\|\le N(x,t)|\phi_1(x,t)-\phi_2(x,t)|$$

where $\|N(x,t)\|_{L_2[a,b]\times C[0,T]}=p$

In other word, we can prove the existence and uniqueness of the solution by the following: we prove that the solution is exist using the successive approximation method, also called the Picard method, that we pick up any real continuous function $\phi_0(x,t)$ in $\mathrm{L}_2[a,b]\times\mathrm{C}[0,\mathrm{T}]$, then construct a sequence ϕ_n defined by

$$\phi_n(x,t)=f(x,t)+\lambda\int_a^b K(x,y)\gamma(y,t,\phi_{n-1}(y,t))dy+\lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi_{n-1}(x,\tau))d\tau \qquad ,(\mu=1)$$

$$\phi_{n-1}(x,t)=f(x,t)+\lambda\int_a^b K(x,y)\gamma(y,t,\phi_{n-2}(y,t))dy+\lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi_{n-2}(x,\tau))d\tau \quad ,(\mu=1)$$

For ease of manipulation it is convenient to introduce :

$$\begin{aligned}\psi_n(x,t) &= \phi_n(x,t) - \phi_{n-1}(x,t) \\ &= \lambda\int_a^b K(x,y)[\gamma(y,t,\phi_{n-1}(y,t)) - \gamma(y,t,\phi_{n-2}(y,t))]dy \\ &+ \lambda\int_0^t F(t,\tau)[\gamma(x,\tau,\phi_{n-1}(x,\tau)) - \gamma(x,\tau,\phi_{n-2}(x,\tau))]d\tau, \qquad n = 1,2,...\end{aligned}$$

Then:

$$\phi_n(x,t) = \sum_{i=0}^{n}\psi_i(x,t) \qquad (2)$$

Hence

$$\psi_n(x,t) = f(x,t) + \lambda\int_a^b K(x,y)\gamma(y,t,\psi_{n-1}(y,t))dy + \lambda\int_0^t F(t,\tau)\gamma(x,\tau,\psi_{n-1}(x,\tau))d\tau$$

Using the properties of the norm we obtain :

$$\|\psi_n(x,t)\| \le |\lambda|\left\|\int_a^b K(x,y)\gamma(y,t,\psi_{n-1}(y,t))dy\right\| + |\lambda|\left\|\int_0^t F(t,\tau)\gamma(x,\tau,\psi_{n-1}(x,\tau))d\tau\right\|$$

For $n = 1$, we get

$$\begin{aligned}\|\psi_1(x,t)\| &\le |\lambda|\left\|\int_a^b K(x,y)\gamma(y,t,\psi_0(y,t))dy\right\| + |\lambda|\left\|\int_0^t F(t,\tau)\gamma(x,\tau,\psi_0(x,\tau))d\tau\right\| \\ &\le |\lambda|\left\|\int_a^b |K(x,y)||\gamma(y,t,\psi_0(y,t))|dy\right\| + |\lambda|\left\|\int_0^t |F(t,\tau)||\gamma(x,\tau,\psi_0(x,\tau))|d\tau\right\|\end{aligned}$$

Using Cauchy Schwarz inequality and from conditions (i)- (iv-a) with $\psi_0 = f(x,t)$ and $\|f\| = A_3$, we get

$$\begin{aligned}\|\psi_1(x,t)\| &\le |\lambda|A_1\int_a^b \|\gamma(y,t,\psi_0(y,t))\|dy + |\lambda|A_2\int_0^t \|\gamma(x,\tau,\psi_0(\mathrm{x},\tau))\|d\tau \\ &\le |\lambda|A_1B_1\|\psi_0(x,t)\| + |\lambda|A_2B_1A_3\|\psi_0(x,t)\|\int_0^t d\tau \\ &\le |\lambda|A_1B_1A_3 + |\lambda|A_2B_1A_3\|t\|\end{aligned}$$

But, we have $0 \le \tau \le t \le T < \infty$, then $\max|t| = T = L$, then we have:

$$\|\psi_1(x,t)\| \le |\lambda|A_3(B_1A_1 + A_2LB_1)$$

In general, we get:

$$\|\psi_1(x,t)\| \le |\lambda|^n A_3(BA_1 + A_2BL)^n = A_3\alpha^n \quad , \alpha = |\lambda|(A_1B + A_2LB) \qquad (3)$$

this bound makes the sequence $\psi_n(x,t)$ converges if

$$\alpha < 1 \Rightarrow |\lambda| < \frac{1}{(A_1B + A_2LB)} \qquad (4)$$

the result (2.4), leads us to say that the formula (2.2) has a convergent solution. So let $n \to \infty$, we have:

$$\phi(x,t) = \sum_{i=0}^{\infty} \psi_i(x,t) = \frac{A_3}{1-\alpha}, \qquad (\alpha < 1)\ (5)$$

the infinite series of (5) is convergent, and $\phi(x,t)$ represents the convergent solution of equation (1). Also each of ψ_i is continuous, therefore $\phi(x,t)$ is also continuous.

