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Finitely Generated Groups With All Finite Subgroups Subnormal

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Abstract. In this paper, the finitely generated periodic groups with all infinite subgroups subnormal are characterized.

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1. Introduction

Let G be a periodic group in which every infinite subgroup is subnormal. If the Baer radical of G is finite, then G is soluble. Thus, since G is locally finite, G is either Baer group or Chernikov group by Lemma 3.5 [1]. If the Baer radical of G with all infinite subgroups subnormal is finite, can it be the finitely generated subgroups of G finite? It is not known. So it is studied the finitely generated residually finite periodic groups with all infinite subgroups subnormal. In this paper, it is given the proof of the following Theorems.

Theorem 1.1. *Let G be a finitely generated infinite periodic group such that all infinite subgroups are subnormal. If H is a subgroup of G, then H is finite or has finite index in G.*

Theorem 1.2. *Let G be a finitely generated infinite residually finite periodic group such that all infinite subgroups are subnormal. Then $G/Frat(G)$ is finite and $Frat(G) = G'G^p$. Morover G has finitely many maximalsubgroups.*

Theorem 1.3. *Let G be an engel group such that all infinite subgroups are subnormal. Then G is a soluble group. If G is a bounded engel group such that all infinite subgroups are subnormal, then G is a nilpotent group.*

2. Proofs of the theorems

Lemma 2.1. *Let G be a finitely generated non-periodic group such that all infinite subgroups are subnormal. Then G is a nilpotent group.*

Proof. Since G is a non-periodic group, there is an element g of G such that the order of g is infinite. Since $< g >$ is a subnormal subgroup of G, there is a series such that $H_1 =< g > \lhd H_2 \lhd \ldots \lhd H_n = G$. We show that G is soluble by induction on n. If $n = 1$, then the case is trivial. Let H_k be soluble for $k < n$. Since H_k is infinite, H_{k+1}/H_k is soluble by the main corollary of [3].

Thus H_{k+1} is soluble. Then G is soluble by induction. Assume that G is non-nilpotent. Then by 15.5.3 [4], there is a normal subgroup K such that G/K is finite but non-nilpotent. Since K is finite, G/K is nilpotent. This gives a contradiction. □

Lemma 2.2. *Let G be a finitely generated hyperabelian group such that every infinite subgroup is subnormal. Then G is either a non-periodic nilpotent group or finite.*

Proof. If G is a non-periodic, G is nilpotent by Lemma 2.1. Assume that G is periodic. If G is nilpotent, then G is finite. If G is non-nilpotent, then there is a quotient group $\bar{G}$ such that $\bar{G}$ is just non-nilpotent group by Lemma 6.17 [5]. Since $\bar{G}$ is a hyperabelian group, there is an abelian normal subgroup $\bar{A}$ such that $\bar{G}/\bar{A}$ is a nilpotent group. Thus $\bar{G}$ is soluble. Since $\bar{G}$ is non-nilpotent finitely generated, there is a normal subgroup $\bar{K}$ such that $\bar{G}/\bar{K}$ is finite but non-nilpotent by 15.5.3 [4]. Thus $\bar{G}$ must be finite. Since $\bar{G}$ is a quotient group, there is a normal subgroup R of G such that $\bar{G} = G/R$. If R is infinite, then $\bar{G}$ is nilpotent. This is a contradiction. Then G is finite since R is finite. Thus we obtain that if G is periodic, then G is finite. □

Theorem 2.3. *Let G be a hyperabelian group with all infinite subgroups subnormal. Then G is soluble*

Proof. Every finitely generated subgroup is soluble by Lemma 2.2. If G is periodic then G is locally soluble periodic group and thus G is locally finite group has an infinite Baer radical, G is soluble by Lemma 3.5 [1]. If G is non-periodic, then we know that G is soluble by the proof of Lemma 2.1. □

Lemma 2.4. *Let G be a finitely generated periodic group such that every infinite subgroup is subnormal. Then G satisfies maximal condition.*

Proof. Assume that H is an infinite subgroup of G. Then, there is a series such that $H = H_n \lhd H_{n-1} \lhd \ldots \lhd H_2 \lhd H_1 = G$. By induction, we show that H_k/H_{k+1} is finite for $k = 1, \ldots, n-1$. If $k = 1$, then H_1/H_2 is finite by the main Theorem of [3] since H_2 is infinite and G is periodic. So H_2 is finitely generated. Let H_k/H_{k+1} be finite for $k = n-2$. Thus H_{k+1} is finitely generated. Then Since H_{n-1}/H_n is finitely generated, $H_n = H$ is finitely generated. Namely every subgroup of G is finitely generated. By Proposition 1.7.f [6], G satisfy maximal condition. □

Lemma 2.5. *Let G be a finitely generated infinite periodic group. If every infinite subgroup is normal, then there is a quotient group $\bar{G}$ such that it satisfies the following conditions:*
i) $\bar{G}$ is just infinite.
ii) $HR(\bar{G}) = B(\bar{G}) = 1$.
iii) Every subgroup of $\bar{G}$ is finite or finite index.
iv) R is finite normal subgroup for $\bar{G} = G/R$.

v) For every element $\bar{x}$ of $\bar{G}$, $C_{\bar{G}(\bar{x})}$ is finite and for every proper subgroup $\bar{H}$ of $\bar{G}$, $\bar{H}^{\bar{G}}$ is proper subgroup of $\bar{G}$.

