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On Qi Type Integral Inequalities

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Abstract

We present a survey on Qi type integral inequalities.

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1. Introduction

Integral inequalities have been frequently employed in the theory of functional analysis, differential equations and applied sciences such as probability and statistics. There are a lot of types integral inequalities such as Hermite - Hadamard type inequalities, Opial type inequalities, Hardy type inequalities. Especially an integral inequality which is called Qi Inequality by mathematics community, has been studied by many authors in the last two decades. Qi Inequality was actually posed by Feng Qi in the preprint [1] and it was formally published in his paper version [2]. In recent years, it has been used extensively for solution of integral inequalities and in several of their applications.

The main aim of this present paper is to provide a survey of "Qi type integral inequalities". We examine a number of generalized and extended versions of Qi Inequality.

In the section 2, we present Qi's original problem and related problems which are generalizations and special forms of his problem.

In the section 3, we list systematically some articles which give affirmative answers to the problems mentioned in the section 2.

2. Qi's Open Problem and Related Problems

In [2], Qi proved the following interesting integral inequality result. He used analytic method and mathematical induction to prove integral inequalities.

Theorem 2.1 ([2], p.1). Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$.

a) If $0 \leq f'(x) \leq 1$ for $x \in (a,b)$ then

$$\int_a^b [f(x)]^3 dx \leq \left(\int_a^b f(x)dx\right)^2 . \#(2.1)$$

b) If $f'(x) \geq 1$, then inequality (2.1) reverses.

c) The equality in (2.1) is valid only if $f(x) \equiv 0$ or $f(x) = x - a$.

As a generalization of inequality (2.1), Qi also obtained the following more general result in [2].

Theorem 2.2 ([2], p.2). For $n \in \mathbb{N}$, assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$ and $f^{(n)}(x) \geq n!$ then

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x)\, dx\right)^{n+1}. \#(2.2)$$

Qi also proposed an open integral inequality problem at the end of [2] which was actually posed by himself in the preprint version [1].

Remark 2.3 ([2], p.3). (Problem 1) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)\, dx\right)^{t-1} \#(2.3)$$

hold for $t > 1$?

Since then, this Problem 1 (Remark 2.3) has been stimulating much interest of many mathematicians and affirmative answer to it has been established.

Remark 2.4. In [2], Qi phrased that "*the inequality (2.2) is not found in [3,4,5,6] and so maybe it is a new inequality*". In fact, it is a new result so has attracted many mathematicians research interest and many extensions, generalizations and applications of inequality (2.2) or (2.3) have been investigated in recent years and published articles devoted to answering those problems. For more detailed information, please see the references therein.

We propose same integral inequality (2.3) for the case $t < 1$ as open problem.

Remark 2.5. (Problem 2) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)\, dx\right)^{t-1} \#(2.4)$$

hold for $t < 1$?

We propose some integral inequality as open problem. The following are the reverses of Problem 1 (Remark 2.3) and Problem 2 (Remark 2.5), respectively.

Remark 2.6. (Problem 3) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)\, dx\right)^{t-1} \#(2.5)$$

hold for $t > 1$?

Remark 2.7. (Problem 4) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)\, dx\right)^{t-1} \#(2.6)$$

hold for $t < 1$?

The following two open problems are special cases of Problem 1 (Remark 2.3) and Problem 2 (Remark 2.6).

Remark 2.8. (Problem 5) Under what conditions does the inequality

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x)\, dx\right)^{n+1} \#(2.7)$$

hold for $n \in \mathbb{N}$?

Remark 2.9. (Problem 6) Under what conditions does the inequality

$$\int_a^b [f(x)]^{n+2} dx \leq \left(\int_a^b f(x)\, dx\right)^{n+1} \#(2.8)$$

hold for $n \in \mathbb{N}$?

In [7], L. Bougoffa posed the following problem (see Problem 2 in [7]) which is similar to Problem 1 (Remark 2.3), which is Qi's inequality. The following inequality which is called Bougoffa Inequality.

Remark 2.10. (Problem 7) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)\,dx\right)^{1-t} \#(2.9)$$

hold for $t < 1$?

The following inequality is the reverse of Bougoffa Inequality.