To show that $\phi(x,t)$ is unique, we assume that $\bar{\phi}(x,t)$ is also a continuous solution of (1) then, we write

$$\phi(x,t) - \bar{\phi}(x,t) = \lambda \int_a^b K(x,y)[\gamma(y,t,\phi(y,t)) - \gamma(y,t,\bar{\phi}(y,t))]dy$$
$$+ \lambda \int_0^t F(t,\tau)[\gamma(x,\tau,\phi(x,\tau)) - \gamma(x,\tau,\bar{\phi}(x,\tau))]d\tau, \qquad (\mu = 1)$$

Which, leads us to the following:

$$\|\phi(x,t) - \bar{\phi}(x,t)\| \le |\lambda| \left\| \int_a^b |K(x,y)| |\gamma(y,t,\phi(y,t)) - \gamma(y,t,\bar{\phi}(y,t))| dy \right\|$$
$$+ |\lambda| \left\| \int_0^t |F(t,\tau)| |\gamma(x,\tau,\phi(x,\tau)) - \gamma(x,\tau,\bar{\phi}(x,\tau))| d\tau \right\|$$

Using conditions (iv-b) then apply

Cauchy-Schwarz inequality, we have:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\| \le |\lambda|\,|K(x,y)|\,(\int_a^b N^2(x,t)\,|\phi(x,t)-\bar{\phi}(x,t)|^2\,dy)^{\frac{1}{2}}$$
$$+|\lambda|\,|F(t,\tau)|\,(\int_0^t N^2(x,\tau)\,|\phi(x,\tau)-\bar{\phi}(x,\tau)|^2\,d\tau)^{\frac{1}{2}}$$

Finally with the aid of conditions(i),(ii),and(iv-b), we obtain:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\| \le \alpha \left\|\phi(x,t)-\bar{\phi}(x,t)\right\|$$

Then:

$$(1-\alpha)\left\|\phi(x,t)-\bar{\phi}(x,t)\right\| \le 0$$

Since $\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|$ is necessarily non-negative, and $\alpha<1$ we get:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\| = 0 \Rightarrow \phi(x,t)=\bar{\phi}(x,t)$$

It follows that if (2.1) has a solution it must be unique.

2 The system of nonlinear Fredholm integral equations:

Consider the nonlinear integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_a^b K(x,y)\gamma(y,t,\phi(y,t))dy+\lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))d\tau \qquad (6)$$

when $t=0$ equation(6) becomes:

$$\phi_0(x)=f_0(x)+\lambda\int_a^b K(x,y)\gamma(y,\phi_0(y))dy \qquad (7)$$

where $\phi_0(x)=\phi(x,0), f_0(x)=f(x,0)$.

The formula (7) represents a nonlinearFredholm integral equation of the second kind at $t=0$. For representing (6) as a system of nonlinear Fredholm integral equations, we use the following numerical method. Divide the interval $[0,T], 0\le t\le T<\infty$ as $0=t_0\le t_1<\ldots<t_k<\ldots<t_N=T$ i.e. $t=t_k, k=0,1,2\ldots N$. Then using the quadrature formula, the Volterra integral term in (6) becomes :

$$\int_0^{t_k} F(t,\tau)\gamma(x,\tau,\phi(x,\tau))d\tau=\sum_{j=0}^{k} u_j F(t_k,t_j)\gamma(x,t_j,\phi(x,t_j))+o(\hbar_i^{\tilde{p}+1}), \quad (\hbar_k\to 0, \tilde{p}>0) \quad (8)$$

where $\hbar_k=\max_{0\le j\le k} h_j, h_j=t_{j+1}-t_j$

The values of K and constant $\tilde{p}$ depend on the number of derivative of $F(t,\tau)$, for all $\tau \in [0,T]$, with respect to t, and $u_0 = \frac{1}{2}h_0, u_N = \frac{1}{2}h_N, u_i = h_i, (i \neq 0, N)$.