Proof. If G is finite, then G is non-nilpotent. Thus by Lemma 6.17 [5], there is a quotient group $\bar{G}$ such that it is just non-nilpotent. Since G is periodic group, $\bar{G}$ is just finite. Therefore $\bar{G}$ has not a finite normal subgroup. Since $HR(\bar{G})$ and $B(\bar{G})$ are locally nilpotent, $HR(\bar{G}) = B(\bar{G}) = 1$. By [7], if $\bar{K}$ is an infinite subgroup group of $\bar{G}$, then there is a subgroup $\bar{C}$ such that $<\bar{K}, \bar{C}> = \bar{K} \times \bar{C}$ and $|\bar{G} : \bar{K} \cdot \bar{C}|$ is finite. Since $\bar{C}$ is a normal subgroup of $<\bar{K}, \bar{C}>$, $\bar{C} = 1$ or $\bar{C}$ is infinite. Assume that $\bar{C} \neq 1$. Then there is a non-trivial $\bar{x} \in \bar{K}$ such that $\bar{C} < C_{\bar{G}}(\bar{K}) < C_{\bar{G}}(\bar{x})$. Since $C_{\bar{G}}(\bar{x})$ is finite, $<\bar{x}>$ is a subgroup of $\bar{G}$. This is a contradiction. Thus $\bar{C} = 1$ and if $\bar{K}$ is an infinite subgroup of $\bar{G}$, then $|\bar{G} : \bar{K}|$ is finite. If $\bar{G} = G/R$, then R is finite since G is a periodic group such that all infinite subgroups are subnormal. Since $\bar{G}$ is coatomic and $HR(\bar{G}) = B(G) = 1$, the last part is obvious. □

Proof. (of the Theorem 1.1) Assume that H is an infinite subgroup of G. By Lemma 2.5, $\bar{H}$ has finite index in $\bar{G}$. Since $|G : H| = |G : HR||HR : H|$ where $\bar{G} = G/R$ and R is a finite normal subgroup, H has finite index in G. □

Lemma 2.6. *Let G be a finitely generated locally graded infinite periodic group. If all finite subgroups of G are subnormal in G, then G has a quotient group $\bar{G}$ such that it is residually finite.*

Proof. By lemma 2.5, there is a quotient group $\bar{G}$ such that $HR(\bar{G}) = B(\bar{G}) = 1$. Thus $\bar{G}$ has not a finite normal subgroup. Put $\bar{G} = G$. Assume that G is not residually finite. Since G is locally graded group, there is a subgroup H such that $|G : H|$ is finite. Thus there is a natural number n such that $G^{(n)} \leq H$. Namely $G' \neq G$. If G' is finite, G is finite. So G' is an finitely generated group. If G' is residually finite since G/G' is finite and all infinite subgroups are subnormal. So G' can not be residually finite. Assume that we have a series such that $G^{(n)} < G^{(n-1)} < \ldots < G' < G$ where $G^{(n)}, \ldots, G'$ and G are the finitely generated but non-residually finite. Since $G^{(n)}$ is locally graded, $G^{(n+1)} < G^{(n)}$. Since $G^{(n+1)}/G^{(n)}$ is finite, $G^{(n+1)}$ is finitely generated bu not residually finite. Thus we obtain a proper descending series such that $\ldots < G^{(n+1)} < G^{(n)} < G^{(n-1)} < \ldots < G' < G$ where $G^{(n)}$ are finitely generated but not residually finite for every natural number n. It can be $\bigcap_{n \in \mathbb{N}} G^{(n)} = 1$ or infinite. If $\bigcap_{n \in \mathbb{N}} G^{(n)} = 1$, then G is residually finite. This is a contradiction. If $\bigcap_{n \in \mathbb{N}} G^{(n)}$ is finite, $G/\bigcap_{n \in \mathbb{N}} G^{(n)}$ is finite by Theorem 1.1. Thus there is a natural number k such that $G^{(k)} < \bigcap_{n \in \mathbb{N}} G^{(n)}$. But since $G^{(k)} < G^{(k+1)}$, this is a contradiction. Namely G is residually finite. □

Lemma 2.7. *Let G be a residually finite countable infinite group. Then G can not have an finite maximal subgroup.*

