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RESEARCH ORGANISATION

Volume 13, Issue 2
Published online: March 15, 2018|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Some Technique To Show The Boundedness Of Rational Difference Equations

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Abstract

This paper deals with the boundedness character of the solutions of the rational difference equations. We present a few methods that are applied to determine the boundedness behavior of the solutions of rational difference equations which studied in the literature.

2010 Mathematics Subject Classification: 39A10, 39A30.

Keywordsand phrases: Difference equations; solution; boundedness; equilibrium point.

1. Introduction

We consider the difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}) \quad n = 0,1, \dots . \#(1.1)$$

Where I is some interval of real numbers and the function $f: I^{k+1} \to I$is a continuously differentiable function. For every set of initial conditions $x_{-k}, x_{-(k-1)}, \dots, x_0 \in I$, the difference equation(1.1) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A solution $\{x_n\}_{n=-k}^{\infty}$.of Eq. (1.1) is called bounded if, for all $n \geq -k$, there exist m and M positive numbers such that

$$m \leq x_n \leq M.$$

For the last few years the boundedness behaviors of solutions of difference equations are being extensively investigated [1-9]. In the light of it, we concentrate on the boundedness behavior of solutions of rational difference equations. We introduce the methods that specify how the boundedness character of the solutions of rational difference equations which studied in the literature are established. We classify the methods used in six sections and give detailed examples of each method in each section. The proofs given in this paper has been taken from the given references.

2. Contradiction Methods

2.1. The case

$$x_{n+1} = \frac{\alpha + x_{n-1}}{(1 + Bx_n)x_{n-1}}, \quad n = 0,1, \dots . \#(2.1)$$

We take this example from [4], see page [201-202].

Theorem 2.1 Every solution of Eq. (2.1) is bounded.

Proof. Suppose for the sake of contradiction that there exists a solution of Eq. (2.1) which is unbounded. There exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \text{ and } x_{n_i+1} > x_j \text{ for } j < n_i + 1. \#(2.2)$$

Once

$$x_{n_i-1} \to 0,$$

Because

$$x_{n_i+1} = \frac{\alpha + x_{n_i-1}}{(1 + Bx_{n_i})x_{n_i-1}}.$$

From this and from

$$x_{n_i-1} = \frac{\alpha + x_{n_i-3}}{(1 + Bx_{n_i-2})x_{n_i-3}}$$

and

$$x_{n_i+1} = \frac{\alpha + \frac{\alpha + x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}}{(1 + Bx_{n_i})\frac{\alpha + x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}}$$

$$= \frac{\alpha + \frac{\alpha + x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}}{(1 + Bx_{n_i})(\alpha + x_{n_i-3})}(1 + Bx_{n_i-2})x_{n_i-3}$$

$$= \frac{\alpha x_{n_i-3} + \alpha Bx_{n_i-2}x_{n_i-3} + \alpha + x_{n_i-3}}{(1 + Bx_{n_i})(\alpha + x_{n_i-3})}$$

$$= \frac{\alpha x_{n_i-3}}{(1 + Bx_{n_i})(\alpha + x_{n_i-3})}(1 + Bx_{n_i-2}) + \frac{1}{1 + Bx_{n_i}} \#(2.3)$$

we have

$$x_{n_i-2} \to \infty \text{ and } x_{n_i}, x_{n_i-3} \to 1.$$

But then, (2.3) implies that, eventually,

$$x_{n_i+1} < x_{n_i-2}$$

Which contradicts (2.2) and completes the proof.

2.2. The case

$$\boldsymbol{x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_{n-1}}, \quad n = 0, 1, \ldots . \#(2.4)}$$

The main result for Eq. (2.4) is the following. See ([4], pp.215-216).

Theorem 2.2

a) Assume that $\beta \geq 1$. Then every solution of Eq. (2.4) increases to ∞.
b) Assume that $\beta < 1$. Then every solution of Eq. (2.4) converges to the positive equailibrium.