Using (8) in (6) , we have:

$$\phi_k(x) = f_k(x) + \lambda \int_a^b K(x,y)\gamma(y,\phi_k(y))dy + \lambda \sum_{j=0}^{k} u_j F_{kj} \gamma(x,t_j,\phi_j(x)) \qquad (9)$$

where $\phi_k(x) = \phi(x,t_k), f_k(x) = f(x,t_k), F_{kj} = F(t_k,t_j)$.

To formula (9) become:

$$\phi_n(x) = G_n(x) + \lambda \int_0^t K(x,y)\gamma(y,t_n,\phi_n(y))dy \qquad (10)$$

where $G_n(x) = f_n(x) + \lambda \sum_{j=0}^{n} u_j F_{nj} \gamma(x,t_j,\phi_j(x)),\ n = 0,1,...,N$.

The formula (10) represent a nonlinear system of Fredholm integral equations of the second kind, andwe have N unknown functions $\phi_n(x)$ corresponding to time interval $[0,T]$.

3 Some numerical techniques for solving the system of nonlinear Fredholm integral equations:

3.1 Trapezoidal rule :

In this section, we use Trapezoidal rule for solving the nonlinear Fredholm-Volterra integral equation of the second kind. The interval $[a,b]$ is divied in to steps of width $h = \frac{b-a}{N}$, $a = x_0, x_i = a + ih = x_0 + ih, i = 0,1,2,...,N$, then we get :

$$\int_a^b f(x) \approx \frac{b-a}{N}[\frac{1}{2}f(x_0) + f(x_1) + ... + f(x_{N-1}) + \frac{1}{2}f(x_N)] \qquad (11)$$

We can apply the Trapezoidal rule on equation (2.10) we obtain :

$$\phi_n(x) = G_n(x) + \lambda h[\frac{1}{2}K(x,x_0)\gamma(x_0,\phi(x_0)) + \sum_{p=1}^{N-1} K(x,x_p)\gamma(x_p,\phi(x_p)) \qquad (12)$$

$$+ \frac{1}{2}K(x,x_N)\gamma(x_N,\phi(x_N))] + E$$

Where the error term is $E=-\frac{b-a}{12}h^2\frac{d^2}{d\xi^2}(K_{n,\xi}\gamma(\phi_\xi)),\xi\in(a,b)$. For easily simplification, we define in (12), $G_n(x)=G_n, K(x_n,x_p)=K_{np}, \phi(x_n)=\phi_n$. Hence, for $x=x_0,x_1,x_2,...,x_n$; $n=0,1,2,...,N$ in (12), and after negligting the error, we have the system of $(n+1)$ equations:

$$\phi_n=G_n+\lambda h[\frac{1}{2}K_{n0}\gamma(x_0,\phi_0)+\sum_{p=1}^{N-1}K_{np}\gamma(x_p,\phi_p)+\frac{1}{2}K_{nN}\gamma(x_N,\phi_N)] \qquad (13)$$

3.2 Simpson's rule :

In this section, we use Simpson's rule for solve equation (10), we will subdivide the interval of integration $[a,b]$ in to N equal subintervals

$$h=\frac{x_N}{N};N\geq 1,x_N=b,a=0.$$

We can apply the Simpson's rule on equation (2.10), we obtain :

$$\begin{aligned}\phi_n(x)=G_n(x)+\frac{\lambda h}{3}[K(x,x_0)\gamma(x_0,\phi(x_0))+4K(x,x_1)\gamma(x_1,\phi(x_1))\\ +2K(x,x_2)\gamma(x_2,\phi(x_2))+...+K(x,x_N)\gamma(x_N,\phi(x_N))]\end{aligned} \qquad (14)$$

For easily simplification, we define in(14), $G_n(x)=G_n, K(x_n,x_p)=K_{np}, \phi(x_n)=\phi_n$. Hence, for $x=x_0,x_1,x_2,...,x_n, n=0,1,2,...,N$ in(14), we have the system of $(n+1)$ equations:

$$\phi_n=G_n+\frac{\lambda h}{3}[K_{n0}\gamma(x_0,\phi_0)+4K_{n1}\gamma(x_1,\phi_1)+2K_{n2}\gamma(x_2,\phi_2)+...+K_{nN}\gamma(x_N,\phi_N)] \qquad (15)$$

3.3 Romberg integration method :

The Romberg integration is depend on the Trapezoidal rule for integer the function $g(x)$ in the interval$[a,b]$.Consider the interval $[x_{i-1},x_i]$ where $x_i-x_{i-1}=h, i=1,2,...,N$ and put $x_0=a, x_N=b$, then we get :

$$\int_a^b g(x)dx=\frac{h}{2}[g(a)+g(b)+2\sum_{i=1}^{n-1}g(x_i)]-\frac{b-a}{12}h^2g''(\xi) \qquad (16)$$

Where $a<\xi<b, h=\frac{b-a}{N}, x_i=a+ih, i=0,1,2,...,N$.