Remark 2.11. (Problem 8) Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)\,dx\right)^{1-t} \#(2.10)$$

hold for $t < 1$?

The following two open problems are extension of Problems 1, 5, 8 and 3, 6, 7 respectively.

Remark 2.12. (Problem 9) Under what conditions does the inequality

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f(x)\,dx\right)^{\beta} \#(2.11)$$

hold for α and β?

Remark 2.13. (Problem 10) Under what conditions does the inequality

$$\int_a^b [f(x)]^\alpha dx \leq \left(\int_a^b f(x)\,dx\right)^{\beta} \#(2.12)$$

hold for α and β?

We can also propose the following generalizations of above open problems.

Remark 2.14. (Problem 11) Under what conditions does the inequality

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f^\gamma(x)\,dx\right)^{\beta} \#(2.13)$$

hold for α, β and γ?

In the case of $\gamma = 1$, Problem 11 is equivalent to Problem 9.

Remark 2.15. (Problem 12) Under what conditions does the inequality

$$\int_a^b [f(x)]^\alpha dx \leq \left(\int_a^b f^\gamma(x)\,dx\right)^{\beta} \#(2.14)$$

hold for α, β and γ?

In the case of $\gamma = 1$, Problem 12 is equivalent to Problem 10.

Remark 2.16. (Problem 13) Under what conditions does the inequality

$$\int_a^b [f(x)]^{\alpha+\beta} dx \geq \left(\int_a^b g^\alpha(x) f^\beta(x)\,dx\right)^{\lambda} \#(2.15)$$

hold for α, β and λ?

In the case of $g = (x - a)$, Problem 13 is equivalent to Problem 17.

Remark 2.17. (Problem 14) Under what conditions does the inequality

$$\int_a^b [f(x)]^{\alpha+\beta} dx \leq \left(\int_a^b g^\alpha(x) f^\beta(x)\,dx\right)^{\lambda} \#(2.16)$$

hold for α, β and λ?

Remark 2.18. (Problem 15) Under what conditions does the inequality

$$\int_a^b [f(x)]^{\alpha+\beta}\,dx \geq \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x) g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\lambda-1} \#(2.17)$$

hold for α, β and λ?

Remark 2.19. (Problem 16) Under what conditions does the inequality

$$\int_a^b [f(x)]^{\alpha+\beta}\,dx \geq \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x) g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\gamma} \#(2.18)$$

hold for α, β, λ and γ?

In the case of $\lambda = 1$, Problem 16 is equivalent to Problem 13.

Lastly, W. Liu et al. proposed the following three open problems in the end of their paper [26].

Remark 2.20. (Problem 17) Under what conditions does the inequality

$$\int_a^b [f(x)]^{\alpha+\beta}\,dx \geq \left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)\,dx\right)^{\lambda} \#(2.19)$$

hold for α, β and λ?

Remark 2.21. (Problem 18) Under what conditions does the inequality

$$\frac{\int_a^b [f(x)]^{\alpha+\beta}\,dx}{\int_a^b [f(x)]^{\alpha+\gamma}\,dx} \geq \frac{\left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right)^{\delta}}{\left(\int_a^b (x-a)^{\alpha} f^{\gamma}(x)dx\right)^{\lambda}} \#(2.20)$$

hold for α, β, γ, δ and λ?

Remark 2.22. (Problem 19) Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$. Under what conditions does the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\left(\int_a^b \varphi\big(f(x)\big)\, g(x)dx\right)^{\delta}}{\left(\int_a^b \varphi\big(h(x)\big)\, g(x)dx\right)^{\lambda}} \#(2.21)$$

hold for δ and λ?

3. Papers on Qi Type Integral Inequalities from 2001 through 2016

In this section, we list systematically some papers on Qi type integral inequalities which give affirmative answers to Qi's open problem and related problems from 2001 through 2016.