Proof.

a) Obviously

$$x_{n+1} > \beta x_n \geq x_n$$

from which the result follows.
b) The change of variables

$$x_n = \frac{1}{y_n}$$

transforms

$$y_{n+1} = \frac{y_n}{\beta + y_n + \alpha y_n y_{n-1}}, \quad n = 0,1,\ldots. \#(2.5)$$

Clearly,

$$y_n < 1 \; for \; n \geq 1.$$

We also claim that every positive solution of Eq. (2.5) is bounded from below by a positive constant. To see this, suppose for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$y_{n_i+1} \to 0 \; and \; y_{n_i+1} < y_j \; for \; j < n_i + 1.$$

From Eq. (2.5) we have

$$y_{n_i}, y_{n_i-1} \to 0.$$

Then eventually

$$y_{n_i+1} = \frac{y_{n_i}}{\beta + y_{n_i} + \alpha y_{n_i} y_{n_i-1}} > y_{n_i}$$

And this contardiction proves our assertion.
Define

$$I = \lim_{n\to\infty} \inf y_n \; and \; S = \lim_{n\to\infty} \sup y_n \,.$$

Obviously,

$$S \leq \frac{S}{\beta + S + \alpha SI} \; and \; I \geq \frac{I}{\beta + I + \alpha SI}$$

from which it follows that

$$\beta + S + \alpha SI \leq 1 \leq \beta + I + \alpha SI$$

andso

$$S = I.$$

This completes the proof.

2.3. The case

$$\boldsymbol{x_{n+1} = \beta + \frac{x_{n-2}}{x_n}, \quad n = 0, 1, \ldots. \#(2.6)}$$

We consider the difference equation (2.6) with the parameter β positive and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0. See ([5], pp.46-48).

Theorem 2.3 Every solution of Eq (2.6) is bounded.

Proof. First of all, we make the following useful general observations about the solutions of Eq.(2.6):

$$x_{n+1} = \beta \text{ for } n \geq 0. \#(2.7)$$

$$x_{n+1} < \beta + \frac{1}{\beta} x_{n-2}, \text{for } n \geq 1. \#(2.8)$$

$$x_{n+1} < \beta + \frac{1}{\beta}\left(\beta + \frac{x_{n-5}}{x_{n-3}}\right) < \beta + 1 + \frac{1}{\beta^2} x_{n-5}, \text{for } n \geq 4. \#(2.9)$$

$$x_{n_i+1} \to \infty \implies x_{n_i-2} \to \infty. \; \#(2.10)$$

$$x_{n_i+1} \to \beta \implies x_{n_i} \to \infty. \; \#(2.11)$$

Now suppose for the sake of contradiction that Eq. (2.6) has an unbounded solution $\{x_n\}$. Then there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \#(2.12)$$

And for every i,

$$x_{n_i+1} > x_j \; for \; j < n_i + 1. \#(2.13)$$

From (2.12) and (2.10) it follows that

$$x_{n_i-2} \to \infty, x_{n_i-5} \to \infty, and \; x_{n_i-8} \to \infty. \; \#(2.14)$$

Now we claim that the subsequence$\{x_{n_i-4}\}$ is bounded. Otherwise, there would exist a subsequence of $\{n_i\}$ which we still denote by $\{n_i\}$, such that

$$x_{n_i-4} \to \infty, x_{n_i-7} \to \infty, and\ x_{n_i-10} \to \infty. \#(2.15)$$

Note that, for every *i*,

$$x_{n_i-4} = \beta + \frac{x_{n_i-7}}{x_{n_i-5}}$$

and

$$x_{n_i-7} = \beta + \frac{x_{n_i-10}}{x_{n_i-8}}.$$

So, as a result of (2.14) and (2.15), we have eventually

$$x_{n_i-7} > x_{n_i-5} \text{and} x_{n_i-10} > x_{n_i-8}. \#(2.16)$$

and

$$\frac{x_{n_i-7}}{x_{n_i-5}} \to \infty \text{ and } \frac{x_{n_i-10}}{x_{n_i-8}} \to \infty.$$

Hence, from (2.16) and (2.9), we note that eventually

$$\begin{aligned} x_{n_i+1} &< \beta + 1 + \frac{1}{\beta^2} x_{n_i-7} \\ &= \beta + 1 + \frac{1}{\beta^2}\left(\beta + \frac{x_{n_i-10}}{x_{n_i-8}}\right) \\ &= \beta + 1 + \frac{1}{\beta} + \frac{1}{\beta^2}\left(\frac{x_{n_i-10}}{x_{n_i-8}}\right). \end{aligned}$$

Because of (2.15), it follows that the right-hand side of the above inequality is eventually less than x_{n_i-10}, which contradicts (2.13), and proves our claim that $\{x_{n_i-4}\}$ is bounded. From this and (2.14), we see

$$x_{n_i-1} = \beta + \frac{x_{n_i-4}}{x_{n_i-2}} \to \infty.$$

Furthermore,

$$\lim_{i\to\infty} \inf x_{n_i-3} > \beta.$$

Otherwise, a subsequence of $\{x_{n_i-4}\}$ would converge to β and therefore from (2.11), $\{x_{n_i-4}\}$ would be unbounded, which is not true.

