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Pathological Properties of Weak L^p Spaces

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Abstract

Weak L^p spaces, that are shown through this paper by L_w^p, are function spaces that are closed to L^p spaces, but somehow larger. The question that we are going to partially answer in this paper, is that how much it is larger. Actually we prove that $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$ contains an infinitely generated algebra.

Keywords: Algebrability; Lineability; Pathological Properties; Weak L^p.

1. Introduction

Weak L^p spaces are function spaces that are closely related to L^p spaces. We do not know the exact origin of Weak L^p spaces. The Book by Colin Benett and Robert Sharpley [1] contains a good presentation of Weak L^p but from the point of view of rearrangement function. In the present paper we first study the Weak L^p space from the point of view of distribution function. This circumstance motivated us to undertake a preparation of the present paper containing a detailed exposition of these function spaces. Then we proceed to the main theorem that proves the existence of an infinitely generated vector space in $Weak\ L^p[0,1] \setminus L^p[0,1] \cup \{0\}$. A subset $M$of a vector space $X$is called lineable in X if $M \cup \{0\}$ contains an infinite dimensional vector space. In this setting, authors in [2]have proved that $Weak\ L^p[0,1] \setminus L^p[0,1]$ is lineable in $Weak\ L^p[0,1]$. What we are going to prove here is that $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$ contains an infinitely generated algebra. This last property is called Algebrability.

The origin of lineability is due to Gurariy([5, 6])that showed that there exists an infinite dimensional linear space such that everynon-zero element of which is a continuous nowhere differentiable function on$\mathcal{C}[0;1]$. Many examples of vector spaces of functions on $\mathbb{R}$or $\mathbb{C}$enjoying certain special properties have been constructed in the recent years. More recently, many authors got interested in this subject and gave a wide range of examples. For more results on lineability we refer the reader to [3].

Throughout this paper $(X, \Omega, \mu)$is a measure space and $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

1. 1 Definition. For $f: X \to \mathbb{F}$ a measurable function on X, the distribution functionof f is the function D_f defined on $[0, \infty)$ as follows

$$D_f(\lambda) \coloneqq \mu(\{x \in X: |f(x)| > \lambda\}). \quad (1)$$

The distribution function D_f provides information about the size of f butnot about the behavior of f itself near any given point. For instance, a functionon $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from 1. 1 that D_f is a decreasing function of λ(not necessarily strictly).

Let (X, Ω, μ) be a measurable space and f and$g$ be a measurable functions on$X$ then D_f enjoys the following properties.

(1)$|g| \leq |f|\mu$- a. e. implies that $D_g \leq D_f$;

(2)$D_{cf}(\lambda) = D_f\left(\frac{\lambda}{|c|}\right)$ for all $c \in \mathbb{C} \setminus \{0\}$ and $\lambda \in [0, \infty)$;

(3)$D_{f+g}(\lambda_1 + \lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$ for all $\lambda_1, \lambda_2 \in [0, \infty)$;

(4)$D_{fg}(\lambda_1 \lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$ for all $\lambda_1, \lambda_2 \in [0, \infty)$;

For more details on distribution function see [4]and[7].

Next, Let (X, Ω, μ) be a measurable space, for $0 < p < \infty$, we consider

$$Weak\ L^p := \left\{f: X \to \mathbb{F}: \exists c > 0\ \forall \lambda > 0, D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p\right\}.$$

We will use the notation L_w^p to show $Weak\ L^p$. Observe that $L_w^\infty = L^\infty$.

1. 2 Proposition. Let $f \in L_w^p$with $0 < p < \infty$. Then

$$\|f\|_{L_w^p} := \inf\left\{c > 0: D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p for\ all\ \lambda \in (0, \infty)\right\} = \left(\sup_{\lambda > 0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}} = \sup_{\lambda > 0} \lambda \{D_f(\lambda)\}^{\frac{1}{p}}.$$

Proof. Let us define

$$\alpha = \left\{c > 0: D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p for\ all\ \lambda \in (0, \infty)\right\},$$

and

$$\beta = \left(\sup_{\lambda > 0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}}.$$

Since $f \in L_w^p$, so there exists $c > 0$ such that

$$D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p, \lambda \in (0, \infty).$$

Therefore

$$\left\{c > 0: D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p for\ all\ \lambda \in (0, \infty)\right\} \neq \emptyset.$$

On the other hand

$$\lambda^p D_f(\lambda) \leq \beta^p,$$

thus$\left\{\lambda^p D_f(\lambda): \lambda > 0\right\}$ is bounded above by β^p and $\beta \in \mathbb{R}$. Therefore

