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TOR DECOMPOSITION OF $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$

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ABSTRACT. We decompose $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$, the connective unitary $K-$theory with p-adic coefficients of the $n-$fold smash product of the classifying space for the cyclic group of prime order p, as a direct sum of some graded groups, which include the graded groups $bu_{p^*}(B\mathbb{Z}/p)$ and $Tor^1_{\mathbb{Z}_p[v]}(bu_{p^*}(B\mathbb{Z}/p), bu_{p^*}(B\mathbb{Z}/p))[-1]$. We deal with the results in [6, Theorem 3.8] together with the Künneth sequence for $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$, to explain that there is no extension problem for this Künneth sequence, for any finite number n not just for $n = 2$ and therefore the middle term of this sequence is a direct sum of the left and the right side.

Keywords: The connective unitary K-theory;a Künneth formula short exact sequence.

1. Introduction

Let bu_* denote connective unitary K-homology on the stable homotopy category of CW spectra [1] so that if X is a space without a basepoint its unreduced bu-homology is $bu_*(\Sigma^\infty X_+)$, the homology of the suspension spectrum of the disjoint union of X with a base-point. In particular $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$ where $\deg(u) = 2$.

For a prime number p, we have bu_p, the connective unitary K-theory with p-adic integer coefficients $\mathbb{Z}_p$, where $bu_p \simeq \vee_{i=1}^{p-1}\Sigma^{2i-2}lu$, lu the Adams summand such that $bu_{p^*}(S^0) \cong \bigoplus_{i=1}^{p-1} lu_{*-2i+2}(S^0)$, $lu_*(S^0) \cong \mathbb{Z}_p[u^{p-1}] \cong \mathbb{Z}_p[v]$ and $\deg(v) = 2(p-1)$.

In §2 we deal with the results in [6, Lemma 3.4], together with the Künneth sequence for $bu_*(B\mathbb{Z}/2)^{\wedge n}$, to explain that there is no extension problem for this Künneth sequence, for any finite number n not just for $n = 2$ and therefore the middle term of this sequence is a direct sum of the left and the right side. From this we will decompose $bu_*(B\mathbb{Z}/2)^{\wedge n}$ as a direct sum of some graded groups.

For any prime p, In §3 we use the splitting $bu_p \simeq \vee_{i=1}^{p-1}\Sigma^{2i-2}lu$ and the Holzsager splitting [3] $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$ to decompose $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$ as a direct sum of some graded groups. This decomposition agreed with the result in [6, Theorem 3.8] and both also yield that there is no extension problems in the Künneth sequence for $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$.

In this section we fix some notations that we will use for this paper and introduce some binomial coefficient identities which will support our calculation.

Notation 1.1.

- For $n \geq 1$, in §2 ,we write P_n for $(B\mathbb{Z}/2)^{\wedge n}$, the $n-$fold smash product of $B\mathbb{Z}/2$. In particular, $P_1 = B\mathbb{Z}/2$, whereas in §3 ,we write P_n for $(B\mathbb{Z}/p)^{\wedge n}$
- we write A_* for $bu_*(P_1)$.
- For a $\mathbb{Z}$-graded group B_*, we write $B_*[n]$ for the graded group with $B_j[n] = B_{j+n}$, so that $bu_*(X)[-1] = bu_{*-1}(X)$.

Lemma 1.2. [2]. For any $j, k, m, n \in \mathbb{N}_0$, we have

(i) $\binom{n}{k} = 0$ if n and k both are not integers or if $n < k$,

(ii) $\sum_{0\le k\le n}\binom{n}{k} = 2^n$,
(iii) $\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$, and
(iv) $\sum_{0\le k\le n}\binom{k}{j}\binom{n-k}{m-j} = \binom{n+1}{m+1}$, where $0 \le j \le m \le n$. □

2. Tor decomposition of $bu_*(B\mathbb{Z}/2)^{\wedge n}$

2.1. For $p = 2$, in this section we will decompose $bu_*(P_n)$ as a direct sum of some graded groups, which include the graded groups $Tor^1_{\mathbb{Z}_2[u]}(bu_*(P_1), bu_*(P_1))[-1]$ and $bu_*(P_1)$.

Definition 2.2.
Let X be a graded group, and $r \ge 0$. We define $T^r(X)_*$ as

$$T^r(X)_* = T(T^{r-1}(X)_*)_*$$

where $T^0(X)_* = X$ and $T^1(X)_* = T(X)_* = Tor^1_{\mathbb{Z}_2[u]}(A_*, X)[-1]$.

From this definition we can deduce that:

(1) $T^r(X)_* = T^m(T^k(X)_*)_*$, for $m + k = r$.
(2) We have $T^r(A_*)_* = \overbrace{T(T(\dots T(A_*)_* \dots)_*)_*}^{r\ times}$, where, by [7] §2.7, $T(A_*)_*$ is non-zero just in degrees $2t + 1 \ge 3$. Then, by applying $T(A_*)_* \otimes_{\mathbb{Z}_2[u]} -$ instead of $A_* \otimes_{\mathbb{Z}_2[u]} -$ to the free resolution of A_*, which is described in [7] Example 2.9, with shifting by (-1) and by using induction on r, we can calculate the graded group $T^r(A_*)_*$. This is non-zero just in degrees $2t + 1$ for $t \ge r$.

Notation 2.3. For the rest of this section, we will write:

- A^r_* for $A^{\otimes^r}_*$, the tensor of A_* with itself over $\mathbb{Z}_2[u]$ $r-$times,
- $A_* \otimes B_*$ for $A_* \otimes_{\mathbb{Z}_2[u]} B_*$, for a $\mathbb{Z}_2[u]-$module B_*, and
- $T^{j_r,j_{r-1},\dots,j_1}_*$ for $T^{j_r}(A_* \otimes T^{j_{r-1}}(A_* \otimes T^{j_{r-2}}(\dots T^{j_2}(A_* \otimes T^{j_1}(A_*)_*)_* \dots)_*)_*)_*$, where $j_i \in \mathbb{N}_0$.

Definition 2.4. Let $0 \le k \le n-1$, we define the weight k iterated T as

$$W^k_n = \bigoplus_{\sum j_i = k} T^{j_{n-k},j_{n-k-1},\dots,j_1}_*$$

where $j_i \in \mathbb{N}_0$.

We will see later, in 3.8, that $bu_*(P_n)$ decomposes as a sum over the W^k_n's. It is easy to check that:

(i) $W^k_n = 0$, for $k \ge n$,
(ii) $W^{n-1}_n = T^{n-1}_*$, $W^0_n = A^n_*$, and
(iii) $W^k_{n+1} = (A_* \otimes W^k_n) \oplus T(W^{k-1}_n)$ for $0 \le k \le n$.

Since W^k_n is constructed from the graded groups $T^{j_{n-k},j_{n-k-1},\dots,j_1}_*$, then to calculate W^k_n as a group we need to calculate each summand as a group. Let us start with the higher 2−torsion when $k = n-1$.

Notation 2.5. In this section to simplify the indices in some formulas, we need to change the indexing conventions in the resolution in [7] to be in the form

$$0 \longrightarrow \oplus_{j_i>0}\mathbb{Z}_2[u]\langle a_{j_i}\rangle \xrightarrow{d} \oplus_{j_i>0}\mathbb{Z}_2[u]\langle b_{j_i}\rangle \longrightarrow A_* \longrightarrow 0$$

where a_{j_i} and b_{j_i} are in degree $2j_i - 1$ and $d(a_{j_i}) = 2b_{j_i} - ub_{j_i-2}$.

