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On Approximation Properties of Multivariate Class of Nonlinear Singular Integral Operators

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Abstract. In the present paper, we study the pointwise approximation of nonlinear multivariate singular integral operators having convolution type kernels of the form:

$$T_\lambda(f;x) = \int_D K_\lambda(t-x, f(t))dt\,, x \in D\,, \lambda \in \Lambda\,,$$

where $D = \prod_{i=1}^{n} \langle a_i, b_i \rangle$ is open, semi-open or closed multidimensional arbitrary bounded box in $\mathbb{R}^n$ or $D = \mathbb{R}^n$ and Λ is non-empty the set of non-negative indices, at a μ-generalized Lebesgue point of $f \in L_P(D)$. Also, we investigate the corresponding rates of convergences at this point.

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1. Introduction

In [15], Taberski handled the problem of pointwise approximation of functions $f \in L_1\langle -\pi, \pi \rangle$ and their derivatives by harnessing the family of convolution type linear singular integral operators depending on two parameters of the form:

$$(1.1) \qquad L_\lambda(f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda(t-x)dt\,, x \in \langle -\pi, \pi \rangle, \qquad \lambda \in \Omega \subset \mathbb{R}_0^+,$$

where the symbol $\langle -\pi, \pi \rangle$ stands for closed, semi-closed or open interval and $K_\lambda(t)$ is the kernel enriched with special properties. The pointwise convergence of the operators of type (1.1) was later examined by Gadjiev [5] and Rydzewska [11] at generalized Lebesgue points and $\mu-$ generalized Lebesgue points of functions which belong to $L_1\langle -\pi, \pi \rangle$ respectively. Then, in [8], Karsli and Ibikli extended the findings of [15, 5] and [11] by studying the convergence of the operators of type (1.1) in the space $L_1\langle a, b \rangle$. For some further studies of linear singular operators in many different settings, the reader may see also, e.g., [12]-[17] and [4].

Approximately four decades ago, Musielak [9] dealt with finding the conditions under which the following nonlinear integral operators of the form:

$$(1.2) \qquad T_w f(y) = \int_G K_w(x-y, f(x))dx, \;\; y \in G, \;\; w \in \Lambda,$$

where G is a locally compact Abelian group with Haar measure and $\Lambda \neq \emptyset$ is an index set with some topology, became convergent. He replaced the linearity property of the operators by an assumption of Lipschitz condition for K_w with

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respect to second variable. Later on, Swiderski and Wachnicki [14] investigated the pointwise convergence of the operators of type (1.2) at Lebesgue points of the functions on different Abelian groups. For further reading, we refer the reader to [1, 2, 7].

As a continuation and generalization of [17], this work presents some pointwise approximation theorems for nonlinear multivariate singular integral operators at a μ-generalized Lebesgue point of the functions $f \in L_p(D)$ (see also, [5]) and the rate of pointwise convergence of these operators in the following form:

$$T_\lambda(f;x) = \int_D K_\lambda(t-x, f(t))\,dt, \quad x \in D,\ \lambda \in \Lambda, \tag{1.3}$$

where $D = \prod_{i=1}^{n} \langle a_i, b_i \rangle$ is open, semi-open or closed arbitrary bounded box in $\mathbb{R}^n$ or $D = \mathbb{R}^n$, $\Lambda \neq \emptyset$ is the set of non-negative indices with accumulation point λ_0 or $\lambda_0 = \infty$.

The paper is organized as follows: In Section 2, we introduce the fundamental definitions. In Section 3, we present two theorems concerning the pointwise convergence of $T_\lambda(f;x)$ whenever x is a μ-generalized Lebesgue point of the function $f \in L_p(D)$. In Section 4, we establish the rate of pointwise convergence of operators of type (1.3). In Section 5, we give conclusion.

2. Preliminaries

Definition 2.1. [6, 12] Denoting unit sphere by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, the polar coordinates transformation on $\mathbb{R}^n$ is given by $G : \mathbb{R}^n \to \mathbb{R}^n$, $G(r, \theta_1, ..., \theta_{n-1}) = (x_1, .., x_n)$, where

$$\begin{aligned} x_1 &= r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2, \quad x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, ..., \\ x_k &= r\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k, ..., x_n = r\sin\theta_1 ... \sin\theta_{n-2}\sin\theta_{n-1}. \end{aligned} \tag{2.1}$$

Here, $k = 2, ..., n-1$, $0 \le \theta_k \le \pi$, $0 \le \theta_{n-1} \le 2\pi$, $r = |x| \neq 0$, $x' = \frac{x}{r} \in S^{n-1}$ and the Jacobian of the transformation is $J = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}...(\sin\theta_{n-2})$.

