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A generalization of Pythagorean triples for desirable quadrilaterals

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Dedicated to Professor Terence Brian Sheil-Small (30 May 1937 – ...) for his mathematical wisdom that the third author enjoyed during the early 1980s at the University of York, England.

Abstract.

We explore the generalization of famous Pythagorean triples (a, b, c) for triangles to Pythagorean quadruples (a, b, c, d) for desirable quadrilaterals. Using number theory and geometrical techniques including Diophantine equations and Ptolemy's Theorem, we show that there are infinite number of such quadrilaterals with specific relations between their sides and diagonals. We conclude our paper with an open question for further investigation.

Keywords:

Pythagorean Triples, Desirable Quadrilaterals, Diophantine equations, Ptolemy's Theorem, Heronian Triangles.

Introduction

At least one thousand years before Pythagoras was born, the Old Babylonians knew of the rule of right triangles, now known as "Pythagorean Theorem [1]. Such Babylonian influence on Greek mathematics is in accordance with the tales that Pythagoras spent more than twenty years of his life acquiring knowledge from the wise men of Egypt and Mesopotamia [1].

Arithmetic and geometry were the main fields studied by ancient Greek mathematicians. Euclid, in his inspiring book "Elementa" devoted chapters to both arithmetic and geometry and made the case for the claim that there is an infinite number of primitive Pythagorean triples, that is a Pythagorean triple all of whose components are relatively prime [2].

It was known to ancient Greek that if $m > 1$ is an odd integer, then the integer triple m, $(m^2 - 1)/2$ and $(m^2 + 1)/2$ satisfy the Pythagorean theorem $(m)^2 + ((m^2 - 1)/2)^2 = ((m^2 + 1)/2)^2$.

There are many positive integers a, b and c that satisfy $a^2 + b^2 = c^2$. Fermat in 1637 conjectured that there are no number n other than 2 that $a^n + b^n = c^n$. This conjecture was known as Fermat's Last Theorem which was proved in affirmative by Wiles and Taylor in 1994 [3]. For a proof of Fermat's Last Theorem using Diophantine geometry see Ossicini [4].

The subject of geometrical shapes with integer sides and diagonals engaged many generations of mathematicians. One example of such a case is the investigation of Heronian triangles. The Pythagorean triples and Diophantine geometry's applications in cryptography, cryptographic coding and population transfer [5, 6, 7] makes them so relevant to the recent development of modern sciences. This motivated us to further investigate and explore the geometric aspects of the Pythagorean triples and their related problems.

From a geometric perspective, there are infinite number of primitive Pythagorean triples e.g. an infinite number of different rectangles (which are not similar), where the lengths of their sides and diagonals are natural numbers. We therefore ask "are there parallelograms, trapezoids, and other quadrilaterals which share the same property? If so, what specific relations must exist between the sides and diagonals of such geometric shapes?" In the present paper we shall explore these questions and state an open problem for further research in this direction.

Main Results

Using the construction of a certain triangle and a parallelogram whose sides and diagonals are natural numbers, we begin with stating and proving two related problems.

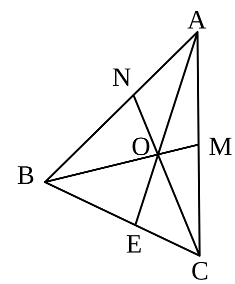

Figure 1 – Perpendicular medians in a triangle

Observation 1

If in the triangle ABC the medians CN and BM are perpendicular to each other (Figure 1), then there holds

(1) $$AB^2 + AC^2 = 5BC^2.$$

Observation 2

If in the parallelogram ABCD there holds $AD = 3 \cdot BC$ (Figure 2), then $AB^2 + AC^2 = 5BC^2$.

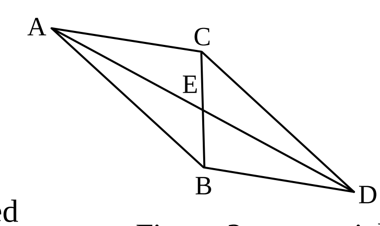

Figure 2 – a special Parallelogram

Proof for Observation 1

Let O denote the intersection of the medians of the triangle ABC.

It is known that $AO = 2OE$. Since the triangle BOC is right-angled at O, $BC = 2OE$ and so $BC = AO$. Known is the formula for calculating the length of the median in a triangle given the lengths of its sides.

In our case $AE^2 = \frac{2AB^2 + 2AC^2 - BC^2}{4}$.

Therefore $AO^2 = \left(\frac{2}{3}\right)^2 AE^2 = \frac{2AB^2 + 2AC^2 - BC^2}{9} = BC^2$.