Papers answering the Problem 1 (Remark 2.3):

- In 2001 (and 2007), work of F. Qi and K. W. Yu (see Theorem 1 in [9] and Theorem 1.1 in [10]).
- In 2001, work of N. Towghi (see Proposition 1 in [11]).
- In 2002, work of T. K. Pogány (see Corollary 2.2 in [12]).
- In 2003, work of M. Akkouchi.(see Theorem C in [13]).
- In 2004, work of J. Pečarić and T. Pejković (see Theorem 2 in [14]).
- In 2004, work of J. S. Sun (see Theorem 1 in [15]).
- In 2006, work of Y. Chen and J. Kimball (see Theorem 3 and Theorem 6 in [16]).
- In 2006, work of P. Yan and G. Gyllenberg (see Theorem 4 and Theorem 6 in [17]).
- In 2007, work of H. Yong (see Corollary 0.2 in [18]).

Paper answering the Problem 3 (Remark 2.6):

- In 2004, work of J. S. Sun (see Theorem 1 in [15]).
- In 2006, work of Y. Chen and J. Kimball (see Theorem 3 in [16]).

- In 2006, work of P. Yan and G. Gyllenberg (see Theorem 4 in [17]).

Paper answering the Problem 4 (Remark 2.7):

- In 2004, work of J. S. Sun (see Theorem 1 in [15]).

Paper answering the Problem 5 (Remark 2.8):

- In 2003, work of S. Mazouzi and F. Qi (see Corollary 3.6 in [19]).
- In 2004, work of J. Pečarić and T. Pejković (see Corollary 3 in [14]).
- In 2006, work of Y. Chen and J. Kimball (see Theorem 4 and Theorem 5 in [16]).
- In 2006, work of P. Yan and G. Gyllenberg (see Theorem 4 in [20]).
- In 2007, work of H. Yong (see Corollary 0.4 in [18]).
- In 2007, work of Q. A. Ngô and P. H. Tung (see Theorem 2.2 in [21]).
- In 2008, work of W. T. Sulaiman (see Theorem 2.8 in [22]).

Papers answering the Problem 6 (Remark 2.9):

- In 2006, work of Y. Chen and J. Kimball (see Theorem 4 in [16]).
- In 2006, work of P. Yan and G. Gyllenberg (see Theorem 4 in [20]).

Paper answering the Problem 7 (Remark 2.10):

- In 2005, work of L. Bougoffa (see Proposition 2 in [7]).
- In 2006, work of W. J. Liu et al. (see Theorem 2.1 in [8]).

Paper answering the Problem 8 (Remark 2.11):

- In 2006, work of W. J. Liu et al. (see Theorem 2.1 in [8]).

Papers answering the Problem 9 (Remark 2.12):

- In 2002, work of T. K. Pogány (see Theorem 2.1 and Theorem 4.1 in [12]).
- In 2003, work of S. Mazouzi and F. Qi (see Corollary 3.5 in [19]).
- In 2006, work of F. Qi et al. (see Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5 in [23]).
- In 2007, work of H. Yong (see Theorem 0.1 in [18]).

Paper answering the Problem 10 (Remark 2.13):

- In 2002, work of T. K. Pogány (see Theorem 3.1, Theorem 3.2 and Theorem 4.2 in [12]).
- In 2006, work of F. Qi et al. (see Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5 in [23]).

Paper answering the Problem 11 (Remark 2.14):

- In 2008, work of W. T. Sulaiman (see Theorem 2.5 in [22]).

Paper answering the Problem 12 (Remark 2.15):

- In 2004, work of J. Pečarić and T. Pejković (see Theorem 3.3 and Theorem 3.5 in [24]).
- In 2008, work of W. T. Sulaiman (see Theorem 2.5 in [22]).

Paper answering the Problem 13 (Remark 2.16):

- In 2010, work of X. Chai, Y. Zhao and H. Du (see Theorem 2.3 and Theorem 2.4 in [25]).

Paper answering the Problem 14 (Remark 2.17):

- In 2010, work of X. Chai, Y. Zhao and H. Du (see Theorem 2.3 in [25]).

Paper answering the Problem 15 (Remark 2.18):

- In 2009, work of W. Liu et al. (see Theorem 1 in [26]).

Paper answering the Problem 16 (Remark 2.19):

- In 2009, work of W. Liu et al. (see Theorem 2 in [26]).

Paper answering the Problem 17 (Remark 2.20):

- In 2016, work of A. Kashuri and R. Liko (see Theorem 2.1 in [27]).