Definition 2.2. [3] A function $\Phi \in L_1(\mathbb{R}^n)$, is said to be radial, if there exists a function $\Psi(|t|)$, defined on $0 \le |t| < \infty$ such that $\Phi(t) = \Psi(|t|)$ almost everywhere.

Definition 2.3.($Class\ A$) Let us suppose that the function $K_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is integrable with respect to second variable with $K_\lambda(.,0) = 0$ $\forall t \in \mathbb{R}^n$ and satisfies the following assumptions:

$a)$ Let $L_\lambda(|t|)$ be a radial integrable function on $\mathbb{R}^n$ as a function of t for each $\lambda \in \Lambda$ such that the following inequality holds:

$$|K_\lambda(t,u) - K_\lambda(t,v)| \le L_\lambda(|t|)\,|u-v|\,, \forall t \in \mathbb{R}^n, \forall u, v \in \mathbb{R} \text{ and for each } \lambda \in \Lambda.$$

$b)$ $\|L_\lambda\|_{L_1(\mathbb{R}^n)} \le M < \infty, \quad \forall \lambda \in \Lambda$, where M is positive real number.

$c)$ $\lim\limits_{\lambda \to \lambda_0} \left| \int_{\mathbb{R}^n} K_\lambda(t,u)dt = u \right|, \forall u \in \mathbb{R}.$

$d)$ $\lim\limits_{\lambda \to \lambda_0} \left[\sup\limits_{\xi \le |t| < \infty} L_\lambda(|t|) \right] = 0, \quad \forall \xi > 0.$

$e)$ $\lim\limits_{\lambda\to\lambda_0}\left[\int\limits_{\xi\le|t|<\infty} L_\lambda(|t|)\,dt\right]=0,\ \ \forall\xi>0.$

$f)$ $L_\lambda(|t|)$ is non-increasing function with respect to $|t|$ on $[0,\infty)$.

In this definition the Lipschitz condition idea presented in [9] is used. Throughout this paper, we suppose that K_λ belongs to $Class\ A$.

3. Convergence at Characteristic Points

In Theorem 3.1, we prove the pointwise convergence of the operators of type (1.3) for the case $D=\prod\limits_{i=1}^{n}\langle a_i,b_i\rangle$, where D is open, semi open or closed multidimensional arbitrary bounded box in $\mathbb{R}^n$.

Theorem 3.1. Let $x\in D$ be a $\mu-$generalized Lebesgue point of the function $f\in L_p(D)$ $(1\le p<\infty)$ such that the following equality holds:

$$\lim_{h\to 0}\left(\frac{1}{\mu(h)}\int\limits_{0}^{h}\int\limits_{S^{n-1}}|f(rt'+x)-f(x)|^p\,r^{n-1}dt'dr\right)^{\frac{1}{p}}=0,\ \ 1\le p<\infty,$$

where $\mu(|t|)$ is defined, increasing, absolutely continuous on $[0,b]$ as a function of $|t|$ for the finite real number b and $\mu(0)=0$. Then, one has

$$\lim_{\lambda\to\lambda_0}|T_\lambda(f;x)-f(x)|=0,$$

on any set Z on which the function

$$\int\limits_{0}^{\delta}\mu'(r)L_\lambda(r)dr,\quad 0<\delta<b$$

is bounded as λ tends to λ_0.

Proof. Set

$$g(t)=\begin{cases} f(t), & t\in D,\\ 0, & t\in\mathbb{R}^n\backslash D.\end{cases}$$

At this stage, there are two cases: $p=1$ and $1<p<\infty$.