Hence $\boxed{AB^2 + AC^2 = 5BC^2}$.

Proof for Observation 2

Applying the property of parallelograms (e.g. see [8]) to the parallelogram ABDC,

in Figure 2, there holds $2AB^2 + 2AC^2 = AD^2 + BC^2$.

In particular, for $AD = 3BC$ we have $2AB^2 + 2AC^2 = 9BC^2 + BC^2$.

In other words $\boxed{AB^2 + AC^2 = 5BC^2}$.

The above two observations give rise to the question that whether finding a triangle whose sides and one of the medians are natural numbers (or rational numbers) is the same as finding a parallelogram with the desired property. We shall call the quadrilateral whose sides and diagonals are natural numbers a "*desirable quadrilateral*."

Based on the second observation, we construct an example of a desirable quadrilateral whose longer diagonal AD is three times its shorter diagonal BC (when testing a hypothesis it is customary to "overload" the problem with another datum to form a particular case).

The task now belongs to the realm of number theory. Using natural numbers, we must solve the Diophantine equation

$\boxed{(2) \qquad b^2 + c^2 = 5a^2}$ and make sure that a, b and c form a triangle.

Solution of the Diophantine equation $b^2 + c^2 = 5a^2$**:**

We first remove the coefficient 5 from the right-hand side.

Letting $b = 2m + n$ and $c = 2n - m$ yields

$\boxed{(3) \qquad m^2 + n^2 = a^2}$.

Thus we formed the Pythagorean triple (m, n, a) from equation (2).

The most common choice of m = 3, n = 4 and a = 5 for the above Pythagorean triple (3) yields

b = 10, c = 5 and a = 5.

On the other hand, we must also make sure that these values for b, c and a form a triangle. Obviously, for the above choices of b, c and a we have $a + c = b$ and

so a triangle cannot be formed. The choice m = 4 and n = 3 is also not suitable.

Constructing an infinite number of desirable parallelograms

The next obvious choice for Pythagorean triple would be (8, 15, 17) which leads to a = 17, c = 22 and b = 31, for which we obtain a desirable parallelogram (see Figure 3).

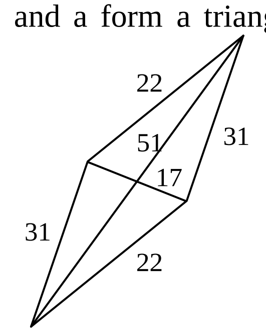

Figure 3 – parallelogram with integer segment lengths

The Pythagorean triple (8, 15, 17) has the form

$(4k\ ,\ 4k^2-1\ ,\ 4k^2+1)$; $k = 2, 3 \dots$.

A Pythagorean triple of this type forms the triple

$b = 4k^2 + 8k - 1$

$c = 8k^2 - 4k - 2$

$a = 4k^2 + 1$.

Obviously, for a, b and c as above, the sum of any two is greater than the third one and so the triple (b, c, a) forms a triangle. It is also not hard to prove that for $k > 1$, the equality $\frac{4k^2+1}{8k^2-4k-2} = \frac{4t^2+1}{8t^2-4t-2}$ holds if and only if $k = t$. Therefore the triples (b, c, a) forms non-similar triangles and the desirable parallelograms derived from the triples have different angles.

Thus we have proven that there is an infinite number of desirable parallelograms with the property that the longer diagonal is three times as long as the shorter diagonal.

In the sequel we identify the desirable parallelogram by (a, b, c, d) whose lengths are a and b and the diagonals are c and d.

Constructing desirable parallelograms where d = tc.

Case 1: d = tc and t is a natural number.

Consider a parallelogram with sides a and b and diagonals c and d so that d = tc.

So, by the property of parallelograms, there holds the relation $2a^2 + 2b^2 = c^2 + t^2c^2$.

To check if such a desirable parallelogram indeed exists, one must solve the Diophantine equation

(4) $\quad 2a^2 + 2b^2 = (t^2+1)c^2$.

For t = 1 we obtain a Pythagorean triple (known as Pythagorean Theorem for right triangles).

For t = 2 we obtain $2a^2 + 2b^2 = 5c^2$.

The case for t = 3 was presented in the previous section above.

In general, let us substitute $a = m + n$, $b = m - n$ in (4) to obtain

$4m^2 + 4n^2 = (t^2+1)c^2$.

Letting $2m = x$, $2n = y$, we obtain,

$x^2 + y^2 = (t^2+1)c^2$.