Thus, eventually,

$$x_{n_i} = \beta + \frac{x_{n_i-3}}{x_{n_i-1}} > \beta + 1$$

And hence, for *i* sufficiently large,

$$x_{n_i+1} = \beta + \frac{x_{n_i-2}}{x_{n_i}} < \beta + \frac{x_{n_i-2}}{\beta + 1} < x_{n_i-2},$$

Which contradicts (2.13). This completes the proof.

3. Invariant Interval Methods

3.1 The case

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0,1, \dots . \#(3.1)$$

We consider the difference equation (3.1). We take this example from [3], see page [15-17].

Theorem 3.1 Every positive solution of Eq. (3.1) is bounded.

Proof. When

$$\gamma \leq 1,$$

we have

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}} \leq x_{n-1}$$

and thus the solutions of Eq. (3.1) are bounded. Now suppose that

$$\gamma > 1$$

And assume that $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (3.1). Choose a positive number *m* such that

$$x_{-1}, x_0 \in \left(m, \frac{\gamma - 1}{m}\right).$$

Define

$$f(x,y)=\frac{\gamma y}{1+xy}.$$

$f(x,y)$is decreasing in x. In fact,

$$f_x=\frac{0-\gamma y.y}{(1+xy)^2}=\frac{-\gamma y^2}{(1+xy)^2}$$

and thus f is decreasing due to $\gamma>1$.

$f(x,y)$ is increasing in y. In fact,

$$f_y=\frac{\gamma(1+xy)-\gamma y.x}{(1+xy)^2}=\frac{\gamma}{(1+xy)^2}$$

and so f is increasing because of $\gamma>1$.

Therefore, by using the increasing character of $f$we find that

$$m=\frac{\gamma m}{1+\frac{\gamma-1}{m}m}<x_1=\frac{\gamma x_{-1}}{1+x_0x_{-1}}<\frac{\gamma\frac{\gamma-1}{m}}{1+m\frac{\gamma-1}{m}}=\frac{\gamma-1}{m}$$

and hence by induction

$$x_n\in\left(m,\frac{\gamma-1}{m}\right), for\ all\ n\geq -1.$$

Consequently, $\{x_n\}_{n=-1}^{\infty}$ is bounded.

3.2 The case

$$x_{n+1}=\beta x_n+\frac{1}{x_{n-1}},\quad n=0,1,\dots.\#(3.2)$$

We consider the difference equation (3.2). See ([3], p.22).

Theorem 3.1 Eq.(3.2) has bounded solutions, if and only if

$$\beta<1.\#(3.3)$$

Proof. We see

$$x_{n+1}>\beta x_n$$

From which it follows that Eq. (3.2) has unbounded solutions for

$$\beta\geq 1.$$

On the other hand when (3.3) holds, we claim that every positive solution of Eq. (3.2) is bounded. Infact, if $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq. (3.2) and if we choose positive numbers m and M such that

$$x_{-1},x_0\in[m,M]\ and\ mM=\frac{1}{1-\beta},$$

then

$$m=\frac{1}{(1-\beta)M}=\beta m+\frac{1}{M}\leq x_1=\beta x_0+\frac{1}{x_{-1}}\leq\beta M+\frac{1}{m}=\frac{1}{(1-\beta)m}=M$$

and inductively,

$$x_n\in[m,M], for\ all\ n\geq -1$$

which proves our claim.

4. Min-Max Methods

4.1. The case

$$x_{n+1}=\frac{\alpha+\beta x_nx_{n-1}+x_{n-1}}{A+x_nx_{n-1}},\quad n=0,1,\dots.\#(4.1)$$

We consider the difference equation (4.1). See ([4], pp.217-218).