Now, let $p=1$. Besides, suppose that $x\in D$ is a fixed $\mu-$generalized Lebesgue point of $f\in L_1(D)$. Using condition (c) of class A, we have the following equality:

$$\begin{aligned}|T_\lambda(f;x)-f(x)|&=\left|\int\limits_{D}K_\lambda(t-x,f(t))dt-f(x)\right|\\ &=\left|\int\limits_{\mathbb{R}^n}K_\lambda(t-x,g(t))dt-\int\limits_{\mathbb{R}^n}K_\lambda(t-x,f(x))dt\right.\\ &\quad\left.+\int\limits_{\mathbb{R}^n}K_\lambda(t-x,f(x))dt-f(x)\right|.\end{aligned}$$

Now, using condition (a) we obtain

$$\begin{aligned} |T_\lambda(f;x)-f(x)| &\leq \int\limits_D |f(t)-f(x)|\, L_\lambda(|t|)dt \\ &+\left|\int\limits_{\mathbb{R}^n} K_\lambda(t-x,f(x))dt - f(x)\right| \\ &+\int\limits_{\mathbb{R}^n\setminus D} |g(t)-f(x)|\, L_\lambda(|t-x|)dt \\ &= I_1+I_2+I_3. \end{aligned}$$

Our aim is to show that $I_1\to 0$, $I_2\to 0$ and $I_3\to 0$ as $\lambda\to\lambda_0$.

Let $B_\delta=\{t:|(t-x|<\delta\}$ and $B_\delta\subset D$. In view of (2.1), we have the following equality for I_1:

$$\begin{aligned} I_1 &= \int\limits_{B_\delta} |f(t)-f(x)|\, L_\lambda(|t-x|)dt + \int\limits_{D\setminus B_\delta} |f(t)-f(x)|\, L_\lambda(|t-x|)dt \\ &= \int\limits_0^\delta \int\limits_{S^{n-1}} |f(rt'+x)-f(x)|\, L_\lambda(r) r^{n-1}dt'dr + \int\limits_{D\setminus B_\delta} |f(t)-f(x)|\, L_\lambda(|t-x|)dt \\ &= I_{11}+I_{12}. \end{aligned}$$

Let us show that $I_{11}\to 0$ as $\lambda\to\lambda_0$. If $x\in D$ is a $\mu-$generalized Lebesgue point of the function $f\in L_1(D)$ then for every $\varepsilon>0$ there exists $\delta_0>0$ such that the following inequality is satisfied

$$\int\limits_0^\delta \int\limits_{S^{n-1}} |f(rt'+x)-f(x)|\, r^{n-1}dt'dr < \varepsilon\mu(\delta),$$

where $0<\delta\leq\delta_0$.

It is easy to see that the following inequality holds for I_{11} :

$$\begin{aligned} I_{11} &\leq \varepsilon \int\limits_0^\delta \left[\underset{r\leq s\leq\delta}{var} L_\lambda(s)+L_\lambda(\delta)\right]\mu'(r)dr \\ &= \varepsilon\int\limits_0^\delta L_\lambda(r)\mu'(r)dr. \end{aligned}$$

Since the following expression

$$\int\limits_0^\delta \mu'(r)L_\lambda(r)dr$$

stays bounded as $\lambda\to\lambda_0$ and $\varepsilon>0$ is arbitrarily small, $I_{11}\to 0$ as $\lambda\to\lambda_0$.

Let us show that $I_{12} \to 0$ as $\lambda \to \lambda_0$. Since the following inequality holds:

$$I_{12} \leq \sup_{\delta \leq r < \infty} L_\lambda(r) \left(\|f\|_{L_1(D)} + |f(x)| \int_D dt \right)$$

and in view of condition (d) of class A, $I_{12} \to 0$ as $\lambda \to \lambda_0$ and by condition (c) of class A, $I_2 \to 0$ as $\lambda \to \lambda_0$. Finally, since

$$I_3 \leq |f(x)| \int_\delta^\infty \int_{S^{n-1}} L_\lambda(r) r^{n-1} dt' dr,$$

by condition (e) of class A, $I_3 \to 0$ as $\lambda \to \lambda_0$. Thus the proof is completed for the case $p = 1$.

Now, let $1 < p < \infty$. Further, assume that $x \in D$ is a fixed $\mu-$generalized Lebesgue point (for one dimensional analogue, see [5]) of $f \in L_p(D)$. Using condition (c) of class A, we have the following equality:

$$\begin{aligned} |T_\lambda(f;x) - f(x)| &= \left| \int_D K_\lambda(t-x, f(t)) dt - f(x) \right| \\ &= \left| \int_{\mathbb{R}^n} K_\lambda(t-x, g(t)) dt - \int_{\mathbb{R}^n} K_\lambda(t-x, f(x)) dt \right. \\ &\quad \left. + \int_{\mathbb{R}^n} K_\lambda(t-x, f(x)) dt - f(x) \right|. \end{aligned}$$