If $h_m=\frac{b-a}{2^{m-1}}$, the Trapezoidal rule is become:

$$\int_a^b g(x)dx = \frac{h_m}{2}[g(a)+g(b)+2\sum_{i=1}^{2^{m-1}-1} g(n,ih_{m-1})] - \frac{b-a}{12}h_m^2 g''(\xi_m) \quad (17)$$

Let :

$$R_{1,1} = \frac{h_1}{2}[f(a)+f(b)] = \frac{b-a}{2}[f(a)+f(b)]$$

$$R_{2,1} = \frac{h_2}{2}[f(a)+f(b)+2f(a+h_2)] = \frac{1}{2}[R_{1,1}+h_1 f(n+\frac{h_1}{2})]$$

$$R_{3,1} = \frac{h_3}{2}[f(a)+f(b)+2[f(a+h_3)+f(a+2h_3)+f(a+3h_3)]]$$

$$R_{1,3} = \frac{1}{2}[R_{1,2}+h_2[f(a+\frac{h_2}{2})+f(a+\frac{3h_2}{2})]]$$

In general, we get:

$$R_{m,1} = \frac{1}{2}[R_{m-1,1}+h_m\sum_{i=1}^{2^{m-2}} f(a+(i-\frac{1}{2})h_{m-1})], \qquad m=2,3,\ldots,N$$

Then using (17) in (10), we get :

$$\phi_n(x) = G_n(x)+\lambda\int_a^b K(x,y)\gamma(y,\phi_n(y))dy \approx G_n(x)+\frac{\lambda h_m}{2}[K(x,x_0)\gamma(x_0,\phi(x_0))$$
$$+2\sum_{i=1}^{2^{m-1}-1} K(x,x_i)\gamma(x_i,\phi(x_i))+K(x,x_N)\gamma(x_N,\phi(x_N))], \quad n=0,1,2,\ldots,N,\; m=1,2,\ldots,N \quad (18)$$

Where equation (18) is the approximate solutions for equation (10). For easily simplification, we define in (18) $G_n(x)=G_n, K(x_n,x_i)=K_{ni}, \phi(x_n)=\phi_n$. Hence, for $x=x_0,x_1,x_2,\ldots,x_n$, $n=0,1,2,\ldots,N$, we have the system of $(n+1)$ of equations:

$$\phi_n = G_n+\lambda\frac{h_m}{2}[K_{n0}\gamma(x_0,\phi_0)+2\sum_{i=1}^{2^{m-1}-1} K_{ni}\gamma(x_i,\phi_i)+K_{nN}\gamma(x_N,\phi_N)], \quad n=0,1,2,\ldots,N,\; m=1,2,\ldots,N \quad (19)$$

4 Numerical Examples :

We solve two examples in system of nonlinear Fredholm integral equations by Trapezoidal rule, Simpson's rule , and Romberg integration method at $N=20,50$,
$T=0.01, 0.1, 0.3, \lambda=1$, and $\mu=1$.
In table (1)-(12) :
$\phi_{Exact} \to$ exact solution, $\phi_{T.R.\to}$ approximate solution of Trapeziodal rule, $E_{T.R} \to$ the absolute error of Trapezoidal rule, $\phi_{S.R.} \to$ approximate solution of Simpson's rule, $E_{S.R.} \to$

the absolute error of Simpson's rule , $\phi_{R.I.} \rightarrow$ approximate solution of Romberg Integration method, $E_{.R.I.} \rightarrow$ the absolute error of Romberg Integration method.

Example 1

Consider the nonlinear Fredholm-Volterra integral equation :-

$$\phi(x,t) = xt - \frac{xt^2}{4} - \frac{x^2t^5}{5} + \int_0^1 xy\,\phi^2(y,t)dy + \int_0^t \tau^2\phi^2(x,\tau)d\tau$$