For Eq.(4.1) it can be seen that, for $n \geq 1$, every positive solution is bounded from below and from above by positive constants. Infact,

$$x_{n+1} \geq \frac{\alpha + \beta x_n x_{n-1}}{A + x_n x_{n-1}} \geq \frac{min\{\alpha, \beta\}}{max\{A, 1\}}$$

which shows that every solution of Eq. (4.1) is bounded from below, for $n \geq 1$, by the positive number

$$m = \frac{min\{\alpha, \beta\}}{max\{A, 1\}}.$$

So,

$$x_{n+1} < \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}} = \frac{\alpha}{x_n x_{n-1}} + \beta + \frac{1}{x_n} \leq \frac{\alpha}{m^2} + \beta + \frac{1}{m}$$

and thus every solution of Eq. (4.1) is also bounded from above, for $n \geq 2$, by the positive number

$$M = \frac{\alpha}{m^2} + \beta + \frac{1}{m}.$$

4.2. The case

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}, \quad n = 0,1, \ldots \#(4.2)$$

We consider the difference equation (4.2). See ([5], pp.41-42).

Theorem 4.1 Assume that $\alpha, \beta \in [0, \infty[$ and $\gamma, \delta, C, D \in (0, \infty)$. Then every positive solution of Eq.(4.2) is bounded from above and from below by positive numbers.

Proof. We see that

$$x_{n+1} \geq \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} \geq \frac{min\{\gamma, \delta\}}{max\{C, D\}}$$

and then $\{x_n\}$ is bounded from below by the positive number

$$m = \frac{min\{\gamma, \delta\}}{max\{C, D\}}.$$

Moreover, for $n \geq 1$

$$x_{n+2} = \frac{\alpha + \beta x_{n+1} + \gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}}$$

$$= \frac{\alpha}{C x_n + D x_{n-1}} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta}{C x_n + D x_{n-1}} \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}$$

$$\leq \frac{\alpha}{(C + D)m} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta}{C x_n + D x_{n-1}} \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}$$

$$= \frac{\alpha}{(C + D)m} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta\alpha}{(C x_n + D x_{n-1})(C x_{n-1} + D x_{n-2})}$$

$$+ \frac{\beta}{C x_n + D x_{n-1}} \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} + \frac{\beta^2 x_n}{(C x_n + D x_{n-1})(C x_{n-1} + D x_{n-2})}$$

$$\leq \frac{\alpha}{(C + D)m} + \frac{max\{\gamma, \delta\}}{min\{C, D\}} + \frac{\beta\alpha}{(C + D)^2 m^2} + \frac{\beta}{(C + D)m} \frac{max\{\gamma, \delta\}}{min\{C, D\}} + \frac{\beta^2}{CDm}$$

and therefore the solution is also bounded from above.

4.3. The case

$$x_{n+1}=\frac{\alpha+\beta x_n x_{n-1}+x_{n-1}}{Bx_n x_{n-1}+x_{n-1}},\quad n=0,1,\dots .\#(4.3)$$

We consider the difference equation (4.3). See ([4], p.419).

Eq. (4.3) is bounded from below and from above by positive constants. In fact for $n\geq 1$,

$$x_{n+1}\geq\frac{\beta x_n x_{n-1}+x_{n-1}}{Bx_n x_{n-1}+x_{n-1}}=\frac{\beta x_n+1}{Bx_n+1}\geq\frac{min\{\beta,1\}}{max\{B,1\}}.$$

Hence, for $n\geq 1$, every positive solution is bounded from below by

$$m=\frac{min\{\beta,1\}}{max\{B,1\}}.$$

So, for $n\geq 2$,

$$x_{n+1}=\frac{\alpha}{Bx_n x_{n-1}+x_{n-1}}+\frac{\beta x_n+1}{Bx_n+1}$$

$$\leq\frac{\alpha}{Bm^2+m}+\frac{min\{\beta,1\}}{max\{B,1\}}$$

Which establishes our claim.

4.4. The case

$$x_{n+1}=\frac{\alpha+x_n x_{n-1}}{A+x_n x_{n-1}},\quad n=0,1,\dots .\#(4.4)$$

We consider the difference equation (4.4). See ([3], p.26).

Every solution of Eq.(4.4) is bounded from above and from below by positive constants. In fact for all $n\geq 0$,

$$\frac{min\{\alpha,1\}}{max\{A,1\}}<x_{n+1}=\frac{\alpha+x_n x_{n-1}}{A+x_n x_{n-1}}<\frac{max\{\alpha,1\}}{min\{A,1\}}.$$

The following case is an example both min-max method and invariant interval methods.