Proposition 2.6. For $n \geq 0$, $t \geq n$,

$$T^n_{2t+1} \cong \mathbb{Z}/2^{t+1-n}\langle \sum_{i+\sum_{k=1}^n j_k = t} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}\rangle,$$

where we have the short exact sequence

$$0 \longrightarrow \oplus_{j_i>0}\mathbb{Z}_2[u]\langle a_{j_i}\rangle \xrightarrow{d} \oplus_{j_i>0}\mathbb{Z}_2[u]\langle b_{j_i}\rangle \longrightarrow A_* \longrightarrow 0$$

as in 2.5, and $v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}$ refers to
$v_{2i+1}\otimes a_{j_1}\otimes a_{j_2}\cdots\otimes a_{j_n} \in A_*\otimes\big(\oplus_{j_1>0}\mathbb{Z}_2[u]\langle a_{j_1}\rangle\big)\otimes\big(\oplus_{j_2>0}\mathbb{Z}_2[u]\langle a_{j_2}\rangle\big)\cdots\otimes\big(\oplus_{j_n>0}\mathbb{Z}_2[u]\langle a_{j_n}\rangle\big)$.

Proof The proof is by induction on n, where the case $n = 1$ is described in [7] and agrees with the above statement. For $n = 2$, let us consider the same free resolution of A_* after applying $(T_*\otimes -)$ and shifting by (-1). We have $T^2_* \subset T_*\otimes\big(\oplus_{j_2>0}\mathbb{Z}_2[u]\langle a_{j_2}\rangle\big)$, where $T_{2\ell+1} \cong \mathbb{Z}/2^{\ell}\langle\sum_{i+j_1=\ell} v_{2i+1}a_{j_1}\rangle$ for $\ell \geq 1$. Then by the same calculation as for T_*, see [7], we can calculate that, for $t \geq 2$, T^2_{2t+1} is a group with generator $\sum_{i+j_1=\ell} v_{2i+1}a_{j_1}a_{j_2}$, where $\ell + j_2 = t$. This generator can be written as $\sum_{i+j_1+j_2=t} v_{2i+1}a_{j_1}a_{j_2}$, which contains a summand $v_{2(t-2)+1}a_{j_1}a_{j_2}$, with $j_1 = j_2 = 1$ and $v_{2(t-2)+1} \in A_{2(t-2)+1} \cong \mathbb{Z}/2^{t-1}$, see [7] for $p = 2$. Therefore $T^2_{2t+1} \cong \mathbb{Z}/2^{t-1}\langle\sum_{i+j_1+j_2=t} v_{2i+1}a_{j_1}a_{j_2}\rangle$. Now assume that the statement is true for n. Again using the same free resolution of A_*, and applying $(T^n_*\otimes -)$ with shifting by (-1), we have,$T^{n+1}_* \subset T^n_*\otimes\oplus_{j_{n+1}>0}\mathbb{Z}_2[u]\langle a_{j_{n+1}}\rangle$, where $T^n_{2\ell+1} \cong \mathbb{Z}/2^{\ell+1-n}\langle\sum_{i+\sum_{k=1}^n j_k=\ell} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}\rangle$ for $\ell \geq n$. Similarly to the case $n = 2$ above, where $T^{n+1}_* = T(T^n_*)$, we can calculate the generator of the group T^{n+1}_{2t+1}, for $t \geq n+1$, to be $\sum_{i+\sum_{k=1}^n j_k=\ell} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}a_{j_{n+1}}$, where $\ell + j_{n+1} = t$. This generator can be written as $\sum_{i+\sum_{k=1}^{n+1} j_k=t} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}a_{j_{n+1}}$ where, for $j_k = 1$, $k = 1, 2, \dots, n+1$, this sum has a summand of the form $v_{2(t-n-1)+1}a_{j_1}a_{j_2}\dots a_{j_{n+1}}$. Again [7], for $p = 2$, implies that $T^{n+1}_{2t+1} \cong \mathbb{Z}/2^{t-n}\langle\sum_{i+\sum_{k=1}^{n+1} j_k=t} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_{n+1}}\rangle$.

□

Lemma 2.7.
Let X_* be a $\mathbb{Z}_2[u]-$module such that u acts trivially on $A_*\otimes X_*$. Then u also acts trivially on $T(A_*\otimes X_*)_*$.

Proof From 2.6, we have an inclusion of $\mathbb{Z}_2[u]-$modules, $T(A_*\otimes X_*)_* \subset A_*\otimes X_*\otimes\big(\oplus_{j>0}\mathbb{Z}_2[u]\langle a_{j_1}\rangle\big)$, where the right hand side is a graded group generated by $\{v\otimes x\otimes a_{j_1}\}$, for $v\otimes x \in A_*\otimes X_*$. By the action of u on $A_*\otimes X_*$, we deduce that u also acts trivially on $A_*\otimes X_*\otimes\big(\oplus_{j>0}\mathbb{Z}_2[u]\langle a_{j_1}\rangle\big)$. □

Lemma 2.8. Let $0 \leq k \leq n-1$. With the exceptions of A_* and T^n_*, each summand of W^k_n is a graded $\mathbb{F}_2-$vector space, on which u acts trivially.

Proof By 2.6 and [7], we know that A_* and T^n_* consist of higher 2−torsion groups with non-trivial action of u, where $uv_{2i-1} = 2v_{2i+1}$ and $v_{2i+1} \in A_*$ of degree $2i+1$. Now let us consider the summand $T^{0,n}_*$ of W^k_n. By 2.7, the results for the other summands are analogous.

By 2.6, any $x \in T^{0,n}_{2t} = \oplus_{t=\ell+s+1}A_{2\ell+1}\otimes T^n_{2s+1}$ can be written as a linear combination of

$$v_{2\ell+1}\otimes\sum_{i+\sum_{k=1}^n j_k=s} v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n} = \sum_{i+\ell+\sum_{k=1}^n j_k=t-1} v_{2\ell+1}\otimes v_{2i+1}a_{j_1}a_{j_2}\dots a_{j_n}.$$

And by [7], we have

$$2v_{2\ell+1}\otimes v_{2i+1} = uv_{2\ell-1}\otimes v_{2i+1} = 2v_{2\ell-1}\otimes v_{2i+3} = \cdots = 2v_1\otimes v_{2i+2\ell+1} = 0,$$

and

$$uv_{2\ell+1} \otimes v_{2i+1} = 2v_{2\ell+3} \otimes v_{2i+1} = uv_{2\ell+3} \otimes v_{2i-1} = \cdots = 2v_{2\ell+2i+3} \otimes v_1 = 0.$$

This shows that $T_*^{0,n}$ is an $\mathbb{F}_2-$vector space with a trivial action of u.

From the definition of W_n^k, we can recognise that any element of the summands of W_n^k can be written as a linear combination of a product of $(n-k)$ v_i's with some a_{j_i}'s. Therefore, by the action of u on A_*, we can prove the required result for the other summands of W_n^k in exactly the same way. □

Notation 2.9. For the rest of our calculations we will write $(\mathbb{Z}/2)^a$ for $a-$times the direct sum of $\mathbb{Z}/2$ with itself.

By 2.6 and 2.8, we can deduce the following result.

Corollary 2.10. Let $n \geq 0$ and $s \geq 2n+2$. Then

$$T_s^{0,n} \cong \begin{cases} (\mathbb{Z}/2)^{t-n}, & \text{when } s = 2t, \\ 0, & \text{otherwise .} \end{cases}$$

Proof $T_{2t}^{0,n}$ is an $\mathbb{F}_2-$vector space with basis $\{v_{2i+1} \otimes x_{2(t-i-1)+1} : \ i = 0, 1, \ldots, t-n-1\}$, where $x_{2(t-i-1)+1} = \sum_{\ell+\sum_{k=1}^n j_k = t-i-1} v_{2\ell+1} a_{j_1} a_{j_2} \ldots a_{j_n} \in T_{2(t-i-1)+1}^n$. The number of summands $t-n$ comes from the dimension of this basis. □

By 2.8, for $n > 1$, we know that the only summand of W_n^k with higher 2−torsion is calculated in 2.6. Now by using some binomial identities given in 1.2 we are going to calculate the rest of the summands, where all of them are graded $\mathbb{F}_2-$vector spaces.