Exact solution $\phi(x,t) = xt$

Case 1 : $N = 20, h = 0.05$

At $T = 0.01$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{.R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00010	0.00010	2.475×10^{-6}	0.00010	1.649×10^{-7}	0.00010	6.27×10^{-11}
0.2	0.00040	0.00040	4.805×10^{-7}	0.00040	3.198×10^{-7}	0.00040	5.007×10^{-10}
0.3	0.00090	0.00090	6.841×10^{-7}	0.00090	4.537×10^{-7}	0.00090	1.689×10^{-9}
0.4	0.00160	0.00160	8.440×10^{-7}	0.00160	5.531×10^{-7}	0.00160	4.003×10^{-9}
0.5	0.00250	0.00250	9.453×10^{-7}	0.00250	5.989×10^{-7}	0.00250	7.812×10^{-10}
0.6	0.00360	0.00360	9.735×10^{-7}	0.00360	5.622×10^{-7}	0.00360	1.350×10^{-8}
0.7	0.00490	0.00490	9.139×10^{-7}	0.00490	3.989×10^{-7}	0.00490	2.144×10^{-8}
0.8	0.00640	0.00640	7.520×10^{-7}	0.00640	4.310×10^{-8}	0.00640	3.200×10^{-8}
0.9	0.00810	0.00810	4.730×10^{-7}	0.00809	4.460×10^{-7}	0.00810	4.556×10^{-8}
1	0.01000	0.00999	8.331×10^{-7}	0.00999	8.229×10^{-6}	0.01000	6.25×10^{-8}

Table(1)

At $T = 0.1$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{.R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00100	0.00102	2.477×10^{-5}	0.00101	1.651×10^{-5}	0.00100	2.383×10^{-8}
0.2	0.00400	0.00404	4.812×10^{-7}	0.00403	3.205×10^{-5}	0.00400	1.203×10^{-7}
0.3	0.00900	0.00906	6.857×10^{-5}	0.00904	4.553×10^{-5}	0.00900	3.269×10^{-7}
0.4	0.01600	0.01608	8.468×10^{-5}	0.01605	5.559×10^{-5}	0.01600	6.801×10^{-7}
0.5	0.02500	0.02509	9.496×10^{-5}	0.02506	6.032×10^{-5}	0.02500	1.213×10^{-6}
0.6	0.03600	0.03609	9.795×10^{-5}	0.03605	5.682×10^{-5}	0.03600	1.950×10^{-6}
0.7	0.04900	0.04909	9.214×10^{-5}	0.04904	4.064×10^{-5}	0.04900	2.899×10^{-6}
0.8	0.06400	0.06407	7.603×10^{-5}	0.06400	5.146×10^{-5}	0.06400	4.036×10^{-6}
0.9	0.08100	0.08104	4.803×10^{-5}	0.08096	3.874×10^{-5}	0.08100	5.289×10^{-6}
1	0.10000	0.09916	8.333×10^{-4}	0.09917	8.229×10^{-4}	0.10000	6.258×10^{-6}

Table (2)

At $T = 0.3$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00300	0.00322	2.227×10^{-4}	0.00315	1.527×10^{-4}	0.00300	4.329×10^{-6}
0.2	0.01200	0.01244	4.495×10^{-4}	0.01230	3.049×10^{-4}	0.01201	1.754×10^{-5}
0.3	0.02700	0.02765	6.542×10^{-4}	0.02744	4.468×10^{-4}	0.02703	3.995×10^{-5}
0.4	0.04800	0.04882	8.276×10^{-4}	0.04856	5.659×10^{-4}	0.04807	7.166×10^{-5}
0.5	0.07500	0.07595	9.557×10^{-4}	0.07564	6.439×10^{-4}	0.07511	1.119×10^{-4}
0.6	0.10800	0.10902	1.022×10^{-3}	0.10865	6.519×10^{-4}	0.10815	1.580×10^{-4}
0.7	0.14700	0.14800	1.006×10^{-3}	0.14754	5.427×10^{-4}	0.14720	2.029×10^{-4}
0.8	0.19200	0.19288	8.803×10^{-4}	0.19224	2.422×10^{-4}	0.19223	2.322×10^{-4}
0.9	0.24300	0.24360	6.041×10^{-4}	0.24263	3.625×10^{-4}	0.24321	2.192×10^{-4}
1	0.30000	0.29251	7.485×10^{-3}	0.29259	7.405×10^{-4}	0.30018	1.828×10^{-4}

Table (3)