Now, using condition (a) we obtain

$$\begin{aligned} |T_\lambda(f;x) - f(x)| &\leq \int_D |f(t) - f(x)| L_\lambda(|t-x|) dt \\ &\quad + \left| \int_{\mathbb{R}^n} K_\lambda(t-x, f(x)) dt - f(x) \right| \\ &\quad + \int_{\mathbb{R}^n \setminus D} |g(t) - f(x)| L_\lambda(|t-x|) dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Since

$$I_3 \leq |f(x)| \int_\delta^\infty \int_{S^{n-1}} L_\lambda(r) r^{n-1} dt' dr,$$

by condition (e) of class A, $I_3 \to 0$ as $\lambda \to \lambda_0$. Next, using Hölder's inequality [10] for the integral I_1 we have the following:

$$I_1 + I_2 \leq \left(\int_D |f(t) - f(x)|^p L_\lambda(|t-x|)dt \right)^{\frac{1}{p}} \times \left(\int_D L_\lambda(|t|)dt \right)^{\frac{1}{q}} + \left| \int_{\mathbb{R}^n} K_\lambda(t-x, f(x))dt - f(x) \right|.$$

Since for positive numbers m, n the inequality: $(m+n)^p \leq 2^p(m^p + n^p)$ holds [10], by taking the $p-$th power of both sides we have:

$$\begin{aligned} (I_1 + I_2)^p &\leq 2^p \int_D |f(t+x) - f(x)|^p L_\lambda(|t|)dt \times \left(\int_{\mathbb{R}^n} L_\lambda(|t|)dt \right)^{\frac{p}{q}} \\ &\quad + 2^p \left| \int_{\mathbb{R}^n} K_\lambda(t-x, f(x))dt - f(x) \right|^p \\ &= 2^p I^* II^* + 2^p III^*. \end{aligned}$$

By conditions (b) and (c) of class A, $II^* \leq M^{\frac{p}{q}} < \infty$ and $III^* \to 0$ as $\lambda \to \lambda_0$, respectively.

Suppose that $B_\delta = \{t : |(t-x| < \delta\}$ and $B_\delta \subset D$. In view of (2.1) and (2.2) we have the following equality for I^*

$$\begin{aligned} I^* &= \int_{B_\delta} |f(t) - f(x)|^p L_\lambda(|t-x|)dt + \int_{D \setminus B_\delta} |f(t) - f(x)|^p L_\lambda(|t-x|)dt \\ &= \int_0^\delta \int_{S^{n-1}} |f(rt'+x) - f(x)|^p L_\lambda(r) r^{n-1} dt' dr + \int_{D \setminus B_\delta} |f(t) - f(x)|^p L_\lambda(|t-x|)dt \\ &= I_{11} + I_{12}. \end{aligned}$$

Let us show that $I_{11} \to 0$ as $\lambda \to \lambda_0$. If $x \in D$ is a $\mu-$generalized Lebesgue point of the function $f \in L_p(D)$ then, for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that the following inequality holds:

$$\int_0^\delta \int_{S^{n-1}} |f(rt'+x) - f(x)|^p r^{n-1} dt' dr < \varepsilon^p \mu(\delta),$$

where $0 < \delta \leq \delta_0$.

It is easy to see that the following inequality holds for I_{11} :

$$\begin{aligned} I_{11} &\leq \varepsilon^p \int_0^\delta \left[\underset{r \leq s \leq \delta}{var} L_\lambda(s) + L_\lambda(\delta) \right] \mu'(r) dr \\ &= \varepsilon^p \int_0^\delta L_\lambda(r) \mu'(r) dr. \end{aligned}$$

Since the following expression

$$\int_0^\delta \mu'(r)L_\lambda(r)dr$$

remains bounded as $\lambda \to \lambda_0$ and $\varepsilon > 0$ is arbitrarily small, $I_{11} \to 0$ as $\lambda \to \lambda_0$.

Let us show that $I_{12} \to 0$ as $\lambda \to \lambda_0$. Since the following inequality holds:

$$|I_{12}| \leq 2^p \sup_{\delta \leq r < \infty} L_\lambda(r) \left(\|f\|^p_{L_p(D)} \ + \ |f(x)|^p \int_D dt \right),$$

and in view of condition (c) of class A, $I_{12} \to 0$ as $\lambda \to \lambda_0$. Thus the proof is completed for $1 < p < \infty$.