4.5. The case

$$x_{n+1}=\frac{\alpha+\sum_{i=0}^{k}\beta_i x_{n-i}}{A+\sum_{i=0}^{k}B_i x_{n-i}},\quad n=0,1,\dots .\#(4.5)$$

We consider the difference equation (4.5). See ([5], pp.34-37).

Theorem 4.2 Consider the $(k+1)^{st}$-order rational difference equation (4.5) with non-negative parameters

$$\alpha,A,\beta_0,\dots,\beta_k,B_0,\dots,B_k$$

and with arbitrary non-negative initial conditions $x_{-k},\dots,x_0$ such that the denominator is always positive. Suppose that for every $i\in\{0,1,\dots,k\}$ for which the parameter β_i in the numerator is positive, the corresponding parameter B_i in the denominator is also positive. Then every solution of Eq. (4.5) is bounded.

Proof. We denote by I and I_0 the following subsets of $\{0,1,\dots,k\}$:

$$I=\{i\in\{0,1,\dots,k\}:\beta_i>0\text{ and}B_i>0\}$$

and

$$I_0=\{i\in\{0,1,\dots,k\}:\beta_i=0\text{ and}B_i>0\}.$$

Hence

$$I \cup I_0 \subset \{0,1,\dots,k\}$$

and Eq. (4.5) is equivalent to

$$x_{n+1} = \frac{\alpha + \sum_{i \in I} \beta_i x_{n-i}}{A + \sum_{i \in I} B_i x_{n-i} + \sum_{i \in I_0} B_i x_{n-i}}, \qquad n = 0,1,\dots. \#(4.6)$$

with $\beta_i, B_i \in (0,\infty)$ for every $i \in I$ and with $B_i > 0$ for every $i \in I_0$. Of course, I or I_0, or both, may be empty sets.

First of all, we show that when

$$A > 0 \; or \; \alpha = 0,$$

every solution of Eq. (4.5) is bounded. In fact, when$A > 0$

$$x_{n+1} \le \frac{\max_{i \in I}(\alpha, \beta_i)\left(1 + \sum_{i \in I} x_{n-i}\right)}{\min_{i \in I}(A, B_i)\left(1 + \sum_{i \in I} x_{n-i}\right)} \le \frac{\max_{i \in I}(\alpha, \beta_i)}{\min_{i \in I}(A, B_i)}$$

and thus every solution of Eq. (4.5) is bounded.

In the above inequality by $\max_{i \in I}(\alpha, \beta_i)$, we mean α if $I = \emptyset$ and the maximum of α and $\max_{i \in I} \beta_i$ otherwise. Similarly for the minimum. Moreover, if $I = \emptyset$ we define

$$\sum_{i \in I} x_{n-i} = 0$$

Next suppose that $\alpha = 0$. Hence the set I must be nonempty and

$$x_{n+1} \le \frac{\sum_{i \in I} \beta_i x_{n-i}}{\sum_{i \in I} B_i x_{n-i}} \le \frac{\max_{i \in I} \beta_i \sum_{i \in I} x_{n-i}}{\min_{i \in I} B_i \sum_{i \in I} x_{n-i}} = \frac{\max_{i \in I} \beta_i}{\min_{i \in I} B_i}$$

and every solution is bounded.

In the remaining part of the proof we suppose that

$$A = 0 \; and \; \alpha > 0$$

Now the proof depends on whether I or I_0 is empty.

Case 1: $I_0 = \emptyset$. So, because $A = 0,\ I \ne \emptyset$ and

$$x_{n+1} = \frac{\alpha + \sum_{i \in I} \beta_i x_{n-i}}{\sum_{i \in I} B_i x_{n-i}} > \frac{\min_{i \in I} \beta_i}{\max_{i \in I} B_i}, for\ n \ge 0.$$

Hence if we set

$$L = \frac{\min_{i \in I} \beta_i}{\max_{i \in I} B_i},$$

note that for $n \ge k$,

$$x_{n+1} \le \frac{\alpha}{L \sum_{i \in I} B_i} + \frac{\max_{i \in I} \beta_i}{\min_{i \in I} B_i}$$

and every solution of Eq. (4.5) is bounded from below and from above. Indeed in this case the equation is permanent.