(2) $$\alpha = \left\{c > 0 : D_f(\lambda) \le \left(\frac{c}{\lambda}\right)^p \; for\ all\ \lambda \in (0, \infty)\right\} \le \beta.$$

Now, let $\varepsilon > 0$, then there exists $c > 0$ such that$\alpha \le c \le \alpha + \varepsilon$and

$$D_f(\lambda) \le \left(\frac{c}{\lambda}\right)^p \; for\ all\ \lambda \in (0, \infty),$$

and thus

$$\lambda^p D_f(\lambda) \le c^p \le (\alpha + \varepsilon)^p, \lambda \in (0, \infty).$$

Then

$$\sup_{\lambda > 0} \lambda^p D_f(\lambda) \le (\alpha + \varepsilon)^p.$$

Therefore

$$\left(\sup_{\lambda > 0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}} \le \alpha + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary so$\beta \le \alpha$. This completes the proof.☐

By this norm, we can redefine L_w^p spaces in the form of L^p spaces.

1. 3 Definition. Let (X, Ω, μ)be a measure space. For $0 < p < \infty$ the space L_w^p is defined as the set of all μ-measurable $\mathbb{F}$-valued functions f such that

$$\inf\left\{c > 0 : D_f(\lambda) \le \left(\frac{c}{\lambda}\right)^p \; for\ all\ \lambda \in (0, \infty)\right\} < \infty.$$

Two functions in L_w^p will be considered equal if they are equal μ-a. e.

For $0 < p < \infty$, L_w^p is larger than L^p. We obtain this result in the next proposition and remark.

1. 4 Proposition. For any $0 < p < \infty$, $L^p \subset L_w^p$ and for any $f \in L^p$we have

$$\|f\|_{L_w^p} \le \|f\|_{L^p}.$$

Proof. If $f \in L^p$, then for any $\lambda > 0, D_f$ we have

$$\lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \le \int_{|f(x)| > \lambda} |f|^p d\mu \le \int_X |f|^p d\mu = \|f\|_{L^p}.$$

Therefore

(3) $$\mu(\{x \in X : |f(x)| > \lambda\}) \le \left(\frac{\|f\|_{L^p}}{\lambda}\right)^p.$$

Next from 3 we have

$$\left(\sup_{\lambda > 0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}} \le \|f\|_{L^p}.$$

This shows that

$$\|f\|_{L_w^p} \leq \|f\|_{L^p}.$$

□

1. 5 Remark. The inclusion in the previous proposition is strict. Indeed let $f(x) = x^{-\frac{1}{p}}$on $(0, \infty)$ with the Lebesgue measure. For any $\lambda > 0$ we have

$$m\left(\left\{x \in (0, \infty): \frac{1}{|x|^{\frac{1}{p}}} > \lambda\right\}\right) = m\left(\left\{x \in (0, \infty): |x| < \frac{1}{\lambda^p}\right\}\right) = 2\lambda^{-p}.$$

Thus $f \in L_w^p(0, \infty)$ but

$$\int_0^\infty \left(\frac{1}{x^{\frac{1}{p}}}\right)^p dx = \int_0^\infty \frac{1}{x} dx \to \infty.$$

So $f \notin L^p$.

2. Main Result

In this section we prove that $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$ contains an infinitely generated algebra. First we present the definition of algebrability and two facts about norm of L_w^p.

2. 1 Definition. Let Abe a subset of an algebra L. A is called algebrable in L if $A \cup \{0\}$ contains an infinitely generated subalgebra W of L.

2. 2 Proposition. Let$f, g \in L_w^p$. Then

(1)$\|cf\|_{L_w^p} = |c|\|f\|_{L_w^p}$, for any constant$c$,

(2)$\|f + g\|_{L_w^p} \leq 2\left(\|f\|_{L_w^p}^p + \|g\|_{L_w^p}^p\right)^{\frac{1}{p}}$.