Proposition 2.11. Let $s \geq 2j_1 + 2j_2 + 2$, we have

$$T_s^{j_2,j_1} \cong \begin{cases} (\mathbb{Z}/2)^{\binom{t-j_1}{j_2+1}}, & \text{when } s = 2t, \\ 0, & \text{otherwise.} \end{cases}$$

Proof The proof is by induction on j_2, where the case $j_2 = 0$ is considered in 2.10. Let assume that the statement is true for $j_2 - 1$, where $T_*^{j_2,j_1} = T(T_*^{j_2-1,j_1})$. Now 2.8 shows that $T_*^{j_2-1,j_1}$ is an $\mathbb{F}_2-$vector space with a trivial action of u. Then

$$T_*^{j_2,j_1} = \ker(I \otimes d) = T_*^{j_2-1,j_1} \otimes \big(\oplus_{j_k>0} \mathbb{Z}_2[u]\langle a_{j_k} \rangle \big)[-1],$$

where $I = id_{T_*^{j_2-1,j_1}}$ and $d : \oplus_{j_k>0}\mathbb{Z}_2[u]\langle a_{j_k}\rangle \to \oplus_{j_k>0}\mathbb{Z}_2[u]\langle b_{j_k}\rangle$ is described in 2.5. Therefore, for $t \geq j_1 + j_2 + 1$, $i \geq j_1 + j_2$ and $j > 0$, and by the binomial identity in 1.2(ii), we have

$$T_{2t}^{j_2,j_1} = \oplus_{t=i+j_k} T_{2i}^{j_2-1,j_1} \otimes_{\mathbb{Z}} \mathbb{Z}\{a_{j_k}\} \cong \oplus_{t=i+j_k} (\mathbb{Z}/2)^{\binom{i-j_1}{j_2}} = (\mathbb{Z}/2)^{\binom{t-j_1}{j_2+1}}.$$

From 2.10, we know that T_*^{0,j_1} is non-zero only in even degrees and also we know that $T_*^{j_2,j_1} = T^{j_2}(T_*^{0,j_1})$. Therefore $T_{2t+1}^{j_2,j_1} = 0$. □

After we have calculated W_n^k for $k = n-1, n-2$, let us now consider the cases for W_n^k when $k = 0, 1$.

Proposition 2.12. Let $t, \ell > 0$, $t \geq \ell$. Then

$$A_s^r \cong \begin{cases} (\mathbb{Z}/2)^{\binom{t+\ell}{2\ell}}, & \text{when } r = 2\ell+1,\ s = 2t+1, \\ (\mathbb{Z}/2)^{\binom{t+\ell-1}{2\ell-1}}, & \text{when } r = 2\ell,\ s = 2t, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proof The proof is by induction on r, where the case $r=2$ is described in [7]. That is, A^2_{2t} is the $\mathbb{F}_2-$vector space with basis $\{v_1\otimes v_{2t-1}, v_3\otimes v_{2t-3}, v_5\otimes v_{2t-5},\dots,v_{2t-1}\otimes v_1\}$. Therefore $A^2_{2t}\cong(\mathbb{Z}/2)^t\cong(\mathbb{Z}/2)^{\binom{t}{1}}$, and $A^2_{2t+1}=0$. Now let us assume that the statement is true for $r=2\ell$, that is, $A^{2\ell}_{2j}\cong(\mathbb{Z}/2)^{\binom{j+\ell-1}{2\ell-1}}$ for $j\le t$. Since $A^{2\ell+1}_*\cong A_*\otimes A^{2\ell}_*$ is an $\mathbb{F}_2-$vector space with trivial action of u, then, for $i\ge 0$ and $j\ge\ell$, we have

$$A^{2\ell+1}_{2t+1}\cong\oplus_{i+j=t}A_{2i+1}\otimes_{\mathbb{Z}}A^{2\ell}_{2j}\cong\oplus_{i+j=t}\mathbb{Z}/2^{i+1}\otimes_{\mathbb{Z}}(\mathbb{Z}/2)^{\binom{j+\ell-1}{2\ell-1}}=(\mathbb{Z}/2)^{\sum_{j=\ell}^{t}\binom{j+\ell-1}{2\ell-1}}\cong(\mathbb{Z}/2)^{\binom{t+\ell}{2\ell}}.$$

Similarly, for the case $r=2\ell+1$, we have $A^{2\ell+1}_{2j+1}\cong(\mathbb{Z}/2)^{\binom{j+\ell}{2\ell}}$ for $\ell\le j\le t$. So , for $i\ge0$ and $j\ge\ell$, we have

$$A^{2\ell+2}_{2t+2}\cong\oplus_{i+j=t}A_{2i+1}\otimes_{\mathbb{Z}}A^{2\ell+1}_{2j+1}\cong\oplus_{i+j=t}(\mathbb{Z}/2)^{\binom{j+\ell}{2\ell}}=(\mathbb{Z}/2)^{\sum_{j=\ell}^{t}\binom{j+\ell}{2\ell}}\cong(\mathbb{Z}/2)^{\binom{t+\ell+1}{2\ell+1}}.$$

Finally, let us consider the case when r is odd and s is even or conversely. Since $A^r_*=\overbrace{A_*\otimes A_*\otimes\cdots\otimes A_*}^{r\ times}$, for $s\ge r$, we have $A^r_s=\bigoplus_{s=2\sum_{k=1}^{r}i_k+r}A_{2i_1+1}\otimes_{\mathbb{Z}}A_{2i_2+1}\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}A_{2i_r+1}$. This asserts that r is associated with s, in the sense that both should be odd or even, and otherwise $A^r_s=0$.

□

Lemma 2.13. Let $t,\ell>0$, $t\ge\ell+1$. Then

$$T(A^r_*)_s\cong\begin{cases}(\mathbb{Z}/2)^{\binom{t+\ell}{2\ell+1}}, & \text{when } r=2\ell+1,\ s=2t+1,\\ (\mathbb{Z}/2)^{\binom{t+\ell-1}{2\ell}}, & \text{when } r=2\ell,\ s=2t, \text{ and}\\ 0, & \text{otherwise.}\end{cases}$$

(Note that $T(A^r_*)_*=T^{j_r,j_{r-1},\dots,j_1}_*$, where $j_i=1$ for $i=r$ and other j_k are zero).

Proof We will calculate $T(A^{2\ell}_*)_{2t}$. The other cases are similar. Let us start from the free resolution of A_*, which is described in 2.5, then applying $(A^{2\ell}_*\otimes -)$ and shifting by (-1) allows us to calculate $T(A^{2\ell}_*)_*$. Since $A^{2\ell}_*$ is an $\mathbb{F}_2-$vector space with trivial action of u, see 2.8, then

$$T(A^{2\ell}_*)_*=\ker(I\otimes d)=A^{2\ell}_*\otimes\oplus_{j_i>0}\mathbb{Z}_2[u]\langle a_{j_k}\rangle$$

where $I=id_{A^{2\ell}_*}$ and $d:\oplus_{j_k>0}\mathbb{Z}_2[u]\langle a_{j_k}\rangle\to\oplus_{j_k>0}\mathbb{Z}_2[u]\langle b_{j_k}\rangle$ is described in 2.5. Therefore, by 2.12 for $i\ge\ell$ and by 1.2(ii), we have

$$T(A^{2\ell}_*)_{2t}=\oplus_{t=i+j_k}A^{2\ell}_{2i}\otimes_{\mathbb{Z}}\mathbb{Z}\{a_{j_k}\}=(\mathbb{Z}/2)^{\sum_{i=\ell}^{t-1}\binom{i+\ell-1}{2\ell-1}}=(\mathbb{Z}/2)^{\binom{t+\ell-1}{2\ell}}.$$

□

By 2.12 and 2.13, we can check that $A^{2\ell}_*[-1]\cong T(A^{2\ell-1}_*)$ and $A^{2\ell+1}_*[-1]\cong T(A^{2\ell}_*)$.