Case 2: $I = \emptyset$. Then $I_0 \neq \emptyset$. In this case the Eq. (4.5) reduces to

$$x_{n+1} = \frac{\alpha}{\sum_{i\in I_0} B_i x_{n-i}} \quad n = 0,1, \dots . \#(4.7)$$

with

$$\sum_{i\in I_0} B_i > 0.$$

We will show that every solution of Eq. (4.7) is bounded. To this end, let $\{x_n\}$ be a solution of Eq. (4.7) and suppose, without loss of generality, that the solution is positive for all $n \geq -k$. Let L, U be chosen in such a way that

$$x_{-k}, \dots, x_0 \in (L, U)$$

and

$$LU = \frac{\alpha}{\sum_{i\in I_0} B_i}.$$

Hence

$$L = \frac{\alpha}{U\sum_{i\in I_0} B_i} < x_1 = \frac{\alpha}{\sum_{i\in I_0} B_i x_{-i}} < \frac{\alpha}{L\sum_{i\in I_0} B_i} = U.$$

Then,

$$x_1 \in (L, U)$$

and by induction

$$x_n \in (L, U), for\ n \geq -k.$$

Case 3: Both I and I_0 are nonempty sets. In this case, as in case 2, we will suppose, without loss of generality, that a solution $\{x_n\}$ is positive and show that there exist an interval (L, U) that contains the entire solution.

To see how the interval is found note that

$$x_1 \in (L, U)$$

if and only if

$$L < \frac{\alpha + \sum_{i\in I} \beta_i x_{-i}}{\sum_{i\in I} B_i x_{-i} + \sum_{i\in I_0} B_i x_{-i}} < U$$

if and only if

$$\sum_{i\in I} (LB_i - \beta_i) x_{-i} + \left(L \sum_{i\in I_0} B_i x_{-i} - \alpha \right) < 0$$

and

$$\sum_{i\in I}(UB_i-\beta_i)x_{-i}+\left(U\sum_{i\in I_0}B_ix_{-i}-\alpha\right)>0$$

if

$$L<\frac{\beta_i}{B_i}<U\ for\ all\ i\in I$$

and

$$\frac{\alpha}{U}<\sum_{i\in I_0}B_ix_{-i}<\frac{\alpha}{L}.$$

$$L\sum_{i\in I_0}B_i<\sum_{i\in I_0}B_ix_{-i}<U\sum_{i\in I_0}B_i$$

and hence it suffices to choose L and U such that

$$x_{-k},\dots,x_0\in(L,U),$$

$$L<\min_{i\in I}\left(\frac{\beta_i}{B_i},\frac{B_i}{\beta_i},\frac{\alpha}{\sum_{j\in I_0}B_j}\right),$$

and

$$LU=\frac{\alpha}{\sum_{j\in I_0}B_j}.$$

With the above choice of (L,U), it is now easy to prove that

$$x_1\in(L,U)$$

and then by induction

$$x_n\in(L,U),for\ n\geq -k.$$

This completes the proof.

5. Invariant Product Methods

5.1. The case

$$x_{n+1}=\frac{\alpha+\beta x_n}{Cx_{n-1}},\quad n=0,1,\dots.\#(5.1)$$

We consider the difference equation (5.1). See ([8], pp.70-71).

This equation is called ***Lyness' Equation***. By the change of variables, Eq. (5.1) reduces to the equation

$$y_{n+1}=\frac{p+y_n}{y_{n-1}},\quad n=0,1,\dots.\#(5.2)$$

where$p=\frac{\alpha C}{\beta^2}$.

The special case of Eq. (5.2) where

$$p=1$$

was discovered by Lyness in 1942 while he was working on a problem in Number Theory. In this special case, the equation becomes

$$y_{n+1}=\frac{1+y_n}{y_{n-1}},\quad n=0,1,\dots.\#(5.3)$$

Every solution of which is periodic with period five. Actually the solution of Eq. (5.3) with initial conditions y_{-1} and y_0 is the five-cycle:

$$y_{-1},y_0,\frac{1+y_0}{y_{-1}},\frac{1+y_{-1}+y_0}{y_{-1}y_0},\frac{1+y_{-1}}{y_0},\dots.$$