Paper answering the Problem 19 (Remark 2.22):

- In 2016, work of A. Kashuri and R. Liko (see Theorem 2.2 in [27]).

We now present the main result of above listed papers in the following subsections.

3.1 Work of F. Qi and K. W. Yu Answering the Problem 1

Theorem 3.1 ([10], p.97). Assume that f is a continuous function on $[a,b]$. If

$$\int_a^b f(x)dx \geq (b-a)^{t-1} \text{ for some } t > 1,$$

then the inequality (2.3) is valid.

3.2 Work of N.Towghi Answering the Problem 1

Assume $f^{(0)}(x) = f(x)$, $f^{(-1)}(x) = \int_a^x f(s)ds$, and $[x]$ denote the greatest integer less than or equal to x. For $t \in (n, n+1]$, where n is a positive integer, suppose $\gamma(t) = t(t-1)(t-2)\cdots[t-(n-1)]$. For $t < 1$, let $\gamma(t) = 1$.

Theorem 3.2 ([11], p.1). Suppose $t > 1$, $x \in [a,b]$, and $f^{(i)}(a) \geq 0$ for $i \leq [t-2]$. If

$$f^{[t-2]}(x) \geq \gamma(t-1)(x-a)^{(t-[t])},$$

then

$$(b-a)^{t-1} \leq \int_a^b f(x)dx,$$

and the inequality (2.3) is valid.

3.3 Work of T. K. Pogány Answering the Problems 1, 9 and 10

Theorem 3.3 ([12], p.3). Assume that $\beta > 0$, $\max\{\beta, 1\} < \alpha$ and suppose that f^{α} is integrable on $[a,b]$. For

$$f(x) \geq (b-a)^{\frac{\beta-1}{\alpha-\beta}},$$

we get the inequality (2.11).

Corollary 3.4 ([12], p.3). For all $f(x) \geq (b-a)^{t-2}$, f^t integrable, the inequality (2.3) is valid for $t > 1$.

Theorem 3.5 ([12], p.3). Assume that f is nonnegative, concave and integrable on $[a,b]$, $\beta > 0$ and $\max\{\beta, 1\} < \alpha$. Assume

$$f(x) \leq \left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^{\alpha}(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}}\right)^{\frac{1}{\alpha-\beta}}, \quad x \in [a,b].$$

Then we obtain the inequality (2.12).

3.4 Work of M. Akkouchi Answering the Problem 1

We will present M. Akkouchi's result. Before stating the results, the following Definition 3.6 is needed.

Definition 3.6 ([13], p.122). Suppose that $[a,b]$ is a finite interval of the real line $\mathbb{R}$. For each real number r, we denote $\mathbb{E}_r(a,b)$ the set of real continuous functions f on $[a,b]$ differentiable on (a,b), such that $f(a) \geq 0$, and $f'(x) \geq r$ for all $x \in (a,b)$.

Theorem 3.6 ([13], p.124). Suppose that $[a,b]$ is a closed interval of $\mathbb{R}$. Suppose that $p \geq 1$ is a real number and assume $f \in \mathbb{E}_p(a,b)$. Then we get

$$\int_a^b [f(x)]^{p+2} dx \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(x) dx \right]^{p+1}.$$

3.5 Work of S. Mazouzi and F. Qi Answering the Problems 5 and 9

Corollary 3.7 ([19], p.4). Suppose $f \in \mathbb{L}^1(a,b)$, the space of integrable functions on the interval (a,b) with respect to the Lebesgue measure, such that $|f(x)| \geq k(x)$ a.e. for $x \in (a,b)$, where

$$(b-a)^{\frac{(p-1)}{(p-q)}} \leq \int_a^b k(x) dx < \infty$$

for some $p > q \geq 1$, thus

$$\left(\int_a^b |f(x)| dx \right)^q \leq \int_a^b |f(x)|^p dx.$$

Corollary 3.8 ([19], p.4). Assume that $f \in C^n([a,b])$ satisfies $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ for $x \in [a,b]$, where $0 \leq i \leq n-1$ and $n \in \mathbb{N}$, the set of all positive integers, then the inequality (2.7) holds.