Corollary 2.14. Let $t,\ell>0$, and $s\ge r+3$. Then

$$(A^r_*\otimes T^1_*)_s\cong\begin{cases}(\mathbb{Z}/2)^{\binom{t+\ell-1}{r}}, & \text{when } r=2\ell+1,\ s=2t,\\ & \text{or } r=2\ell,\ s=2t+1, \text{ and}\\ 0, & \text{otherwise.}\end{cases}$$

(Note that $A^r_*\otimes T^1_*=T^{j_{r+1},j_r,\dots,j_1}_*$, where $j_i=1$ for $i=1$ and other j_k are zero).

Proof We will consider the case $r = 2\ell + 1$ and $s = 2t$. The other cases are similar. Fro 2.6, 2.12 and 1.2(ii), for $i \geq \ell$ and $j \geq 1$ we have

$$(A_*^r \otimes T_*^1)_{2t} = \oplus_{t=i+j+1}(A_{2i+1}^{2\ell+1} \otimes_{\mathbb{Z}} T_{2j+1}^1) \cong \oplus_{t=i+j+1}(\mathbb{Z}/2)^{\binom{i+\ell}{2\ell}} \cong (\mathbb{Z}/2)^{\Sigma_{i=\ell}^{t-2}\binom{i+\ell}{2\ell}} = (\mathbb{Z}/2)^{\binom{t+\ell-1}{2\ell+1}}$$

□

By 2.12 and 2.14, for $r > 0$, we can check the following.

Corollary 2.15. Let $r > 0$, then

$$A_*^r \otimes T_*^1 \cong A_*^{r+1}[-2].$$

Definition 2.16. Given $j_i \in \mathbb{N}_0$ for $i \geq 1$, we define $\beta_{i,n} = \Sigma_{k=i}^n j_k$. (of course, $\beta_{i,n}$ depend on $j_i, \dots, j_n$, but the sequence will be clear from the context.)

Corollary 2.17. Let $n > 1$, then

$$T^{j_n, j_{n-1}, \dots, j_1} = T^{0, j_{n-1}, \dots, j_1} \otimes X_*^{\beta_{n,n}}[-\beta_{n,n}]$$

where $X_* = \oplus_{j_k > 0}\mathbb{Z}_2[u]\langle a_{j_k}\rangle$, that is, $X_{2s-1} = \mathbb{Z}_2\{u^m a_{j_k} : s = m + j_k\}$.

Proof The proof follows from $T_*^{j_n, j_{n-1}, \dots, j_1} = \overbrace{T(T \dots T}^{j_n-\ times}(T_*^{0, j_{n-1}, \dots, j_1})_* \dots)_*$, and the fact tha $T_*^{0, j_{n-1}, \dots, j_1}$ is an $\mathbb{F}_2$-vector space with trivial action of u. So $T_*^{1, j_{n-1}, \dots, j_1} = T(T_*^{0, j_{n-1}, \dots, j_1})_* =$ $T_*^{0, j_{n-1}, \dots, j_1} \otimes X_*[-1]$ and $T_*^{2, j_{n-1}, \dots, j_1} = T(T_*^{1, j_{n-1}, \dots, j_1}) = T_*^{0, j_{n-1}, \dots, j_1} \otimes X_*^2[-2]$. Therefor inductively on n we get the required result.

□

Proposition 2.18. Let $r_1 \geq 0$, $r_2 > 1$ and $s \geq r_1 + r_2 + 2$. Then

$$(A_*^{r_1} \otimes T(A_*^{r_2})_*)_s \cong \begin{cases} (\mathbb{Z}/2)^{\binom{t+k}{r_1+r_2}}, & \text{when } s = 2t+1,\ r_1 + r_2 = 2k+1, \\ (\mathbb{Z}/2)^{\binom{t+k-1}{r_1+r_2}}, & \text{when } s = 2t,\ r_1 + r_2 = 2k, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

(Note that $A_*^{r_1} \otimes T(A_*^{r_2})_* = T_*^{j_{r_2+r_1}, \dots, j_{r_2}, \dots, j_1}$ where $j_i = 1$ for $i = r_2$ and other j_k are zero.)

Proof Let us consider the above graded group in degree $s = 2t$. The other cases are simila By 2.12 and 2.13, for $i \geq \ell_1$ and $j \geq \ell_2 + 1$, we have

$$(A_*^{r_1} \otimes T(A_*^{r_2})_*)_{2t} = \begin{cases} \oplus_{t=i+j} A_{2i}^{2\ell_1} \otimes_{\mathbb{Z}} T(A_*^{2\ell_2})_{2j}, & \text{when } r_1 = 2\ell_1,\ r_2 = 2\ell_2, \\ \oplus_{t=i+j+1} A_{2i+1}^{2\ell_1+1} \otimes_{\mathbb{Z}} T(A_*^{2\ell_2+1})_{2j+1}, & \text{when } r_1 = 2\ell_1 + 1,\ r_2 = 2\ell_2 + 1. \end{cases}$$

$$\cong \begin{cases} \oplus_{t=i+j}(\mathbb{Z}/2)^{\binom{i+\ell_1-1}{2\ell_1-1}\binom{j+\ell_2-1}{2\ell_2}}, & \text{when } r_1 = 2\ell_1,\ r_2 = 2\ell_2, \\ \oplus_{t=i+j+1}(\mathbb{Z}/2)^{\binom{i+\ell_1}{2\ell_1}\binom{j+\ell_2}{2\ell_2+1}}, & \text{when } r_1 = 2\ell_1 + 1,\ r_2 = 2\ell_2 + 1. \end{cases}$$

Then, for $r_1 = 2\ell_1$ and $r_2 = 2\ell_2$, 1.2(iii) shows that

$$\sum_{t=i+j}\binom{i+\ell_1-1}{2\ell_1-1}\binom{j+\ell_2-1}{2\ell_2} = \sum_{m=2\ell_1-1}^{t+\ell_1+\ell_2-2}\binom{m}{2\ell_1-1}\binom{t+\ell_1+\ell_2-2-m}{2\ell_2} = \binom{t+k-1}{r_1+r_2},$$

where $2k = r_1 + r_2$. And similarly, for $r_1 = 2\ell_1 + 1$, $r_2 = 2\ell_2 + 1$ and $2k = r_1 + r_2$

$$\sum_{t=i+j+1}\binom{i+\ell_1}{2\ell_1}\binom{j+\ell_2}{2\ell_2+1} = \binom{t+\ell_1+\ell_2}{2\ell_1+2\ell_2+2} = \binom{t+k-1}{r_1+r_2}.$$

Examples 2.19. For $r_1 + r_2 = 5$, and $t \geq 3$, we can calculate that

$$T(A_*^5)_{2t+1} = (A_* \otimes T(A_*^4)_*)_{2t+1} = (A_*^2 \otimes T(A_*^3)_*)_{2t+1} = (A_*^3 \otimes T(A_*^2)_*)_{2t+1} \cong (\mathbb{Z}/2)^{\binom{t+2}{5}}.$$

And for $r_1 + r_2 = 6$, and $t \geq 4$, we can calculate that

$$T(A_*^6)_{2t} = (A_* \otimes T(A_*^5)_*)_{2t} = (A_*^2 \otimes T(A_*^4)_*)_{2t} = (A_*^3 \otimes T(A_*^3)_*)_{2t} = (A_*^4 \otimes T(A_*^2)_*)_{2t} \cong (\mathbb{Z}/2)^{\binom{t+2}{6}}.$$

By 2.6 we know that $T_*^{j_1}$ is concentrated in odd degrees $s \geq 2j_1 + 1$, whereas 2.11 shows that $T_*^{j_2,j_1}$ is concentrated in even degrees $s \geq 2j_1 + 2j_2 + 2$. Similarly to these two cases we can see that $T_*^{j_n,j_{n-1},\ldots,j_1}$ is concentrated in odd degrees $s \geq 2\beta_{1,n} + n$ if n is odd, and in even degrees if n is even. This means $T_*^{j_n,j_{n-1},\ldots,j_1}$ is non-zero just in degrees $s \geq 2\beta_{1,n} + n$ where s and n both are odd or both are even.

