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On The Quotient Function Of Entire Function Represented By Dirichlet Series Of Two Complex Variables

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Abstract.

The object of this paper is study a few result involving the maximum term and the rank of the maximum term of entire function represented by Dirichlet series**.**

1- Introduction

Let E be the set of all entire functions $f(s_1, s_2)$ represented by a double entire Dirichlet series,where $f(s_1, s_2)$ can be written as[4] :-

$$f(s_1, s_2) = \sum_{i,j=0}^{\infty} a_{i,j}\, e^{\lambda_i s_1 + \mu_j s_2} \qquad (1.1)$$

where $a_{i,j} \in C, (s_1, s_2) \in C^2$, and $0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_i \to \infty, 0 \le \mu_0 < \mu_1 < \cdots < \mu_j \to \infty$, and

$$\lim_{i\to\infty} \frac{\log i}{\lambda_i} = 0 = \lim_{i\to\infty} \frac{\log j}{\mu_j} \qquad (1.2)$$

Let[1],

$$M(\sigma_1, \sigma_2, f) = \sup_{-\infty<t_j<\infty}\{|f(\sigma_1 + it_1, \sigma_2 + it_2)|\}, (\sigma_1, \sigma_2) \in R^2 \qquad (1.3)$$

be the maximum modulus of $f(s_1, s_2)$ on the tube Re $s_1 = \sigma_1$, Re $s_2 = \sigma_2$.

and let[1],

$$U(\sigma_1, \sigma_2, f) = \max_{i,j\ge 0}\{|a_{i,j}|e^{\lambda_i\sigma_1 + \mu_j\sigma_2}\}, (\sigma_1, \sigma_2) \in R^2 \qquad (1.4)$$

also if $f(s_1, s_2)$be entire Dirichlet series defined by (1.1), then the order P and lower order $\lambda$of $f(s_1, s_2)$can be defined as[4]:

$$P = \lim_{\sigma_1,\sigma_2\to\infty} \sup \frac{\log\log M\ (\sigma_1,\sigma_2,f)}{\log(e^{\sigma_1}+e^{\sigma_2})} \quad (1.5)$$

$$\lambda = \lim_{\sigma_1,\sigma_2\to\infty} \inf \frac{\log\log M(\sigma_1,\sigma_2,f)}{\log(e^{\sigma_1}+e^{\sigma_2})} \quad (1.6)$$

In this paper the definitions of the rank and quotient function are extended to several complex variables. In the last section of this paper some remarks and theorems are given, where the characterizations of the order and lower order in terms of the rank and quotient function are studied.

2- Some important definitions

In this section two definitions, which are generalized of the definition of the rank and quotient function[2], are given.

Definition (2.1) :- Let $f(s_1,s_2)$be entire Dirichlet series defined by (1.1), and let $U(\sigma_1,\sigma_2)$ be the maximum term of $f(s_1,s_2)$, the rank of the maximum term can be defined as:-

$$V(\sigma_1,\sigma_2,f) = \max_{i,j\in N}\left\{\left(\lambda_i,\mu_j\right) : U(\sigma_1,\sigma_2) = \left|a_{i,j}\right| e^{\lambda_i\sigma_1+\mu_j\sigma_2}\right\} \quad (2.1)$$

Where Ndenote the set of all nature numbers.

Definition (2.2):- Let $f(s_1,s_2)$ be entire Dirichlet series defined by (1.1), and let $P \in Z_+$, (Z_+ be the set of all positive integers),then for every entire function$f(s_1,s_2)$, there exist the quotient function of the P^{th} order A_P, where A_P can be defined as:-

$$A_P(\sigma_1,\sigma_2,f) = \frac{U_P(\sigma_1,\sigma_2,f^{(P)})}{U(\sigma_1,\sigma_2,f)}, (\sigma_1,\sigma_2) \in R^2 \quad (2.2)$$

3-Theorems and Remarks

In this section the following theorems and remarks are given.

Theorem(3.1): For every entire function $f(s_1,s_2) \in E$ that,

$$\lim_{\sigma_1,\sigma_2\to\infty} \sup \frac{(A_P(\sigma_1,\sigma_2,f))^{\frac{1}{P}}}{\lambda_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2}+\mu_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2}} \le 1 \le \lim_{\sigma_1,\sigma_2\to\infty} \inf \frac{(A_P(\sigma_1,\sigma_2,f))^{\frac{1}{P}}}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} \quad (3.1)$$

where $|V(\sigma_1,\sigma_2,f)|^2 = {\lambda_i}^2 + {\mu_j}^2$

Proof

Now from [6] that, for any $P \in Z_+$,

$$\lambda_{|V(\sigma_1,\sigma_2,,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,,f)|^2} \le \left(\frac{U_P(\sigma_1,\sigma_2,f^{(P)})}{U(\sigma_1,\sigma_2,f)}\right)^{\frac{1}{P}} \le \lambda_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2}+\mu_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2} \quad (3.2)$$

dividing both side of the first inequality in (3.2) by $\lambda_{|V(\sigma_1,\sigma_2,,f)|^2+}\mu_{|V(\sigma_1,\sigma_2,f)|^2}$, and proceeding to limits, to get

$$\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{(A_P(\sigma_1,\sigma_2,f))^{\frac{1}{P}}}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2+}\mu_{|V(\sigma_1,\sigma_2,f)|^2}}\geq 1, \tag{3.3}$$

and dividing both side of the second inequality in (3.2) by $\lambda_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2+}\mu_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2}$

to get,

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{(A_P(\sigma_1,\sigma_2,f))^{\frac{1}{P}}}{\lambda_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2+}\mu_{\left|V(\sigma_1,\sigma_2,f^{(P)})\right|^2}}\leq 1 \tag{3.4}$$

Combining (3.3) and (3.4) to get (3.1).