This time, instead of canceling the coefficient t^2+1, we make the substitution $x = tp + q$, $y = tq - p$ and obtain $p^2 + q^2 = c^2$.

Therefore, if (p, q, c) is a Pythagorean triple, then $m=\frac{tq+p}{2}$, $n=\frac{tq-p}{2}$ and consequently

$c=\sqrt{p^2+q^2}$, $b=\frac{t(p-q)+p+q}{2}$, $a=\frac{t(p+q)+q-p}{2}$.

Without loss of generality, we assume p > q to assure that both b and a are positive.

Multiplying c, b and a by 2 we obtain the triple

$$(a,b,c)=(t(p+q)+q-p\ ,\ t(p-q)+p+q\ ,\ 2\sqrt{p^2+q^2})$$

which satisfies the Diophantine equation $\boxed{2a^2+2b^2=(t^2+1)c^2}$.

Testing for t = 2 yields q = 3 and p = 4 and then $(a,b,c)=(13,9,10)$.

The desirable parallelogram is shown in Figure 4.

The interested reader can try other values of t.

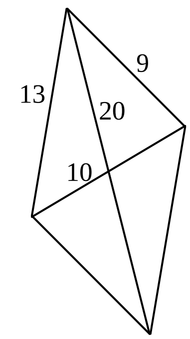

Figure 4 – parallelogram with integer segment lengths

Case 2: d = tc and t is a positive rational number

For the Diophantine equation $2a^2+2b^2=c^2+d^2$ we substitute

$a=x+y$, $b=x-y$ and obtain $4x^2+4y^2=c^2+d^2$ or

$$x^2+y^2=\left(\frac{c}{2}\right)^2+\left(\frac{d}{2}\right)^2.$$

A simple rearrangement yields $x^2-\left(\frac{d}{2}\right)^2=\left(\frac{c}{2}\right)^2-y^2$ or

$$\boxed{(5)\qquad \left(x-\frac{d}{2}\right)\left(x+\frac{d}{2}\right)=\left(\frac{c}{2}-y\right)\left(\frac{c}{2}+y\right)}.$$

From (5) for $x+\frac{d}{2}=pq$, $x-\frac{d}{2}=1$, $\frac{c}{2}+y=q$, $\frac{c}{2}-y=p$ we obtain

$y=\frac{q-p}{2}$, $c=p+q$, $d=pq-1$, $x=\frac{pq+1}{2}$.

Therefore we have the quadruple

$$(a,b,c,d)=\left(\frac{pq+q-p+1}{2},\frac{pq+p-q+1}{2},p+q,pq-1\right).$$

This yields

$$\boxed{\begin{array}{ll}(6) & a = pq + q - p + 1 \\ & b = pq + p - q + 1 \\ & c = 2(p+q) \\ & d = 2(pq-1)\end{array}}.$$

It is easy to verify that a, b, c and d are positive rational numbers if p > (q-1)/(q+1) and q > (p-1)/(p+1).

On the other hand, a, b and c are guaranteed to form a triangle since

$$0 < 2(p-1)(q-1) = 2pq + 2 - 2p - 2q = a + b - c$$

$$0 < 4q = a + c - b$$

$$0 < 4p = b + c - a.$$

Therefore a, b, c and d form the desirable parallelogram where d = tc or

$$\boxed{(7) \quad \frac{c}{d} = \frac{q+p}{pq-1}}.$$

Here we note that c/d = (p+q)/(pq-1) approaches 1/p if q approaches infinity and approaches 1/q if p approaches infinity.

So, we have constructed an infinite number of desirable parallelograms in which the ratio between its diagonals is a rational number.

Table 1 presents the quadruples a, b, c and d for p = 2 and q an integer from 3 to 10.

	$\mathbf{a = 3q - 1}$	$\mathbf{b = q + 3}$	$\mathbf{c = 2q + 4}$	$\mathbf{d = 4q - 2}$	$\mathbf{\frac{c}{d} = \frac{q+2}{2q-1}}$
$q = 3$	8	6	10	10	1
$q = 4$	11	7	12	14	$\frac{6}{7}$
$q = 5$	14	8	14	18	$\frac{7}{9}$
$q = 6$	17	9	16	22	$\frac{8}{11}$
$q = 7$	20	10	18	26	$\frac{9}{13}$
$q = 8$	23	11	20	30	$\frac{2}{3}$
$q = 9$	26	12	22	34	$\frac{11}{17}$
$q = 10$	29	13	24	38	$\frac{12}{19}$

Table 1

For $\frac{1}{2} + \frac{5}{2q-1} = \frac{q+2}{2q-1} = \frac{c}{d}$, we note that the ratio $\frac{c}{d}$ approaches ½ as q approaches infinity.