By all the previous calculations, we can deduce the next proposition, which is about the calculation of $T_*^{j_n,j_{n-1},\ldots,j_1}$ as a graded group, for any $n > 0$, and using this we can calculate W_n^r for any $0 \leq r \leq n-1$.

Proposition 2.20. Let $n > 1$. Then, using the notation of 2.16,

$$T_s^{j_n,j_{n-1},\ldots,j_1} \cong (\mathbb{Z}/2)^{\binom{\frac{s+n-2}{2} - j_1}{\beta_{2,n}+n-1}},$$

for $s \geq 2\beta_{1,n} + n$.

Proof If $j_k = 0$, for all $k \geq 1$, then the left hand side is A_*^n, which is considered in 2.12 and agrees with the above result. And if there is only one k such that $j_k \neq 0$, then the left hand side has the form $A_*^{n-k} \otimes T^{j_k}(A_*^k)_*$, which can be calculated by 2.6, 2.12 and 2.18. Now, if there are at least k_1, k_2 such that j_{k_1} and j_{k_2} are not zero, then in this case we need to use induction on n to calculate the above graded group, where the case $n = 2$ is considered in 2.11. Let us assume that the statement is true for n where $T_*^{j_n,j_{n-1},\ldots,j_1}$ is concentrated in odd degrees. Since

$$T_*^{j_{n+1},j_n,\ldots,j_1} = T_*^{0,j_n,\ldots,j_1} \otimes X_*^{\beta_{n+1,n+1}}[-\beta_{n+1,n+1}],$$

see 2.17, where $X_* = \oplus_{j_i>0}\mathbb{Z}_2[u]\langle a_{j_i}\rangle$ and

$$T_{2t}^{0,j_n,\ldots,j_1} = \oplus_{t=k+s+1} A_{2k+1} \otimes_{\mathbb{Z}} T_{2s+1}^{j_n,j_{n-1},\ldots,j_1} \cong \oplus_{t=k+s+1}(\mathbb{Z}/2)^{\binom{\frac{2s+1+n-2}{2} - j_1}{\beta_{2,n}+n-1}},$$

for $k \geq 0$ and $s \geq \beta_{1,n} + \frac{n-1}{2}$. That is, by 1.2$(ii)$,

$$T_{2t}^{0,j_n,\ldots,j_1} \cong (\mathbb{Z}/2)^{\Sigma_{s=\beta_{1,n}+\frac{n-1}{2}}^{t-1}\binom{\frac{2s+n-1}{2} - j_1}{\beta_{2,n}+n-1}} = (\mathbb{Z}/2)^{\binom{\frac{2t+n-1}{2} - j_1}{\beta_{2,n}+n}}.$$

Then, for $2t \geq 2\beta_{1,n+1} + n + 1$, $i_r \geq \beta_{1,n} + \frac{n+1}{2} + r - 1$ and $m_r > 0$ where $1 \leq r \leq j_{n+1}$, we get

$$T_{2t}^{j_{n+1},j_n,\ldots,j_1} = \Big(\oplus_{t=i_{j_{n+1}}+m_{j_{n+1}}} (\ldots(\oplus_{i_3=i_2+m_2}(\oplus_{i_2=i_1+m_1} T_{2i_1}^{0,j_n,\ldots,j_1} \otimes X_{2m_1-1})_{2i_2} \otimes X_{2m_2-1})_{2i_3} \ldots)_{2i_{j_{n+1}}} \otimes X_{2m_{j_{n+1}}-1}\Big)_{2t}$$

where the right hand side isomorphic to

$$\oplus_{t=i_{j_{n+1}}+m_{j_{n+1}}} \cdots \oplus_{i_3=i_2+m_2} \oplus_{i_2=i_1+m_1} (\mathbb{Z}/2)^{\binom{\frac{2i_1+n-1}{2} - j_1}{\beta_{2,n}+n}} = (\mathbb{Z}/2)^D$$

and

$$D = \sum_{i_{j_{n+1}}=\beta_{1,n}+\frac{n+1}{2}+j_{n+1}-1}^{t-1} \cdots \sum_{i_2=\beta_{1,n}+\frac{n+1}{2}+1}^{i_3-1} \sum_{i_1=\beta_{1,n}+\frac{n+1}{2}}^{i_2-1} \binom{\frac{2i_1+n-1}{2}-j_1}{\beta_{2,n}+n}.$$

By 1.2(ii), we have

$$\sum_{i_1=\beta_{1,n}+\frac{n+1}{2}}^{i_2-1} \binom{\frac{2i_1+n-1}{2}-j_1}{\beta_{2,n}+n} = \binom{\frac{2i_2+n-1}{2}-j_1}{\beta_{2,n}+1+n}$$

and

$$\sum_{i_2=\beta_{1,n}+\frac{n+1}{2}+1}^{i_3-1} \binom{\frac{2i_2+n-1}{2}-j_1}{\beta_{2,n}+1+n} = \binom{\frac{2i_3+n-1}{2}-j_1}{\beta_{2,n}+2+n}.$$

Therefore we can deduce that

$$D = \sum_{i_{j_{n+1}}=\beta_{1,n}+\frac{n+1}{2}+j_{n+1}-1}^{t-1} \binom{\frac{2i_{j_{n+1}}+n-1}{2}-j_1}{\beta_{2,n}+j_{n+1}-1+n} = \binom{\frac{2t+n-1}{2}-j_1}{\beta_{2,n+1}+n} = \binom{\frac{s+(n+1)-2}{2}-j_1}{\beta_{2,n+1}+n}$$

where $s = 2t \geq 2\beta_{1,n+1}+n+1$.

□

Now we have calculated the summands of W_n^r as groups. In the next theorem we will deal with the results in [6, Lemma 3.4], together with the Künneth sequence for P_n, to explain that there is no extension problem for this Künneth sequence, for any finite number n not just for $n = 2$ and therefore the middle term of this sequence is a direct sum of the left and the right side. From this we will decompose $bu_*(P_n)$ as a direct sum of W_n^r, for $0 \leq r \leq n-1$.

Theorem 2.21. Let $n \geq 1$. Then

$$bu_*(P_n) = \bigoplus_{r=0}^{n-1} W_n^r.$$

Proof The proof is by induction on n. Let us start from the Künneth short exact sequence for P_n,

$$0 \to A_* \otimes bu_*(P_{n-1}) \to bu_*(P_n) \to T(bu_*(P_{n-1})) \to 0,$$

and consider the case $n = 3$. The case $n = 2$ was already considered in [7] and there is no extension problem for the Künneth sequence when $n = 2$ because the left hand side is non-zero only in even degrees, whereas the right side is non-zero only in odd degrees. Therefore

$$bu_*(P_2) \cong A_*^2 \oplus T(A_*) = T_*^{0,0} \oplus T_*^1 = W_2^0 \oplus W_2^1.$$

For $n = 3$, the analogous Künneth sequence has the form

$$0 \to T_*^{0,0,0} \oplus T_*^{0,1} \to bu_*(P_3) \to T_*^{1,0} \oplus T_*^2 \to 0.$$

Now 2.12 and 2.14 show that

$$(T_*^{0,0,0} \oplus T_*^{0,1})_s \cong \begin{cases} (\mathbb{Z}/2)^{\binom{t-1}{1}}, & \text{when } s = 2t,\ t \geq 2, \\ (\mathbb{Z}/2)^{\binom{t+1}{2}}, & \text{when } s = 2t+1,\ t \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Whereas 2.13 and 2.6 show that

$$(T_*^{1,0} \oplus T_*^2)_s \cong \begin{cases} (\mathbb{Z}/2)^{\binom{t}{2}}, & \text{when } s = 2t,\ t \geq 2, \\ \mathbb{Z}/2^{t-1}, & \text{when } s = 2t+1,\ t \geq 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $bu_i(P_3) = 0$ for $i < 3$, $bu_3(P_3) \cong \mathbb{Z}/2$ and for $t \geq 2$, we have exact sequences

$$0 \to (\mathbb{Z}/2)^{\binom{t-1}{1}} \to bu_{2t}(P_3) \to (\mathbb{Z}/2)^{\binom{t}{2}} \to 0,$$

and

$$0 \to (\mathbb{Z}/2)^{\binom{t+1}{2}} \to bu_{2t+1}(P_3) \to \mathbb{Z}/2^{t-1} \to 0.$$