Remark1:- If $f(s_1,s_2)$ of order $P\in R_+^*\cup\{0\}$,(R_+^* is the set of extended positive real numbers), and lower order $\lambda\in R_+^*\cup\{0\}$, it follows from (3.2) and the following result [3]

$$P=\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log\log M(\sigma_1,\sigma_2,f)}{\log(e^{\sigma_1}+e^{\sigma_2})}=\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log(\lambda_{|V(\sigma_1,\sigma_2,f)|^2+}\mu_{|V(\sigma_1,\sigma_2,f)|^2})}{\log(e^{\sigma_1}+e^{\sigma_2})}$$

and,

$$\lambda=\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log\log M(\sigma_1,\sigma_2,f)}{\log(e^{\sigma_1}+e^{\sigma_2})}=\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log(\lambda_{|V(\sigma_1,\sigma_2,f)|^2+}\mu_{|V(\sigma_1,\sigma_2,f)|^2})}{\log(e^{\sigma_1}+e^{\sigma_2})}$$

that,

$$P=\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log\left(\frac{U_P(\sigma_1,\sigma_2,f^{(P)})}{U(\sigma_1,\sigma_2,f)}\right)^{\frac{1}{P}}}{\log(e^{\sigma_1}+e^{\sigma_2})} \tag{3.5}$$

and

$$\lambda=\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log\left(\frac{U_P(\sigma_1,\sigma_2,f^{(P)})}{U(\sigma_1,\sigma_2,f)}\right)^{\frac{1}{P}}}{\log(e^{\sigma_1}+e^{\sigma_2})} \tag{3.6}$$

Remark 2: If $f(s_1,s_2)$ of order $P\in R_+^*\cup\{0\}$,(R_+^* is the set of extended positive real numbers) , and lower order $\lambda\in R_+^*\cup\{0\}$, it follows from the result of [5] , that

$$\lim_{\sigma_1,\sigma_2\to\infty} \inf \frac{\log U(\sigma_1,\sigma_2,f)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} \le \frac{1}{P}$$
$$\le \frac{1}{\lambda} \le \lim_{\sigma_1,\sigma_2\to\infty} \sup \frac{\log U(\sigma_1,\sigma_2,f)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} \tag{3.7}$$

Remark 3: For every entire function $f(s_1,s_2) \in E$,it follows from the results of([2] ,[6]) that

$$U(\sigma_1,\sigma_2,f) \le U_1\left(\sigma_1,\sigma_2,f^{(1)}\right) \le \cdots \le U_P\left(\sigma_1,\sigma_2,f^{(P)}\right) \le \cdots \tag{3.8}$$

Next the following theorem is improve of Remark2.

Theorem(3.2): For every entire function $f(s_1,s_2) \in E$ of order $P \in R_+^* \cup \{0\}$,(R_+^* is the set of extended positive real numbers) , and lower order $\lambda \in R_+^* \cup \{0\}$ and for any $P \in N$, that

$$\lim_{\sigma_1,\sigma_2\to\infty} \inf \frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} \le \frac{1}{P} \le$$
$$\frac{1}{\lambda} \le \lim_{\sigma_1,\sigma_2\to\infty} \sup \frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} \tag{3.9}$$

Proof

Now from (3.2) , that

$$\log\{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,,f)|^2}\} \le \frac{1}{P}\{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right) - \log U(\sigma_1,\sigma_2,f)\}$$
$$\le \log\left\{\lambda_{\left|V\left(\sigma_1,\sigma_2,f^{(P)}\right)\right|^2}+\mu_{\left|V\left(\sigma_1,\sigma_2,f^{(P)}\right)\right|^2}\right\} \tag{3.10}$$

from the first inequality in (3.10),it follows

$$P\left(\lim_{\sigma_1,\sigma_2\to\infty} \frac{\log\left\{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,,f)|^2}\right\}}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}\right)$$
$$\le \lim_{\sigma_1,\sigma_2\to\infty} \frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} - \lim_{\sigma_1,\sigma_2\to\infty} \frac{\log U(\sigma_1,\sigma_2,f)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}$$
$$\le \lim_{\sigma_1,\sigma_2\to\infty} \frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}} - \frac{1}{\lambda} \tag{3.11}$$

in view of (3.7). Since $\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}$ tends to infinity with (σ_1,σ_2),it follows

from (3.11), that

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}\geq\frac{1}{\lambda} \tag{3.12}$$

Also, from the second inequality in (3.10), to get

$$P\left(\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}\right)$$

$$\geq\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}-\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log U(\sigma_1,\sigma_2,f)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}$$

$$\geq\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}-P \tag{3.13}$$

in view of (3.7). Since, from remark 1,

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\operatorname{Log}\left\{\lambda_{\left|V\left(\sigma_1,\sigma_2,f^{(P)}\right)\right|}+\mu_{\left|V\left(\sigma_1,\sigma_2,f^{(P)}\right)\right|}\right\}}{\log(e^{\sigma_1}+e^{\sigma_2})}=\frac{1}{P}$$

it follows, from (3.13), that

$$\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}\leq\frac{1}{P} \tag{3.14}$$

Combining (3.12) and (2.14), to get (3.9)

Now the corollaries are immediate from (3.9):-

Corollary 1 :- Iff is of infinite order, then

$$\lim_{\sigma_1,\sigma_2\to\infty}\inf\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}=0. \tag{3.15}$$

Corollary 2 :- If f is of lower order zero, then

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log U_P\left(\sigma_1,\sigma_2,f^{(P)}\right)}{\lambda_{|V(\sigma_1,\sigma_2,f)|^2}+\mu_{|V(\sigma_1,\sigma_2,f)|^2}}=+\infty. \tag{3.16}$$

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