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Existence and uniqueness of solutions of Robin's problem for the anisotropic hyperbolic heat equation with non regular data

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Abstract
We find existence and uniqueness results about solutions of Robin's problem for the general anisotropic hyperbolic heat equation in the case of infinitely differentiable coefficients but irregular distributions data for the internal heat sources and boundary and initial conditions.

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1 Introduction and physical motivation

Hyperbolic heat conduction equation is a fundamental tool in some modern industrial applications such as microelectronics and the processing of materials by irradiation with a laser beam of high intensity and very short application times (see [3], [4] and [9] for instance). Usually the mathematical formulation of these problems leads to the study of mixed boundary Neumann problems with boundary conditions given by irregular distributions such as Heaviside's function or Dirac's δ distribution. From the mathematical point of view it is more interesting to consider the general Robin's problem in order to cover simultaneously Dirichlet and Neumann problems.

Real industrial materials frequently are non isotropic (see [10] for instance for some concrete examples). In this case and assuming the density ρ and the specific heat c to be constant the hyperbolic heat equation in the open set Ω occupied by the body is (see [2])

$$-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}k_{ij}(\mathbf{x})\,\frac{\partial T}{\partial x_j}(\mathbf{x},t)\right)+\rho\,c\left(\frac{\partial T}{\partial t}(\mathbf{x},t)+\tau\,\frac{\partial^2 T}{\partial t^2}(\mathbf{x},t)\right)=$$

$$=\rho\left(S(\mathbf{x},t)+\tau\,\frac{\partial S}{\partial t}(\mathbf{x},t)\right), \tag{1}$$

where $T(\mathbf{x},t)$ is the temperature in point $\mathbf{x}$ and instant t, $(k_{ij}(\mathbf{x}))$ is the symmetric thermal conductivity tensor of the material, τ is the relaxation parameter and $S(\mathbf{x},t)$ denotes the internal heat sources in the body. If $k_{ij}(\mathbf{x})=k$ for every $1\le i,j\le 3$ we obtain the isotropic hyperbolic heat equation for an homogeneous body. Moreover the preservation of the second low of thermodynamics implies that

$$\forall\,\mathbf{x}\in\overline{\Omega}\quad\forall\,(\eta_1,\eta_2,\eta_3)\neq 0\qquad\sum_{i,j=1}^{3}k_{ij}(\mathbf{x})\,\eta_i\,\eta_j>0,$$

which means that the differential operator in (1) is strongly elliptic in $\overline{\Omega}$.

The study of these problems in its full generality is very ambitious and it is expected that a rigorous mathematical treatment will be quite difficult and complex. Some steps have been developed previously. The existence, uniqueness and regularity of solutions of mixed boundary value problems in the case of regular data has been studied in [6]. In [7] the traces of the elements of certain Banach spaces of "irregular functions" containing the solutions of the anisotropic hyperbolic heat equation with irregular data are considered.

In this paper a further step is developed. Specifically the purpose of this paper *is to find existence and uniqueness theorems about solutions of Robin's problem for the anisotropic hyperbolic heat equation in the case of infinitely differentiable coefficients up to the closure* $\overline{\Omega}$ *of the spatial domain in (1) but with non regular data distributions in its right side and in the boundary and initial conditions of the problem* as a necessary step toward future more involved investigations.

Notation is standard in general. Points $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ are denoted by $\mathbf{x}$ in short. We deal with cylindrical open sets $\Omega\times]0,T[\subset \mathbb{R}^{n+1}$ where $T>0$ determines the temporal interval of the problem. Concerning spatial domains, unless otherwise is clearly stated, *all open domains* $\Omega\subset\mathbb{R}^n$, $n>1$ *considered in this paper will be always bounded and having boundary* $\partial\Omega$ *being a* $\mathcal{C}^\infty$ *manifold of dimension* $n-1$ *such that* Ω *is locally on the same side of* $\partial\Omega$. Traces on any type of boundary of $\Omega\times]0,T[$ are always understood as suitable standard extensions of the ordinary restriction map to the boundary of functions $f\in\mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ (details can be found in [5] and [7]). All used functions and vector spaces are assumed to be real. Basic facts about interpolation spaces $[X,Y]_\theta$ necessary to define Sobolev spaces $H^r(\Omega)$, $H^{r,s}(\Omega\times]0,T[)$ and $H^{r,s}_{0,0}(\Omega\times]0,T[)$ with $r,s\in[0,\infty[$ and the main properties of all these spaces can be found for instance in [5].