Desirable Quadrilaterals

First we note that for quadrilaterals whose sides and diagonals are rational numbers, we can multiply them by their common denominator and obtain a similar quadrilateral whose sides and diagonals are natural numbers.

Here we determine a few special cases and conclude the section with an open problem for future investigation.

Set A – **isosceles trapezoids whose base, side and diagonal are integers are desirable.**

Proof:

It is easy to construct the triangle ADC with sides $AD = a$, $DC = b$, $AC = c$ where a, b and c are integers. Then we inscribe ADC in a circle and construct the isosceles trapezoid ABCD (of course inscribed in the same circle) see Figure 6.

From Ptolemy's Theorem [9] we have $AB \cdot b + a \cdot a = c \cdot c$,

and so

$$\boxed{(8) \qquad AB = \frac{c^2 - a^2}{b}}.$$

Hence, if a, b and c are natural numbers, then AB is a rational number.

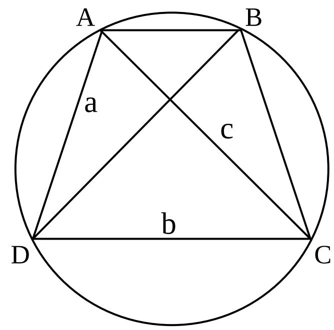

Figure 6 – trapezoid inscribed in a circle

Set B – **kites made from two congruent Heronian triangles are desirable.**

A Heronian triangle is a triangle whose side lengths and area are rational numbers (e.g see [10, 11]). We multiply the sides by their common denominator to obtain the Heronian triangle whose sides and area are natural numbers. See Figure 7.

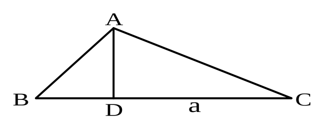

Figure 7 – Heronian triangle

Proof

If ABC is a Heronic triangle, then its altitude AD satisfies $AD = \frac{S_{\Delta ABC} \cdot 2}{a}$.

Therefore the altitude AD is a rational number and the kite ABEC is a desirable quadrilateral.

Next we shall present another set of desirable quadrilateral.

Set C – **quadrilaterals inscribed in a circle with one diagonal bisecting the other are desirable.**

We make use of the method of the transparent proof (e.g. see [12-13]), with the construction illustrated by a particular case from

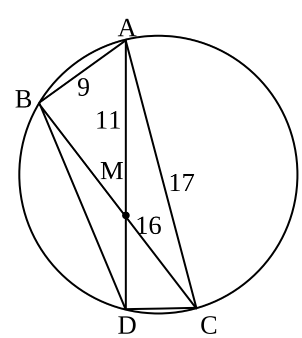

Figure 8 – quadrilateral inscribed in a circle whose one diagonal bisects the other

which one can easily understand how to construct a desirable quadrilateral from any other suitable particular case.

We choose the desirable parallelogram

(a, b, c, d) = (17, 9, 16, 22).

In Figure 8, the sides of the triangle ABC are AB = 9, AC = 17, BC = 16. Consequently, the median of ABC is AM = 11.

We inscribe ABC in the circle and extend AM to intercept the circle at the point D.

Claim

ABCD is a desirable quadrilateral.

Proof

It is clear that $AM \cdot MD = BM \cdot MC$.

We observe that AD = AM + MD is a rational number since both AM and MD are rational numbers.

From the similarity of the triangles BMD and AMC we obtain that $\frac{BM}{BD} = \frac{AM}{AC}$.

Obviously, BD is a rational number since BM, AM and Ac are rational numbers.

There exist other sets of desirable quadrilaterals, but we shall stop here. Let us only point out that no example has been found of a quadrilateral “without special pedigree” the lengths of whose sides and diagonals are natural numbers.

Summary

The paper dealt with methods for finding families of parallelograms the lengths of whose sides and diagonals are natural numbers. This can be seen as a generalization of Pythagorean triples which forms a subset of the desirable parallelograms – the rectangles. We have shown methods for solving Diophantine equations.

The solution of the Diophantine equations and the imposed condition that the triangle inequality holds true show that there exists an infinite number of parallelograms (which are not rectangles) that their sides and diagonals are natural numbers.

Examples of desirable quadrilaterals (which are not parallelograms) are also explored. We conclude the paper with the following

Open Problem:

What other desirable quadrilaterals (which are not parallelograms) there exist whose sides, diagonals and area are natural numbers?

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