3.6 Work of J. Pečarić and T. Pejković Answering the Problem 12

Theorem 3.9 ([24], p.6). Suppose $\alpha > 0$, $1 < \beta \leq 2$ and $\gamma \geq 2\alpha + 1$. The differentiable function $f: [a,b] \to \mathbb{R}$ satisfies $f(a) = 0$ and $0 \leq f'(x) \leq M$ for all $a \leq x \leq b$, where

$$0 < b - a \leq \left(\frac{\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma}}{\gamma - \alpha} \right)^{\frac{1}{\gamma - \alpha\beta - \beta + 1}}.$$

Then the inequality

$$\int_a^b [f(x)]^\gamma dx \leq \left(\int_a^b f^\alpha(x) dx \right)^\beta$$

holds.

3.7 Work of J. Pečarić and T. Pejković Answering the Problems 1 and 5

Theorem 3.10 ([14], p.2). Let $f \in C^1([a,b])$ satisfies$f(a) \geq 0$ and

$$f'(x) \geq (t-2)(x-a)^{t-3} \text{ for } x \in [a,b]$$

and $t \geq 3$. Then the inequality (2.3) holds.

The equality is valid only if $a = b$ or $f(x) = x - a$ and $t = 3$.

Corollary 3.11 ([14], p.2). Let $f \in C^1([a,b])$ satisfies $f(a) \geq 0$ and

$$f'(x) \geq n(x-a)^{n-1} \text{ for } x \in [a,b]$$

and a positive integer n, then the inequality (2.7) is valid.

3.8 Work of J. S. Sun Answering the Problems 1, 3 and 4

Theorem 3.12 ([15], p.1). Suppose that $f(x)$ is differentiable on (a,b) and $f(a) = 0$, we obtain

a) If $f'(x) \geq 0$ and $t \leq 1$, then we get inequality (2.6).

b) If $f'(x) \geq 0$ and $1 \leq t \leq 2$, then we get inequality (2.3).

c) If $0 \leq f^{'(x)} \leq (t-2)(x-a)^{t-3}$ and $2 \leq t \leq 3$, then we get inequality (2.5).

d) If $f'(x) \geq (t-2)(x-a)^{t-3}$ and $t \geq 3$, then we get inequality (2.3).

3.9 Work of L. Bougoffa Answering the Problem 7

Proposition 3.13 ([7], p.2). For a given positive integer $p \geq 2$, if $0 < m \leq f(x) \leq M$ on $[a,b]$ with $M \leq \frac{m^{(p-1)^2}}{(b-a)^p}$, then we get

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}.$$

3.10 Work of Y. Chen and J. Kimball Answering the Problems 1, 3, 5 and 6

Theorem 3.14 ([16], p.2). Suppose that p is a positive number and $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$. If

$$\left[f^{1/p}\right]'(x) \geq (p+1)^{\frac{1}{p}-1} \text{ for } x \in (a,b),$$

then

$$\int_a^b [f(x)]^{p+2} dx \geq \left[\int_a^b f(x)dx\right]^{p+1}. \#(3.1)$$

If

$$0 \leq \left[f^{1/p}\right]'(x) \leq (p+1)^{\frac{1}{p}-1}$$

for $x \in (a,b)$, then the inequality (3.1) reverses.

Theorem 3.15 ([16], p.2). Assume $f(x)$ has derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)=0$ for $i=0,1,2,\cdots,n-1$. If

$$f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$$

and $f^{(n)}(x)$ is increasing, then the inequality (2.7) holds. If

$$0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$$

and $f^{(n)}(x)$ is decreasing, then the inequality (2.7) reverses.

3.11 Work of F. Qi, A. J. Li, W. Z. Zhao, D. W. Niu and J. Cao Answering the Problems 9 and 10

Theorem 3.16 ([23], p.2). Suppose that $f(x)$ is continuous and not identically zero on $[a,b]$, differentiable in (a,b), with $f(a)=0$, and suppose that α, β are positive real numbers such that $\alpha > \beta > 1$. If

$$\left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \gtreqless \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \text{ for all } x \in (a,b),$$

then we get

$$\int_a^b [f(t)]^{\alpha} dt \gtreqless \left[\int_a^b f(t)dt\right]^{\beta}.$$

3.12 Work of W. J. Liu, C. C. Li and J. W. Dong Answering the Problems 7 and 8

Theorem 3.17 ([8], p.5). Suppose that $p>2$ is a positive number and $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$.If

$$[f^{p-2}]'(x) \geq p^p\,(p-2)/(p-1)^{p+1} \text{ for } x \in (a,b),$$

then

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}. \#(3.2)$$

If

$$0 \leq [f^{p-2}]'(x) \leq p^p\,(p-2)/(p-1)^{p+1} \text{ for } x \in (a,b),$$

then the inequality (3.2) reverses.