Case 1 : $N = 50, h = 0.02$

At $T = 0.01$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00010	0.00010	9.901×10^{-8}	0.00010	6.599×10^{-8}	0.00010	1.02×10^{-11}
0.2	0.00040	0.00040	1.920×10^{-7}	0.00040	1.279×10^{-7}	0.00040	8.08×10^{-10}
0.3	0.00090	0.00090	2.273×10^{-7}	0.00090	1.805×10^{-7}	0.00090	2.717×10^{-10}
0.4	0.00160	0.00160	3.366×10^{-6}	0.00160	2.212×10^{-7}	0.00160	6.431×10^{-9}
0.5	0.00250	0.00250	3.762×10^{-7}	0.00250	2.187×10^{-7}	0.00250	1.255×10^{-9}
0.6	0.00360	0.00360	3.861×10^{-7}	0.00360	2.249×10^{-7}	0.00360	2.166×10^{-9}
0.7	0.00490	0.00490	3.604×10^{-7}	0.00490	7.709×10^{-9}	0.00490	3.437×10^{-9}
0.8	0.00640	0.00640	2.931×10^{-7}	0.00640	1.724×10^{-8}	0.00640	5.128×10^{-9}
0.9	0.00810	0.00810	1.782×10^{-7}	0.00809	1.064×10^{-7}	0.00810	7.291×10^{-9}
1	0.01000	0.00999	8.333×10^{-6}	0.00999	8.316×10^{-6}	0.01000	1×10^{-8}

Table (4)

At $T = 0.1$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00100	0.00102	9.920×10^{-6}	0.00101	6.618×10^{-6}	0.00100	2.001×10^{-8}
0.2	0.00400	0.00404	1.928×10^{-5}	0.00403	1.287×10^{-5}	0.00400	8.403×10^{-8}
0.3	0.00900	0.00906	2.749×10^{-5}	0.00904	1.822×10^{-5}	0.00900	1.978×10^{-7}
0.4	0.01600	0.01608	3.396×10^{-5}	0.01605	2.242×10^{-5}	0.01600	3.657×10^{-7}
0.5	0.02500	0.02509	3.808×10^{-5}	0.02506	2.233×10^{-5}	0.02500	5.878×10^{-7}
0.6	0.03600	0.03609	3.925×10^{-5}	0.03605	2.312×10^{-5}	0.03600	8.538×10^{-7}
0.7	0.04900	0.04909	3.683×10^{-5}	0.04904	1.160×10^{-6}	0.04900	1.133×10^{-6}
0.8	0.06400	0.06407	3.016×10^{-5}	0.06400	2.573×10^{-6}	0.06400	1.361×10^{-6}
0.9	0.08100	0.08104	1.851×10^{-5}	0.08096	1.559×10^{-5}	0.08100	1.418×10^{-6}
1	0.10000	0.09916	8.333×10^{-4}	0.09917	8.316×10^{-4}	0.10000	1.001×10^{-6}

Table (5)

At $T = 0.3$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00300	0.00322	9.372×10^{-5}	0.00306	6.401×10^{-5}	0.00300	4.629×10^{-6}
0.2	0.01200	0.01244	1.913×10^{-4}	0.01213	1.336×10^{-4}	0.01201	1.854×10^{-5}
0.3	0.02700	0.02765	1.913×10^{-4}	0.02720	2.039×10^{-4}	0.02704	4.175×10^{-5}
0.4	0.04800	0.04882	2.874×10^{-4}	0.04827	2.724×10^{-4}	0.04807	7.390×10^{-5}
0.5	0.07500	0.07595	4.510×10^{-4}	0.07530	3.093×10^{-4}	0.07511	1.135×10^{-4}
0.6	0.10800	0.10902	5.025×10^{-4}	0.10835	3.574×10^{-4}	0.10815	1.569×10^{-4}
0.7	0.14700	0.10480	5.164×10^{-4}	0.14719	1.955×10^{-4}	0.14719	1.951×10^{-4}
0.8	0.19200	0.19288	4.703×10^{-4}	0.19222	2.220×10^{-4}	0.19221	2.111×10^{-4}
0.9	0.24300	0.24360	3.281×10^{-4}	0.24268	3.125×10^{-4}	0.24317	1.741×10^{-4}
1	0.30000	0.29251	7.500×10^{-3}	0.29251	7.485×10^{-3}	0.30013	1.324×10^{-4}

Table (6)

Example 2

Consider the nonlinear Fredholm-Volterra integral equation:-

$$\phi(x,t) = xt - \frac{t^2 e^{x^2}}{4} - \frac{x^2 t^4}{3} + \int_0^1 e^{x^2} y\,\phi^2(y,t)dy + \int_0^t t\,\phi^2(x,\tau)d\tau$$