Proof. Assume that F is a finite maximal subgroup. We can get $G = \{F, a_1, a_2, \ldots\}$. Since G is residually finite, there is a normal subgroup N_1 such that $F \not\subset N_1$ and G/N_1 is finite. Since F is maximal subgroup, $G/N_1 = FN_1/N_1$. Thus there are normal subgroups N_{n+1} such that $\{F, a_1, a_2, \ldots\} \not\subset N_{n+1}$ and $G/N_{n+1} = FN_{n+1}/N_{n+1}$ is finite. G is isomorph to a subgroup of $G/N_1 \times G/N_2 \times \ldots$ since $\bigcap_{i\in\mathbb{N}} N_i = 1$. It is clear that the derived lengths of G/N_i is bounded. Then G is soluble and therefore G is finite. This gives a contradiction. □

Proof. (of the Theorem 1.2) Since G is finitely generated, G is countable. Since G is finitely generated (so coatomic by Theorem A.10.11 [8]) infinite residually finite, G has an infinite maximal subgroup. Namely $G \neq Frat(G)$. Moreover every maximal subgroup is finite by Lemma 2.7. Since all infinite subgroups of G are subnormal in G, every maximal subgroup is normal in G. Because of $G' < Frat(G)$, $Frat(G)$ is infinite. $G/Frat(G)$ is finite by Theorem 1.1. Since $G/G'G^p$ is an elementary abelian group and an elementary abelian group has a trivial Frattini subgroup, $Frat(G) = G'G^p$. By Theorem A.10.9 [8], G has finitely many maximal subgroups. □

Corollary 2.8. *Let G be a finitely generated residually finite p-group such that every infinite subgroup is subnormal. If G/G' is cyclic, then G is finite*

Proof. Since $G/Frat(G)$ is finite by Theorem 1.2, $Frat(G)$ is finitely generated. Moreover $G/Frat(G)$ is cyclic. Since $Frat(G)$ is a set of the non-generators of G, G is finite. □

Lemma 2.9. *Let G be a finitely generated residually finite periodic group such that every infinite subgroup is subnormal. If the subnormality index is bounded with a natural number n, then G is finite.*

Proof. We now that G is countable since G is finitely generated. Assume that G is infinite. Let F be a finite subgroup of G. Since G is residually finite, there are normal subgroups N_i such that $\bigcap_{n\in\mathbb{N}} N_i = 1$ and G/N_i is finite. Since $[G,_n F] = [G,_n \bigcap_{n\in\mathbb{N}} FN_i] \leq \bigcap_{n\in\mathbb{N}} [G,_n FN_i] \leq \bigcap_{n\in\mathbb{N}} FN_i = F$, F is a subnormal subgroup of G. Namely since every subgroup of G is subnormal, G is nilpotent. Thus G is finite. This is a contradiction. □

Lemma 2.10. *Let G be finitely generated residually finite periodic group such that all infinite subgroups containing the finite subgroup F are the normal subgroups of G. Then G is finite.*

Proof. We can get that G is countable group by Lemma 2.5. Since G is coatomic, there is a maximal subgroup M_1 containing F. Then there is an element a_1 such that $G/M_1 \cong < a_1 >$ and $a_1 \notin M_1$. Assume that there are subgroups $M_1, \ldots, M_{i+1}$ containing F such that $M_i/M_{i+1} \cong < a_{i+1} >$ and $a_1, a_2, \ldots, a_{i+1} \notin M_{i+1}$. Since M_{i+1} is finitely generated and thus coatomic, ther is an infinite maximal subgroup M_{i+2} containing F. By Lemma 2.7, every maximal subgroup is normal. Thus there is the subgroup M_{i+2} such that $M_{i+1}/M_{i+2} \cong < a_{i+2} >$ and $a_1, a_2, \ldots, a_{i+2} \notin M_{i+2}$ for every natural number i. Since $F \leq \bigcap_{n\in\mathbb{N}} M_i$, $\bigcap_{n\in\mathbb{N}} M_i$ is an infinite normal subgroup of G and finite index in G by Lemma 2.5 (i). This is a contradiction since $a_1, a_2, \ldots \notin \bigcap_{n\in\mathbb{N}} M_i$ and $a_i \bigcap_{n\in\mathbb{N}} M_i \neq a_j \bigcap_{n\in\mathbb{N}} M_i$ for every $a_i \neq a_j$ □

Proof. (of the Theorem 1.3) If G is a non-periodic group, then G is soluble. Let G be a periodic group and F be a finitely generated subgroup of G. If F has an infinite abelian subgroup, then F is soluble. If every maximal abelian subgroup of F is finite, then F is nilpotent by Theorem 7.23 [5]. Since G is locally finite group, G has infinite Baer radical. By Lemma 3.5 [1] G is soluble. If G is a bounded engel group, then G is nilpotent by 12.3.3 [4] and the main theorem of [2]. □

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