By [6, Lemma 3.4], we have

$$bu_{2t}(P_3) \cong (\mathbb{Z}/2)^{\sum_{j=0}^{1}\binom{j}{0}\binom{t+j-1}{j+1}} = (\mathbb{Z}/2)^{\binom{t-1}{1}+\binom{t}{2}}$$

and

$$bu_{2t+1}(P_3) \cong \mathbb{Z}/2^{t-1} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{1}\binom{j}{1}\binom{t+j}{j+1}} = \mathbb{Z}/2^{t-1} \oplus (\mathbb{Z}/2)^{\binom{t+1}{2}}.$$

Thus the above calculations tell us that there are no extension problems in the Künneth sequence. Therefore

$$bu_*(P_3) \cong T_*^{0,0,0} \oplus T_*^{0,1} \oplus T_*^{1,0} \oplus T_*^2 = \bigoplus_{r=0}^{2} W_3^r,$$

where

$$W_3^0 = A_*^3 = T_*^{0,0,0},$$
$$W_3^1 = (A_* \otimes T(A_*)_*) \oplus T(A_*^2)_* = T_*^{0,1} \oplus T_*^{1,0}, \text{ and}$$
$$W_3^2 = T^2(A_*)_* = T_*^2.$$

Now, let us assume that there is no extension problem for the above Künneth sequence for $n = 2n_1$, and the statement is true in this case. And let us start again from the Künneth sequence,

$$0 \to A_* \otimes bu_*(P_{2n_1}) \to bu_*(P_{2n_1+1}) \to T(bu_*(P_{2n_1})) \to 0$$

where

$$bu_{2t}(P_{2n_1+1}) = (\mathbb{Z}/2)^{\sum_{j=0}^{2n_1-1}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{t-2n_1+j+i+1}{j+1}},$$

for $t \geq n_1 + 1$.

By 2.8, $A_* \otimes bu_*(P_{2n_1})$ is an $\mathbb{F}_2-$vector space with trivial action of u, where, by [6, Lemma 3.4], we have

$$bu_{2t+1}(P_{2n_1}) = \mathbb{Z}/2^{t-2n_1+2} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-2}\binom{j}{2i+1}\binom{t-2n_1+j+i+3}{j+1}}$$

and $\mathbb{Z}/2^{t-2n_1+2}$ comes from the only summand $T_{2t+1}^{2n_1-1}$ of $bu_*(P_{2n_1})_{2t}$ which consists of a higher $2-$torsion group with non-trivial action of u. Then, by 2.10 for $m \geq n_1$ and $\ell \geq 0$, we have

$$\begin{aligned}(A_* \otimes bu_*(P_{2n_1}))_{2t} &\cong (A_* \otimes T_*^{2n_1-1})_{2t} \oplus \oplus_{t=\ell+m+1} A_{2\ell+1} \otimes (\mathbb{Z}/2)^{\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-2}\binom{j}{2i+1}\binom{m-2n_1+j+i+3}{j+1}} \\ &= (\mathbb{Z}/2)^{t-2n_1+1} \oplus (\mathbb{Z}/2)^{\sum_{m=n_1}^{t-1}\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-2}\binom{j}{2i+1}\binom{m-2n_1+j+i+3}{j+1}}.\end{aligned}$$

In the other side of the sequence, for $t \geq n_1 + 1$, $m \geq n_1$ and $k > 0$, we have

$$T(bu_*(P_{2n_1}))_{2t} = \oplus_{t=m+k} bu_{2m}(P_{2n_1}) \otimes X_{2k-1},$$

where $X_{2k-1} = \mathbb{Z}\{u^m a_{\ell_i} : k = m + \ell_i,\text{ and } a_{\ell_i} \text{ of degree } 2\ell_i - 1\}$. Then [6, Lemma 3.4] shows that

$$\begin{aligned} T(bu_*(P_{2n_1}))_{2t} &\cong \oplus_{t=m+k}(\mathbb{Z}/2)^{\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{m-2n_1+j+i+2}{j+1}} \\ &= (\mathbb{Z}/2)^{\sum_{m=n_1}^{t-1}\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{m-2n_1+j+i+2}{j+1}}. \end{aligned}$$

Inductively on t and n_1, we can see that

$$\begin{aligned} &t - 2n_1 + 1 + \sum_{m=n_1}^{t-1}\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-2}\binom{j}{2i+1}\binom{m-2n_1+j+i+3}{j+1} \\ &+ \sum_{m=n_1}^{t-1}\sum_{j=0}^{2n_1-2}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{m-2n_1+j+i+2}{j+1} = \sum_{j=0}^{2n_1-1}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{t-2n_1+j+i+1}{j+1}. \end{aligned}$$

Therefore,

$$bu_{2t}(P_{2n_1+1}) \cong (A_* \otimes bu_*(P_{2n_1}))_{2t} \oplus T(bu_*(P_{2n_1}))_{2t}.$$

Similarly, we can deduce the same result for degree $2t+1$. This yields that there is no extension problem in the Künneth sequence for P_{2n_1+1}, so $bu_*(P_{2n_1+1}) \cong (A_* \otimes bu_*(P_{2n_1})) \oplus T(bu_*(P_{2n_1}))$. By 3.5, we have

$$\bigoplus_{r=0}^{2n_1} W^r_{2n_1+1} = \bigoplus_{r=0}^{2n_1}(A_* \otimes W^r_{2n_1} \oplus T(W^{r-1}_{2n_1})) = \bigoplus_{r=0}^{2n_1-1}(A_* \otimes W^r_{2n_1} \oplus T(W^r_{2n_1}))$$

where the right side is equal to $(A_* \otimes bu_*(P_{2n_1})) \oplus T(bu_*(P_{2n_1}))$. Thus $bu_*(P_{2n_1+1}) = \bigoplus_{r=0}^{2n_1} W^r_{2n_1+1}$.

Similarly, if we assume the result for $n = 2n_1 + 1$, a similar calculation shows that there is no non-trivial extension in the Künneth sequence for P_{2n_1+2}, that is, $bu_*(P_{2n_1+2}) \cong (A_* \otimes bu_*(P_{2n_1+1})) \oplus T(bu_*(P_{2n_1+1}))$, and again 3.5 gives the required result for $2n_1 + 2$. Thus

$$bu_*(P_{2n_1+2}) = \bigoplus_{r=0}^{2n_1+1} W^r_{2n_1+2}.$$

□

Remark 2.22. Each W^r_n has $\binom{n-1}{r}$ summands, which gives the total number of summands of $bu_*(P_n)$ to be $\sum_{r=0}^{n-1}\binom{n-1}{r} = 2^{n-1}$.