Exact solution $\phi(x,t) = xt$

Case 1: $N = 20, h = 0.05$

At $T = 0.01$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00010	0.00010	2.500×10^{-6}	0.00010	1.666×10^{-6}	0.00010	6.344×10^{-10}
0.2	0.00040	0.00040	2.500×10^{-6}	0.00040	1.664×10^{-6}	0.00040	2.629×10^{-9}
0.3	0.00090	0.00090	2.495×10^{-6}	0.00090	1.655×10^{-6}	0.00090	6.236×10^{-9}
0.4	0.00160	0.00160	2.476×10^{-6}	0.00160	1.623×10^{-6}	0.00160	1.192×10^{-8}
0.5	0.00250	0.00250	2.427×10^{-6}	0.00250	1.538×10^{-6}	0.00250	2.041×10^{-8}
0.6	0.00360	0.00360	2.326×10^{-6}	0.00360	1.343×10^{-6}	0.00360	3.280×10^{-8}
0.7	0.00490	0.00490	2.131×10^{-6}	0.00490	9.309×10^{-7}	0.00490	5.073×10^{-8}
0.8	0.00640	0.00640	1.783×10^{-6}	0.00640	1.030×10^{-7}	0.00640	7.673×10^{-8}
0.9	0.00810	0.00810	1.182×10^{-6}	0.00809	1.024×10^{-6}	0.00810	1.145×10^{-7}
1	0.01000	0.00999	2.265×10^{-5}	0.00997	2.236×10^{-5}	0.01000	1.699×10^{-7}

Table (7)

At $T = 0.1$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00100	0.00125	2.500×10^{-4}	0.00116	1.666×10^{-4}	0.00100	9.402×10^{-8}
0.2	0.00400	0.00425	2.503×10^{-4}	0.00416	1.667×10^{-4}	0.00400	5.066×10^{-7}
0.3	0.00900	0.00925	2.503×10^{-4}	0.00916	1.663×10^{-4}	0.00900	1.436×10^{-6}
0.4	0.01600	0.01624	2.495×10^{-4}	0.01616	1.641×10^{-4}	0.01600	3.064×10^{-6}
0.5	0.02500	0.02524	2.462×10^{-4}	0.02515	1.573×10^{-4}	0.02500	5.501×10^{-6}
0.6	0.03600	0.03623	2.380×10^{-4}	0.03613	1.398×10^{-4}	0.03600	8.727×10^{-6}
0.7	0.04900	0.04922	2.206×10^{-4}	0.04910	1.005×10^{-4}	0.04901	1.250×10^{-5}
0.8	0.06400	0.06418	1.869×10^{-4}	0.06401	1.892×10^{-5}	0.06401	1.629×10^{-5}
0.9	0.08100	0.08112	1.259×10^{-4}	0.08097	2.459×10^{-5}	0.08101	1.912×10^{-5}
1	0.10000	0.09776	2.236×10^{-3}	0.09773	2.265×10^{-3}	0.10001	1.703×10^{-5}

Table (8)

At $T = 0.3$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00300	0.00525	2.252×10^{-3}	0.00450	1.502×10^{-3}	0.00300	3.073×10^{-6}
0.2	0.01200	0.01427	2.270×10^{-3}	0.01351	1.518×10^{-3}	0.12022	2.230×10^{-5}
0.3	0.02700	0.02931	2.312×10^{-3}	0.02855	1.556×10^{-3}	0.02707	7.208×10^{-5}
0.4	0.04800	0.05038	2.381×10^{-3}	0.04961	1.613×10^{-3}	0.04816	1.637×10^{-4}
0.5	0.07500	0.07746	2.468×10^{-3}	0.07666	1.667×10^{-3}	0.07530	3.011×10^{-4}

0.6	0.10800	0.11053	2.539×10^{-3}	0.10965	1.655×10^{-3}	0.10847	4.748×10^{-4}
0.7	0.14700	0.10495	2.527×10^{-3}	0.14844	1.446×10^{-3}	0.14765	6.535×10^{-4}
0.8	0.19200	0.19431	2.311×10^{-3}	0.19279	7.978×10^{-4}	0.19277	7.741×10^{-4}
0.9	0.24300	0.24469	1.692×10^{-3}	0.24227	7.237×10^{-4}	0.24372	7.205×10^{-4}
1	0.30000	0.27960	2.039×10^{-2}	0.27985	2.014×10^{-2}	0.30015	1.563×10^{-4}

Table (9)