3.13 Work of P. Yan and M. Gyllenberg Answering the Problems 5 and 6

Theorem 3.18 ([20], p.4). Suppose $n \in \mathbb{Z}^+$. Assume $f(x)$ has derivative of the n-th order on the interval (a,b) and $f^{(n-1)}(x)$ is continuous on $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\cdots,n-1$.

a) If

$$f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}} \text{ for } x \in (a,b),$$

then the inequality (2.7) is valid.

b) If

$$0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}} \text{ for } x \in (a,b),$$

then the inequality (2.7) is reversed.

3.14 Work of P. Yan and M. Gyllenberg Answering the Problems 1 and 3

Theorem 3.19 ([17], p.2). Suppose that k is a non-negative integer and p is a positive number such that $p > k$. Assume that $f(x)$ has a derivative of the $(k+1)$-th order on the interval (a,b) such that $f^{(k)}(x)$ is continuous on $[a,b]$, $f(x)$ is non-negative on $[a,b]$ and $f^{(i)}(a) = 0$ for $i = 0,1,2,\cdots,k$.

a) If

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b),$$

Then

$$\int_a^b [f(x)]^{p+2} dx \geq \left[\int_a^b f(x)dx\right]^{p+1}. \#(3.3)$$

b) If

$$0 \leq \left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \leq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b),$$

then the inequality (3.3) is reversed.

3.15 Work of H. Yong Answering the Problem 9

Theorem 3.20 ([18], p.1244). Assume $\alpha > \beta \geq 2$, $m = [\beta]$, $f(x) \in C^1[a,b]$, $f'(x) \geq f(x) \geq 0$ and

$$\left[f^{\alpha-\beta}(x)\right]' \geq (\alpha-\beta)\frac{\beta(\beta-1)\cdots(\beta-m+1)}{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}(x-\alpha)^{\beta-m}.$$

Then the inequality (2.11) holds.

Where $[\beta]$ denote the integer part of β.

3.16 Work of Q. A. Ngô and P. H. Tung Answering the Problem 5

Theorem 3.21 ([21], p.3). Suppose that n is a positive integer. Assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $0 \leq i \leq n-1$, and

$$f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}},$$

then the inequality (2.7) is valid.

3.17 Work of W. T. Sulaiman Answering the Problems 5, 11 and 12

Theorem 3.22 ([22], p.891). Let f be positive and has continuous 2nd derivative on the interval $[a,b]$ such that $f(a) = 0$, $f'(a) = 0$, and suppose $\gamma > \alpha > 0$, $\beta > 1$, $\beta(\alpha+1) > (\gamma+1)$. If

$$\frac{f(t)f''(t)}{\left(f'(t)\right)^2} \geq \beta(\alpha+1)-(\gamma+1), \#(3.4)$$

then

$$\int_a^b f^{\gamma}(x)dx \geq \left(\int_a^b f^{\alpha}(x)dx\right)^{\beta}. \#(3.5)$$

If the inequality (3.4) reverses, then the inequality (3.5) reverses as well.

Theorem 3.23 ([22], p.893). Suppose that n is a positive integer. Assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)=0$, where $0 \leq i \leq n-1$. Suppose that α, β, γ are positive numbers such that $\alpha\beta > \gamma$. If

$$\left(f^{(n)}(x)\right)^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)n!}{(n\alpha+1)\beta}(b-a)^{\beta(n\alpha+1)-(n\gamma+1)},$$

then the inequality (3.5) holds true.

In particular, for $\gamma=n+2, \beta=n+1, \alpha=1$, the following inequality,

$$\int_a^b f^{n+2}(x)dx \geq \left(\int_a^b f(x)dx\right)^{n+1},$$

which is given as inequality (2.7) in Problem 5 (Remark 2.8), is valid.