Example 2.23. For $n = 5$, we have $bu_*(P_5) = \bigoplus_{r=0}^{4} W^r_5$, where

$$\begin{aligned} W^0_5 &= T^{0,0,0,0,0}_* \\ W^1_5 &= T^{1,0,0,0}_* \oplus T^{0,1,0,0}_* \oplus T^{0,0,1,0}_* \oplus T^{0,0,0,1}_* \\ W^2_5 &= T^{2,0,0}_* \oplus T^{0,2,0}_* \oplus T^{0,0,2}_* \oplus T^{1,1,0}_* \oplus T^{1,0,1}_* \oplus T^{0,1,1}_* \\ W^3_5 &= T^{3,0}_* \oplus T^{2,1}_* \oplus T^{1,2}_* \oplus T^{0,3}_*,\text{ and} \\ W^4_5 &= T^4_*. \end{aligned}$$

Hence $bu_*(P_5)$ has $2^4 = 16$ summands. In degree $2t$, we have $W_5^0 = W_5^2 = W_5^4 = 0$, whereas

$$W_5^1 = T_{2t}^{1,0,0,0} \oplus T_{2t}^{0,1,0,0} \oplus T_{2t}^{0,0,1,0} \oplus T_{2t}^{0,0,0,1} \cong (\mathbb{Z}/2)^{3\binom{t+1}{4}+\binom{t}{3}} \text{ and}$$
$$W_5^3 \cong (\mathbb{Z}/2)^{\binom{t}{4}+\binom{t-1}{3}+\binom{t-2}{2}+\binom{t-3}{1}},$$

so $bu_{2t}(P_5) \cong (\mathbb{Z}/2)^{\sum_{j=0}^{3}\sum_{i=0}^{1}\binom{j}{2i}\binom{t+j+i-3}{j+1}}$ and this result agrees with the result in [6, Lemma 3.4]. Similarly, in degree $2t+1$ we have $W_5^1 = W_5^3 = 0$ whereas

$$W_5^0 = T_{2t+1}^{0,0,0,0,0} \cong (\mathbb{Z})^{\binom{t+2}{4}}$$
$$W_5^2 = T_{2t+1}^{2,0,0} \oplus T_{2t+1}^{0,2,0} \oplus T_{2t+1}^{0,0,2} \oplus T_{2t+1}^{1,1,0} \oplus T_{2t+1}^{1,0,1} \oplus T_{2t+1}^{0,1,1} \cong (\mathbb{Z}/2)^{3\binom{t+1}{4}+\binom{t-1}{2}+2\binom{t}{3}} \text{ and}$$
$$W_5^4 = T_{2t+1}^{4} \cong \mathbb{Z}/2^{t-3}.$$

Thus $bu_{2t+1}(P_5) \cong \mathbb{Z}/2^{t-3} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{3}\sum_{i=0}^{1}\binom{j}{2i+1}\binom{t+j+i-2}{j+1}}$ and this result also agrees with the result in [6, Lemma 3.4].

3. Tor decomposition of $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$

3.1. In 1972, Holzsager [3] split the space $\Sigma B\mathbb{Z}/p$ with p-adic coefficients into the wedge of $p-1$ spaces B_i, where B_i has homology only in dimensions $2k(p-1)+2i$, for all natural numbers k. So the spectrum $\Sigma^\infty B\mathbb{Z}/p$ splits as $\Sigma^\infty B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1}\Sigma^\infty B_i$, see also [4]. Here the spectrum B_i has stable cells in dimension $2k(p-1)+2i-\epsilon$, for $\epsilon = 0,1$ such that $2k(p-1)+2i-\epsilon \geq 0$. The splitting of $B\mathbb{Z}/p$ as a spectrum is also written as $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$.

By [5], for the case $E = lu$ the Adams summand and $X = B\mathbb{Z}/p$, we have the Thom isomorphism $lu_{q+2}(T(\xi)) \cong lu_q(B\mathbb{Z}/p)$, that is, $lu_*(T(\xi)) \cong lu_*(\Sigma^2 B\mathbb{Z}/p)$. This isomorphism is induced by a homotopy equivalence $lu \wedge T(\xi) \simeq lu \wedge \Sigma^2 B\mathbb{Z}/p$. By applying the splitting of $B\mathbb{Z}/p$ and substituting $T(\xi) = \frac{B\mathbb{Z}/p}{B^1}$ in this homotopy equivalence we get

$$lu \wedge (B_1 \vee B_2 \vee \cdots \vee B_{p-1})/(B^1) \simeq lu \wedge \Sigma^2(B_1 \vee B_2 \vee \cdots \vee B_{p-1}).$$

Both sides of the last equivalence are wedges of $p-1$ pieces, and by comparing the dimensions of bottom cells we deduce the following homotopy equivalence $lu \wedge \Sigma^2 B_i \simeq lu \wedge B_{i+1}$ for $1 \leq i < p-1$. Inductively on i, we get $lu \wedge \Sigma^{2(i-1)} B_1 \simeq lu \wedge B_i$.

It would be more interesting if we can carrying on for any prime p using the splitting $bu_p \simeq \vee_{i=1}^{p-1}\Sigma^{2i-2} lu$ and the Holzsager splitting $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$ to decompose $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$ as a direct sum of some graded groups. This decomposition agreed with the result in [6, Theorem 3.8] and both yield that there is no extension problems in the Künneth sequence for $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$.

The purpose of this section is the composition of $lu_*(B_1)^n$ first and using the above splitting to deduce the composition of $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$.

Notation 3.2.

- In order to exploit certain splittings of spectra and at the same time to simplify the writing, we will write bu for bu_p, the connective unitary K-theory with p-adic integer coefficients $\mathbb{Z}_p$, where $bu_p \simeq \vee_{i=1}^{p-1}\Sigma^{2i-2} lu$.
- Here we write A_* for $lu_*(B_1)$.

By the Atiyah-Hirzebruch spectral sequence, see [1], for $X = B_1$ and $E = lu$ we have $lu_j(B_1) = \mathbb{Z}/p^{k+1}$ when $j = 2k(p-1)+1$ and it is zero otherwise.

Example 3.3. As in [7, 2.9], for $X = B_1$, the Künneth sequence has the form

$$0 \to lu_*(B_1) \otimes_{\mathbb{Z}_p[v]} lu_*(B_1) \to lu_*(B_1 \wedge B_1) \to \mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(B_1), lu_*(B_1))[-1] \to 0.$$

So, in degree $2k(p-1)+2$, the left-hand side is the graded $\mathbb{F}_p$−vector space spanned by

$$\{v_1 \otimes v_{2k(p-1)+1}, v_{2(p-1)+1} \otimes v_{2(k-1)(p-1)+1}, \dots, v_{2k(p-1)+1} \otimes v_1\}.$$

which is concentrated in even degrees.

To calculate the graded group $\mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(B_1), lu_*(B_1))[-1]$, we can consider the following free $\mathbb{Z}_p[v]$−resolution of $lu_{2(p-1)*+1}(B_1)$

$$0 \longrightarrow \oplus_{j\geq 0}\mathbb{Z}_p[v]\langle a_{2j(p-1)+1}\rangle \xrightarrow{\ d\ } \oplus_{j\geq 0}\mathbb{Z}_p[v]\langle b_{2j(p-1)+1}\rangle \xrightarrow{\ \varepsilon\ } lu_{2(p-1)*+1}(B_1) \longrightarrow 0$$

where $\varepsilon(b_{2j(p-1)+1}) = v_{2j(p-1)+1}$ for all $j \geq 0$ and $d(a_{2j(p-1)+1}) = pb_{2j(p-1)+1} - vb_{2(j-1)(p-1)+1}$ for $j \geq 0$.

After applying $(lu_{2(p-1)*+1}(B_1) \otimes_{\mathbb{Z}_p[v]} -)$ to the above resolution, we can calculate

$$\ker(I \otimes d) = \ker(\oplus_{j\geq 0} lu_{2(p-1)*+1}(B_1)\langle a_{2j(p-1)+1}\rangle \to \oplus_{j\geq 0} lu_{2(p-1)*+1}(B_1)\langle b_{2j(p-1)+1}\rangle).$$

In degree $2k(p-1)+2$, this graded group has a generator of the form

$$v_1 \otimes a_{2k(p-1)+1} + v_{2(p-1)+1} \otimes a_{2(k-1)(p-1)+1} + \cdots + v_{2k(p-1)+1} \otimes a_1.$$

Since this generator has a summand $v_{2k(p-1)+1}$, so in degree $2k(p-1)+2$, the group $\mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_{2*+1}(B_1), lu_{2*+1}(B_1))$ is

$$\mathbb{Z}/p^k\langle v_1 \otimes a_{2k(p-1)+1} + v_{2(p-1)+1} \otimes a_{2(k-1)(p-1)+1} + \cdots + v_{2k(p-1)+1} \otimes a_1\rangle.$$

So $\mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_{2*+1}(B_1), lu_{2*+1}(B_1))[-1]$, in degree $2k(p-1)+3$, is the finite cyclic group of order p^k. this group is concentrated in odd degrees. So the middle group $lu_*(B_1 \wedge B_1)$ in any given degree is isomorphic to the one on the left or the one on the right side.