The natural framework to study (1) is to consider a general operator

$$\mathcal{A}:=\mathcal{X}+\lambda\frac{\partial}{\partial t}+\eta\,\frac{\partial^2}{\partial t^2}\quad \lambda\neq 0,\ \eta\neq 0$$

where

$$\mathcal{X}:=\sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|\leq 1}(-1)^{|\boldsymbol{\alpha}|}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\,\mathbf{x}^{\boldsymbol{\alpha}}}\left(a_{\boldsymbol{\alpha\beta}}(\mathbf{x})\,\frac{\partial^{|\boldsymbol{\beta}|}}{\partial\,\mathbf{x}^{\boldsymbol{\beta}}}\right)$$

is a selfadjoint strongly elliptic operator in $\overline{\Omega}$ with real coefficients $a_{\boldsymbol{\alpha\beta}}(\mathbf{x})\in\mathcal{C}^\infty(\overline{\Omega})$. To cover the desired generality we shall consider differential boundary operators of order $o_{\mathfrak{B}}\leq 1$ of type

$$\mathfrak{B}: v(\mathbf{x})\longrightarrow\mathfrak{B}(v(\mathbf{x})):=f(\mathbf{x})\,Tr(v(\mathbf{x}))+\sum_{i=1}^{n}g_i(\mathbf{x})\,Tr\Big(\frac{\partial v}{\partial x_i}(\mathbf{x})\Big)\tag{2}$$

with real coefficients $f(\mathbf{x}), g_j(\mathbf{x}), 1\leq j\leq n$ in $\mathcal{C}^\infty(\overline{\partial\Omega})$ and where Tr denotes the trace operator on the boundary $\partial\Omega$). Usually, to simplify, traces of distributions $v(\mathbf{x},t)$ will be denoted by the same symbol but making advise of the variability of $(\mathbf{x},t)$. For example, $v(\mathbf{x},0)$ will denote the trace on the section $\big\{(\mathbf{x},0)\mid\mathbf{x}\in\Omega\big\}$.

The formal adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is the operator

$$\mathcal{A}^*:=\mathcal{X}-\lambda\frac{\partial}{\partial t}+\eta\,\frac{\partial^2}{\partial t^2}.$$

Proposition 1 ***Green's formula.*** *There are boundary differential operators* $\mathfrak{R}_{\mathcal{A}}$, $\mathfrak{U}_{\mathcal{A}}$ *and* $\mathfrak{V}_{\mathcal{A}}$ *of type (2) of orders* $o_{\mathfrak{R}}, o_{\mathfrak{U}}$ *and* $o_{\mathfrak{V}}$ *such that* $o_{\mathfrak{B}}+o_{\mathfrak{U}}=1$, $o_{\mathfrak{B}}+o_{\mathfrak{R}}=1$, $o_{\mathfrak{U}}+o_{\mathfrak{V}}=1$ *and Green's formula*

$$\int_0^T\left(\int_\Omega\left(v\,\mathcal{A}(u)-u\,\mathcal{A}^*(v)\right)\,d\mathbf{x}\right)\,dt=$$
$$=\int_0^T\left(\int_{\partial\Omega}\left(\mathfrak{U}_{\mathcal{A}}(u)\,\mathfrak{V}_{\mathcal{A}}(v)-\mathfrak{R}_{\mathcal{A}}(v)\,\mathfrak{B}(u)\right).\,d\sigma\right)\,dt+$$
$$+\lambda\left(\int_\Omega\Big[v(\boldsymbol{x},t)u(\boldsymbol{x},t)\Big]_{t=0}^{t=T}\,d\boldsymbol{x}\right)+\eta\Big(\int_\Omega\Big[v(\boldsymbol{x},t)\frac{\partial u}{\partial t}(\boldsymbol{x},t)-u(\boldsymbol{x},t)\frac{\partial v}{\partial t}(\boldsymbol{x},t)\Big]_{t=0}^{t=T}\Big)\tag{3}$$

holds for every $v\in H^{s,s}(\Omega\times]0,T[)$, $s\geq 2$ *and* $u\in\mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ *(the boxes have the usual meaning for the Barrow's rule of integration).*