Case 2: $N=50, h=0.02$

At $T=0.01$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00010	0.00010	1.000×10^{-6}	0.00010	6.66×10^{-7}	0.00010	1.042×10^{-10}
0.2	0.00040	0.00040	1.040×10^{-6}	0.00040	6.659×10^{-7}	0.00040	4.420×10^{-10}
0.3	0.00090	0.00090	9.967×10^{-7}	0.00090	6.585×10^{-7}	0.00090	1.069×10^{-9}
0.4	0.00160	0.00160	9.878×10^{-7}	0.00160	6.493×10^{-7}	0.00160	2.073×10^{-9}
0.5	0.00250	0.00250	9.665×10^{-7}	0.00250	5.621×10^{-7}	0.00250	3.567×10^{-9}
0.6	0.00360	0.00360	9.230×10^{-7}	0.00360	5.378×10^{-7}	0.00360	5.717×10^{-9}
0.7	0.00490	0.00490	8.412×10^{-7}	0.00490	9.051×10^{-9}	0.00490	8.749×10^{-9}
0.8	0.00640	0.00640	6.957×10^{-7}	0.00640	4.170×10^{-8}	0.00640	1.298×10^{-8}
0.9	0.00810	0.00810	4.460×10^{-7}	0.00809	2.387×10^{-7}	0.00810	1.891×10^{-8}
1	0.01000	0.00999	2.265×10^{-5}	0.00999	2.260×10^{-5}	0.01000	2.718×10^{-7}

Table (10)

At $T=0.1$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00100	0.00110	1.000×10^{-4}	0.00101	6.669×10^{-5}	0.00100	4.242×10^{-8}
0.2	0.00400	0.00410	1.002×10^{-4}	0.00403	6.684×10^{-5}	0.00400	2.988×10^{-7}
0.3	0.00900	0.00910	1.005×10^{-4}	0.00904	6.670×10^{-5}	0.00900	9.522×10^{-7}
0.4	0.01600	0.01610	1.007×10^{-4}	0.01605	6.686×10^{-5}	0.01600	2.141×10^{-6}
0.5	0.02500	0.02510	1.002×10^{-4}	0.02506	5.976×10^{-5}	0.02500	3.905×10^{-6}
0.6	0.03600	0.03609	9.783×10^{-5}	0.03605	5.931×10^{-5}	0.03600	6.101×10^{-6}
0.7	0.04900	0.04909	9.155×10^{-5}	0.04904	8.337×10^{-6}	0.04900	8.307×10^{-6}
0.8	0.06400	0.06407	7.796×10^{-5}	0.06400	1.256×10^{-5}	0.06400	9.689×10^{-6}
0.9	0.08100	0.08105	5.157×10^{-5}	0.08096	3.274×10^{-5}	0.08100	8.863×10^{-6}
1	0.10000	0.09773	2.265×10^{-3}	0.09917	2.260×10^{-3}	0.10000	2.725×10^{-6}

Table (11)

At $T = 0.3$

x	ϕ_{Exact}	$\phi_{T.R.}$	$E_{T.R.}$	$\phi_{S.R.}$	$E_{S.R.}$	$\phi_{R.I.}$	$E_{R.I.}$
0	0	0	0	0	0	0	0
0.1	0.00300	0.00390	9.026×10^{-4}	0.00360	6.025×10^{-4}	0.00300	4.329×10^{-6}
0.2	0.01200	0.01292	9.204×10^{-4}	0.01262	6.201×10^{-4}	0.01202	2.121×10^{-5}
0.3	0.02700	0.02796	9.661×10^{-4}	0.02766	6.617×10^{-4}	0.02707	7.004×10^{-5}
0.4	0.04800	0.04904	1.047×10^{-3}	0.04874	7.425×10^{-4}	0.04803	1.599×10^{-4}
0.5	0.07500	0.07616	1.160×10^{-3}	0.07575	7.959×10^{-4}	0.07529	2.932×10^{-4}
0.6	0.10800	0.10928	1.282×10^{-3}	0.10893	9.360×10^{-4}	0.10845	4.571×10^{-4}
0.7	0.14700	0.14836	1.364×10^{-3}	0.14761	6.168×10^{-4}	0.14761	6.153×10^{-4}
0.8	0.19200	0.19331	1.312×10^{-3}	0.19272	7.234×10^{-4}	0.19269	6.976×10^{-4}
0.9	0.24300	0.24397	9.715×10^{-4}	0,24363	6.582×10^{-4}	0.24358	5.870×10^{-4}
1	0.30000	0.27960	2.039×10^{-2}	0.27964	2.035×10^{-2}	0.30055	5.501×10^{-4}

Table (12)

5 The conclusion :

From the previous discussions we conclude the following :

1. As N is increasing the error are decreasing.

2. As x and t are increasing in $[0,1]\times[0,1]$, the error due to Trapezoidal rule, Simpson's rule, and Romberg integration method are also increasing.

3. The errors due to the Romberg integration method less than the errors due to the Trapezoidal rule and Simpson's rule. (i.e. Romberg integration method the best methods to solve nonlinear Fredholm-Volterra integral equation with continuous kernel).

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