By applying $T(A_*)_* \otimes_{\mathbb{Z}_p[v]} -$ instead of $A_* \otimes_{\mathbb{Z}_p[v]} -$ to the previous free resolution of A_* with shifting by (-1) and by using induction on n, we can calculate the graded group T^n_*. This is non-zero just in degrees $2t(p-1)+2n+1$.

Proposition 3.4. For $n, t \geq 0$,

$$T^n_{2t(p-1)+2n+1} \cong \mathbb{Z}/p^{t+1}\langle \sum_{i+\sum_{k=1}^n j_k = t} v_{2i(p-1)+1} a_{j_1} a_{j_2} \dots a_{j_n}\rangle,$$

where a_{j_k} is in degree $2j_k(p-1)+1$.

Definition 3.5. Let $0 \leq k \leq n-1$, we define the weight k iterated T as

$$W^k_n = \bigoplus_{\sum j_i = k} T^{j_{n-k}, j_{n-k-1}, \dots, j_1}_*$$

where $j_i \in \mathbb{N}_0$, and $T^{j_{n-k}, j_{n-k-1}, \dots, j_1}$ as in 2.3.

Lemma 3.6. Let $0 \leq k \leq n-1$. With the exceptions of A_* and T^n_*, each summand of W^k_n is a graded $\mathbb{F}_p$−vector space, on which v acts trivially.

By all the previous calculations, we can deduce the next result, which is about the calculation of $T^{j_n, j_{n-1}, \dots, j_1}_*$ as a graded group, which is non-zero just in degrees $2t(p-1)+2\beta_{1,n}+n$ for $t \geq 0$, and again using this to calculate W^r_n for any $0 \leq r \leq n-1$.

Proposition 3.7. Let $n > 1$. Then, using the notation of 2.16,

$$T^{j_n, j_{n-1}, \dots, j_1}_{2t(p-1)+2\beta_{1,n}+n} \cong (\mathbb{Z}/p)^{\binom{t+\beta_{2,n}+n-1}{\beta_{2,n}+n-1}}.$$

For $p = 2$, we can get the same result in 2.20. By 3.7, if $j_k = 0$ for all $k = 1, 2, \dots n$, we can calculate the graded group A^n_* which is non-zero just in degrees $2t(p-1)+n$

$$A^n_{2t(p-1)+n} = T_{2t(p-1)+n}^{\overbrace{0,0,\dots,0}^{n\ times}} \cong (\mathbb{Z}/p)^{\binom{t+n-1}{n-1}}.$$

This result also agree with 2.12 when $p = 2$.

By the splitting $bu \simeq \vee_{i=1}^{p-1}\Sigma^{2i-2}lu$, the Holzsager splitting $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$ and $lu \wedge \Sigma^{2(i-1)} B_1 \simeq lu \wedge B_i$ we have

$$bu \wedge \overbrace{B\mathbb{Z}/p \wedge B\mathbb{Z}/p \wedge \cdots \wedge B\mathbb{Z}/p}^{n\ times} \simeq \bigvee_{i_1,i_2,\dots,i_{n+1}=0}^{p-2} \sum^{2\Sigma_{k=1}^{n+1} i_k} lu \wedge \overbrace{B_1 \wedge B_1 \wedge \cdots \wedge B_1}^{n\ times}.$$

Applying the homotopy group π_* we get that

$$bu_*(B\mathbb{Z}/p)^{\wedge n} \cong \bigoplus_{i_1,i_2,\dots,i_{n+1}=0}^{p-2} lu_{*-2\Sigma_{k=1}^{n+1} i_k}(B_1)^{\wedge n}.$$

Again we have calculated the summands of W^r_n for $lu_*(B_1)$ as graded groups. In the next theorem we will deal with the results in [6, 3.8], together with the Künneth sequence for $lu_*(B_1)$ and using the above discutient to explain that there is no extension problem for this Künneth sequence for $bu_*(B\mathbb{Z}/p)^{\wedge n}$, for any finite number n, and decompose $bu_*(B\mathbb{Z}/p)^{\wedge n}$ as a direct sum of some graded groups. The proof is similar to 3.8, so it is enough to consider some spacial cases as examples.

Theorem 3.8. Let $n \geq 1$. Then

$$bu_*(B\mathbb{Z}/p)^{\wedge n} = \bigoplus_{i_1,i_2,\dots,i_{n+1}=0}^{p-2} \bigoplus_{r=0}^{n-1} W^r_n$$

where $W^r_n = \bigoplus_{\Sigma j_i = r} T^{j_{n-r}, j_{n-r-1},\dots,j_1}_{*-2\Sigma_{k=1}^{n+1} i_k}$.

Example 3.9. For $n = p = 3$, $bu_9(B\mathbb{Z}/3)^{\wedge 3} = \bigoplus_{i_1,i_2,i_3,i_4=0}^{1} \bigoplus_{r=0}^{2} W^r_3$ where $W^r_3 = \bigoplus_{\Sigma j_i = r} T^{j_{3-r}, j_{2-r},\dots,j_1}_{9-2\Sigma_{k=1}^{4} i_k}$. By 3.4 and 3.7 we have $bu_9(B\mathbb{Z}/3)^{\wedge 3} = T^2_9 \oplus (T^{0,0,0}_7)^4 \oplus (T^2_5)^6 \oplus (T^{0,0,0}_3)^4 = \mathbb{Z}/3^2 \oplus (\mathbb{Z}/3)^{22}$. And by [6, 3.8], we have $bu_9(B\mathbb{Z}/3)^{\wedge 3} = \Gamma(k,4) \oplus (\mathbb{Z}/3)^{\sum_{j=0}^{1} \sum_{\lambda_1,\lambda_2,\dots,\lambda_{2-j}=0}^{1} \binom{j}{1}\binom{4+j-\Sigma_{a=1}^{2-j}\lambda_a}{j+1}} = \mathbb{Z}/3^2 \oplus (\mathbb{Z}/3)^{22}$, where $\Gamma(k,4) = \mathbb{Z}/3^2 \oplus (\mathbb{Z}/3)^6$ and $(\mathbb{Z}/3)^{\sum_{j=0}^{1} \sum_{\lambda_1,\lambda_2,\dots,\lambda_{2-j}=0}^{1} \binom{j}{1}\binom{4+j-\Sigma_{a=1}^{2-j}\lambda_a}{j+1}} = (\mathbb{Z}/3)^{16}$.

References

[1] J.F. Adams: *Stable Homotopy and Generalised Homology*; University of Chicago Press (1974).

[2] L. Graham, E.Knuth and O. Patashnik: Concrete mathematics; Addison-Wesley Publishing Company (1994).

[3] Holzsager,Richard: Stable splitting of $K(G,1)$; Proc. Amer. Math. Soc. Proceedings of the American Mathematical Society. vol 31 (1972) 305–306.

[4] E.Ossa: Connective K-theory of elementary abelian groups; Transformation groups (Osaka, 1987). Lecture Notes in Math. vol 1375, 269–275. Springer,Berlin (1987).

[5] R. Switzer: Algebraic topology—homotopy and homology; Classics in Mathematics, (note: Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)]). Springer-Verlag, Berlin (2002) xiv+526.

[6] Mira, Khairia M: $bu_*(B\mathbb{Z}/p)^n$ as a Graded Group; Journal of Progressive Research in Mathematics(JRM) ISSN: 2395-0218, SCITECH RESEARCH ORGANISATION, , 1981-1988, Volume 12, Issue 3 available at www.scitecresearch.com/journals/index.php/jprm.

[7] Bruner, Robert R. and Mira, Khairia M. and Stanley, Laura A. and Snaith, Victor P: Ossa's theorem via the Künneth formula; Mathematics and Statistics 3(3): 58-64, 2015, DOI: 10.13189/ms.2015.030302, http://www.hrpub.org/download/20150620/MS2-13403792.pdf