In Theorem 3.2 , we prove the pointwise convergence of the operators of type (1.3) for the case $D = \mathbb{R}^n$.

Theorem 3.2. Suppose that the hyphothesis of Theorem 3.1 is satisfied. Then, one has

$$\lim_{\lambda \to \lambda_0} |T_\lambda(f;x) - f(x)| = 0,$$

whenever $x \in \mathbb{R}^n$ is a $\mu-$generalized Lebesgue point of the function $f \in L_p(\mathbb{R}^n)$.

Proof. Following the proof technic used in Theorem 3.1, we have

$$|T_\lambda(f;x) - f(x)|^p \leq 2^{2p} \left\{ L_\lambda(\delta) \|f\|^p_{L_p(\mathbb{R}^n)} \ + \ |f(x)|^p \int_\delta^\infty \int_{S^{n-1}} L_\lambda(r) r^{n-1} dt' dr \right\}$$

$$\times \left(\int_{\mathbb{R}^n} L_\lambda(|t|)dt \right)^{\frac{p}{q}} + 2^p \varepsilon^p \int_0^\delta \mu'(r)L_\lambda(r)dr \times \left(\int_{\mathbb{R}^n} L_\lambda(|t|)dt \right)^{\frac{p}{q}}$$

$$+ 2^p \left| \int_{\mathbb{R}^n} K_\lambda(|t-x|, f(x))dt - f(x) \right|^p.$$

Since K_λ belongs to class A, the remaining part of the proof is clear.

4. Rate of Convergence

Theorem 4.1. Suppose that the hypotheses of Theorem 3.2 are satisfied. Let

$$\Delta(\lambda, \delta) = \int_0^\delta \mu'(r)L_\lambda(r)dr,$$

where $0 < \delta \leq \delta_0$, and the following assumptions are satisfied:

(i) $\Delta(\lambda, \delta) \to 0$ as $\lambda \to \lambda_0$ for some $\delta > 0$.

(ii) For every $\xi > 0$,

$$L_\lambda(\xi) = o(\Delta(\lambda, \delta))$$

as $\lambda \to \lambda_0$.

(iii) For every $\xi > 0$,

$$\int_\xi^\infty \int_{S^{n-1}} L_\lambda(r) r^{n-1} dt' dr = o(\Delta(\lambda, \delta))$$

as $\lambda \to \lambda_0$.

(iv)

$$\left| \int\limits_{\mathbb{R}^n} K_\lambda(|t-x|, f(x))dt - f(x) \right|^p = o(\Delta(\lambda,\delta)).$$

Then, at each $\mu-$generalized Lebesgue point of $L_p(\mathbb{R}^n)$ we have

$$|T_\lambda(f;x) - f(x)| = o(\Delta(\lambda,\delta)^{\frac{1}{p}})$$

as $\lambda \to \lambda_0$.

Proof. Using Theorem 3.2., we may write

$$|T_\lambda(f;x) - f(x)|^p \leq 2^{2p} \left\{ L_\lambda(\delta) \|f\|^p_{L_p(\mathbb{R}^n)} + |f(x)|^p \int\limits_\delta^\infty \int\limits_{S^{n-1}} L_\lambda(r) r^{n-1} dt' dr \right\}$$

$$\times \left(\int\limits_{\mathbb{R}^n} L_\lambda(|t|)dt \right)^{\frac{p}{q}} + 2^p \varepsilon^p \int\limits_0^\delta \mu'(r) L_\lambda(r) dr \times \left(\int\limits_{\mathbb{R}^n} L_\lambda(|t|)dt \right)^{\frac{p}{q}}$$

$$+ 2^p \left| \int\limits_{\mathbb{R}^n} K_\lambda(|t-x|, f(x))dt - f(x) \right|^p.$$

From $(i)-(iv)$ and using class A conditions the desired result is easily obtained, that is

$$|T_\lambda(f;x) - f(x)| = o(\Delta(\lambda,\delta)^{\frac{1}{p}}).$$

Thus the proof is completed.

Remark 1. *Note that the similar result can be obtained for the case D is arbitrary bounded box in $\mathbb{R}^n$ by using Theorem 3.1.*

5. Conclusion

In this paper, the pointwise convergence of the convolution type nonlinear multidimensional singular integral operators is investigated. For this aim, we defined a special class of kernel functions. Therefore, the main results are presented as Theorem 3.1 and Theorem 3.2. Also, by using main results, the rates of pointwise convergences of the indicated type operators are discussed.

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