3.18 Work of W. Liu, Q. A. Ngô and V. N. Huy Answering the Problems 15 and 16

Theorem 3.24 ([26], p.203). Suppose that $f(x), g(x) \geq 0$ are continuous functions on $[a,b]$, g is non decreasing. Suppose that λ, α, β are positive constants with $\alpha+\beta \geq \lambda > 1$. If

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a,b]$$

and

$$\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(t)dt \geq (b-a)^{\lambda-1},$$

then the inequality (2.17) is valid.

Theorem 3.25 ([26], p.204). Suppose that $f(x), g(x) \geq 0$ are continuous functions on $[a,b]$, g is non decreasing. Suppose that $\lambda, \gamma, \alpha, \beta$ are positive constant with $\alpha+\beta \geq \lambda > \gamma > 1$. If

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a,b]$$

and

$$\left(f^{\frac{\alpha+\beta}{\lambda}\cdot\frac{\lambda-\gamma}{\gamma-1}}(x)\right)' \geq \frac{(\lambda-\gamma)\gamma^{\frac{1}{\gamma-1}}}{\lambda-1}, \quad \forall x \in (a,b),$$

then the inequality (2.18) is valid.

3.19 Work of X. Chai, Y. Zhao and H. Du Answering the Problem 13 and 14

Theorem 3.26 ([25], p.1815). Let $f(x), g(x) > 0$ be continuous functions on $[a,b]$.

a) Assume that $\lambda \geq 1$, $\alpha \neq 0$, $\beta \in \mathbb{R}$ and

$$g^{\lambda}(x) \leq \left(\int_a^b f^{\beta}(x)dx\right)^{(1-\lambda)/\alpha} f(x) \text{for all} \quad x \in [a,b], \#(3.6)$$

then we can get the inequality (2.15).

b) Assume that $\lambda < 0$, $\alpha \neq 0$, $\beta \in \mathbb{R}$ and (3.6), then the inequality (2.15) holds.

c) Assume that $0 < \lambda \leq 1$, $\alpha \neq 0$, $\beta \in \mathbb{R}$ and

$$g^{\lambda}(x) \geq \left(\int_a^b f^{\beta}(x)dx\right)^{(1-\lambda)/\alpha} f(x) \text{for all} \quad x \in [a,b],$$

then the inequality in (2.15) reverses hence the inequality (2.16) holds.

3.20 Work of A. Kashuri and R. Liko Answering the Problems 17 and 19

Theorem 3.28 ([27], p.878). Suppose that $f(x) \geq 0$ is a continuous function on $[a,b]$ satisfying

$$\int_x^b (t-a)^{min\{1,\beta\}} dt \leq \int_x^b f^{min\{1,\beta\}}(t)dt, \forall x \in [a,b].$$

Then for all $\lambda \geq 1$ the inequality (2.19) is valid under each of the following conditions:

a) For all $\beta > 1$ and $\alpha > 0$ such that

$$\frac{(b-a)^{\alpha+2}}{\alpha+2} \leq 1.$$

b) For $\beta \in (0,1]$ and $\alpha > 0$ such that

$$\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \leq 1.$$

Theorem 3.29 ([27], p.879). Assume that $f(x), g(x), h(x) > 0$ are continuous functions on $[a,b]$ with $f(x) \leq h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x), g(x)$ are increasing. Suppose that $\varphi(x)$ is positive and convex function with $\varphi(0) = 0$.

Then the inequality (2.21) is valid under each of the following conditions:

a) $\lambda = \delta = 0$ and $f(x) = h(x)$, for all $x \in [a,b]$;

b) $\lambda = \delta \in [1,+\infty)$, for all $x \in [a,b]$;

c) $\varphi(f(a)) \geq \frac{1}{(b-a)g(a)}$ for $1 \leq \delta < \lambda$;

d) $\varphi(f(b)) \leq \frac{1}{(b-a)g(a)}$ for $1 \leq \lambda < \delta$.

4. Future Work

We plan to publish a paper on a review of applications of Qi type integral inequalities, see [28-59].

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