Proof.(1) For $c > 0$we have

$$\mu(\{x \in X: |cf(x)| > \lambda\}) = \mu\left(\{x \in X: |f(x)| > \frac{\lambda}{c}\}\right),$$

thus

$$D_{cf}(\lambda) = D_f\left(\frac{\lambda}{c}\right).$$

Therefore

$$\|cf\|_{L_w^p} = \left(\sup_{\lambda>0} \lambda^p D_{cf}(\lambda)\right)^{\frac{1}{p}} = \left(\sup_{\lambda>0} \lambda^p D_f\left(\frac{\lambda}{c}\right)\right)^{\frac{1}{p}} = \left(\sup_{cw>0} c^p w^p D_f(w)\right)^{\frac{1}{p}}$$
$$= c\left(\sup_{cw>0} w^p D_f(w)\right)^{\frac{1}{p}}.$$

So

$$\|cf\|_{L_w^p} = c\|f\|_{L_w^p}.$$

(2) Note that

$$\{x \in X: |f(x)+g(x)| > \lambda\} \subseteq \left\{x \in X: |f(x)| > \frac{\lambda}{2}\right\} \cup \left\{x \in X: |g(x)| > \frac{\lambda}{2}\right\}.$$

Hence

$$\mu\{x \in X: |f(x)+g(x)| > \lambda\} \leq \mu\left(\left\{x \in X: |f(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x \in X: |f(x)| > \frac{\lambda}{2}\right\}\right).$$

Thus

$$\lambda^p D_{f+g}(\lambda) \leq \lambda^p D_f\left(\frac{\lambda}{2}\right) + \lambda^p D_g\left(\frac{\lambda}{2}\right) \leq 2^p\left[\sup_{\lambda>0} \lambda^p D_f(\lambda) + \sup_{\lambda>0} \lambda^p D_g(\lambda)\right]$$
$$= 2^p\left[\|f\|_{L_w^p}^p + \|g\|_{L_w^p}^p\right].$$

Therefore

$$\|f+g\|_{L_w^p} = \left(\sup_{\lambda>0} \lambda^p D_{f+g}(\lambda)\right)^{\frac{1}{p}} \leq 2^p\left[\|f\|_{L_w^p}^p + \|g\|_{L_w^p}^p\right]^{\frac{1}{p}}.$$

□

The previous proposition shows that L_w^p is a vector space and that $\|.\|_{L_w^p}$ is a quasi-norm on it (which is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $c \geq 1$, that is $\|f+g\| \leq c(\|f\| + \|g\|)$.)

To proceed to the main theorem, we need the following lemma.

2. 3 Lemma. If $I \subseteq \mathbb{R}$has finite Lebesgue measure, thenχ_I, the characteristic function of$I$, belongs to L_w^p for all $0 < p < \infty$.

Proof. For all $\lambda > 0$we have

$$m(\{x \in \mathbb{R}: |\chi_I(x)| > \lambda\}) = \begin{cases} 0, & \lambda > 1 \\ m(I), & 0 < \lambda \leq 1 \end{cases}.$$

Thus

$$\|\chi_I\|_{L_w^p} = \left(\sup_{\lambda>0} \lambda^p D_{\chi_I}(\lambda)\right)^{\frac{1}{p}} \leq (m(I))^{\frac{1}{p}}.$$

So $\chi_I \in L_w^p$.□

2. 4 Theorem. $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R})$is algebrable in $L_w^p(\mathbb{R})$, for all $0 < p < \infty$.

Proof. Let $0 < p < \infty$ be fixed. Define $f: \mathbb{R} \to \mathbb{R}$as

$$f(x) = \begin{cases} |x|^{-\frac{1}{p}}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases},$$

For each $n \in \mathbb{N}$let $g_n = \chi_{[0,\frac{1}{n}]}$. Let $n \in \mathbb{N}$ be fixed. By the argument after Proposition 2. 2, $(f.g_n)^m \in L_w^p(\mathbb{R})$ for all $n, m \in \mathbb{N}$.

On the other hand by remark 1. 5, $(f.g_n)^m \notin L^p(\mathbb{R})$. Since for each $i, j \in \mathbb{N}$with $i < j$, $\mathrm{m}\left(\left[0,\frac{1}{i}\right] \setminus \left[0,\frac{1}{j}\right]\right) \neq 0$, so for each $i, j \in \mathbb{N}$ with $i < j$, $f.g_j \neq f.g_i$.

Therefore $\{(f.g_n)^m : m,n \in \mathbb{N}\}$ is an infinite set. Again by the argument after Proposition 2.2 and above reasoning, the algebra generated by $\{(f.g_n)^m : m,n \in \mathbb{N}\}$ is a subset of $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$. So $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R})$ is algebrable.☐

2. 5 Remark. This trend of research seems to be very broad. One can ask about the cardinality of the generators of the algebra contained in $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$. Another question that can be considered is that whether there exists an infinitely generated free algebra of functions contained in $L_w^p(\mathbb{R}) \setminus L^p(\mathbb{R}) \cup \{0\}$ or not. Also changing the space $\mathbb{R}$ by a general measure space would contribute to new results.

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