Eq. (5.2) possesses the invariant

$$I_n=(p+y_{n-1}+y_n)\left(1+\frac{1}{y_{n-1}}\right)\left(1+\frac{1}{y_n}\right)=constant\quad\#(5.4)$$

from which it follows that every solution of Eq. (5.2) is bounded from above and from below by positive constants. In fact for $n\geq 0$

$$(p+y_n+y_{n+1})\left(1+\frac{1}{y_n}\right)\left(1+\frac{1}{y_{n+1}}\right)=\left(p+y_n+\frac{p+y_n}{y_{n-1}}\right)\left(1+\frac{1}{y_n}\right)\left(1+\frac{y_{n-1}}{p+y_n}\right)$$

$$=\left(\frac{p+y_n}{p+y_n}+\frac{1}{y_{n-1}}\right)\left(1+\frac{1}{y_n}\right)(p+y_n+y_{n-1})$$

$$=(p+y_{n-1}+y_n)\left(1+\frac{1}{y_{n-1}}\right)\left(1+\frac{1}{y_n}\right).$$

The proof follows by induction.

5.2. The case

$$x_{n+1}=\frac{\alpha}{(1+x_n)x_{n-1}},\quad n=0,1,\dots.\#(5.5)$$

We consider the difference equation (5.5). See ([3], p.8).

This equation has some similarities with ***Lyness's*** Equation,

$$x_{n+1}=\frac{\alpha+x_n}{x_{n-1}},\quad n=0,1,\dots.\#(5.6)$$

which is gifted with the invariant (see(5.4)):

$$(\alpha+x_{n-1}+x_n)\left(1+\frac{1}{x_{n-1}}\right)\left(1+\frac{1}{x_n}\right)=constant,\forall n\geq 0.$$

In fact, as for Eq. (5.6), Eq.(5.5) possesses an invariant, namely,

$$x_{n-1}+x_n+x_{n-1}x_n+\alpha\left(\frac{1}{x_{n-1}}+\frac{1}{x_n}\right)=constant,\forall n\geq 0.\#(5.7)$$

By using (5.7) we see that every positive solution of Eq. (5.7) is bounded from above and from below by positive constants. In fact for $n\geq 0$

$$x_n+x_{n+1}+x_nx_{n+1}+\alpha\left(\frac{1}{x_n}+\frac{1}{x_{n+1}}\right)=x_n+\frac{\alpha}{(1+x_n)x_{n-1}}+x_n\frac{\alpha}{(1+x_n)x_{n-1}}$$

$$+\alpha\left(\frac{1}{x_n}+\right)\frac{(1+x_n)x_{n-1}}{\alpha}$$

$$=x_n+\frac{\alpha(1+x_n)}{(1+x_n)x_{n-1}}+\frac{\alpha}{x_n}+x_{n-1}+x_nx_{n-1}$$

$$=x_{n-1}+x_n+x_{n-1}x_n+\alpha\left(\frac{1}{x_{n-1}}+\frac{1}{x_n}\right).$$

The proof follows by induction.

6. Initial Conditions Methods

6.1. The case

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0,1, \dots . \#(6.1)$$

We consider the difference equation (6.1). See ([3], pp.15-16).

When one of the initial conditions of a solution of Eq. (6.1) is zero, Eq. (6.1) reduces to the linear equation

$$x_{n+1} = \gamma x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution φ is , then the solution of the equation is

$$\cdots, 0, \varphi, 0, \gamma\varphi, 0, \gamma^2\varphi, \cdots.$$

Therefore the solution converges to zero when

$$\gamma < 1.$$

When

$$\gamma = 1,$$

the solution is the period-two sequence:

$$\cdots, 0, \varphi, 0, \varphi, 0, \varphi, \cdots$$

and when

$$\gamma > 1 \text{ and } \varphi > 0,$$

the solution is unbounded.

6.2. The case

$$x_{n+1} = \frac{(1 + \beta x_n) x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0,1, \dots . \#(6.2)$$

We consider the difference equation (6.2). See ([4], pp.202-204).

When one of the initial conditions of a solution of Eq. (6.2) is zero, Eq. (6.2)reduces to the linear equation

$$x_{n+1} = \frac{1}{A} x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution is φ, then the solution of the equation is

$$\cdots, 0, \varphi, 0, \frac{1}{A}\varphi, 0, \frac{1}{A^2}\varphi, \cdots.$$

So the solution converges to zero when

$$A > 1.$$

When

$$A = 1,$$

the solution is the (not necessarily prime) period-two sequence:

$$\cdots, 0, \varphi, 0, \varphi, \cdots$$

and when

$$A < 1 \text{ and } \varphi > 0,$$

the solution is unbounded.

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