Proof. If $v\in\mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ the result is straightforward from Aronszajn-Milman's theorem [1] or $\big[$ [5], chapter 2, theorem 2.1 $\big]$, Fubini's theorem, and two consecutive integration by parts with respect to t. By density of $\mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ in $H^{s,s}(\Omega\times]0,T[)$, $s\geq 2$ ($\big[$[5], chapter 4, remark 2.2 $\big]$) the result follows easily for $v\in H^{s,s}(\Omega\times]0,T[)$, $s\geq 2$. ∎

We need to consider some new spaces in order to precise the degree of irregularity of the data. Let $\rho : \Omega \longrightarrow [0, \infty[$ be the continuous function defined by $\rho(\mathbf{x}) := d(\mathbf{x}, \partial\Omega) := \inf_{\mathbf{y}\in\partial\Omega} \|\mathbf{x}-\mathbf{y}\|$ for every $\mathbf{x} \in \Omega$. Given $s \in \mathbb{N} \cup \{0\}$ we define

$$\Sigma^s(\Omega) := \Big\{ f \in L^2(\Omega) \;\Big|\; \|f\|_{\Sigma^s(\Omega)} := \Big(\sum_{|\boldsymbol{\alpha}|\leq s} \Big\| \rho^{|\boldsymbol{\alpha}|} \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \Big\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} < \infty \Big\}.$$

$\Sigma^s(\Omega)$ endowed with the norm $\|.\|_{\Sigma^s(\Omega)}$ turns out to be a Banach space. We extend the latter definition to the case $s \in]0, \infty[$ by complex interpolation defining $\Sigma^s(\Omega) = \big[\Sigma^{E[r]+1}(\Omega), \Sigma^{E[r]+1}(\Omega)\big]_{1-\theta}$ for $s = E[s] + \theta$, $0 < \theta < 1$, where $E[s]$ is the integral part of s, endowed with any canonical norm of the interpolated space. The dual space will be represented by $\Sigma^{-s}(\Omega)$. Clearly

$$\forall\, s > 0 \quad H^s(\Omega) \subset \Sigma^s(\Omega) \tag{4}$$

with continuity. As above, it can be shown (see [5], chapter 2) that $\mathcal{D}(\Omega)$ is dense in $\Sigma^r(\Omega)$ for $r \geq 0$. In consequence $\Sigma^{-r}(\Omega)$ is a space of distributions on Ω, $\Sigma^r(\Omega) \subset L^2(\Omega) \subset \Sigma^{-r}(\Omega)$ becomes a Gelfand triple and $\mathcal{D}(\Omega)$ is dense in $\Sigma^{-r}(\Omega)$ too.

In order to distinguish the behavior of temporal and spatial variables we introduce another space. Given $0 < T$ we fix a number $T_0 < \frac{T}{2}$. Consider the function $\varphi_{T_0,T} \in \mathcal{C}^\infty(\mathbb{R})$ with compact support contained in $[0, T]$ defined by

$$\varphi_{T_0,T}(t) := \begin{cases} e^{-\frac{T_0^2}{T_0^2-(x-T_0)^2}} & \text{if } 0 < x \leq T_0 \\ \frac{1}{e} & \text{if } T_0 \leq x \leq T - T_0 \\ e^{-\frac{T_0^2}{T_0^2-(x-T+T_0)^2}} & \text{if } T - T_0 \leq x < T \\ 0 & \text{if } x \in]-\infty, 0] \cup [T, \infty[\end{cases}$$

For every $s \in \mathbb{N} \cup \{0\}$ we define

$$\Sigma^{s,s}(\Omega\times]0, T[) :=$$
$$= \Big\{ f \in L^2(]0, T[, \Sigma^s) \Big|\; \|f\|_{\Sigma^{s,s}(\Omega\times]0,T[)} := \Big(\sum_{j=0}^{s} \Big\| |\varphi_{T_0 T}(t)|^j \frac{\partial^j f}{\partial t^j} \Big\|^2_{L^2(]0,T[,\Sigma^{s-j}(\Omega))} \Big)^{\frac{1}{2}} < \infty \Big\}$$

which turns out to be a Banach space under the norm $\|f\|_{\Sigma^{s,s}(\Omega\times]0,T[)}$.

As above, the definition is extended to non negative real numbers by interpolation

$$\forall\, s \geq 0 \quad \Sigma^{s,s}(\Omega\times]0, T[) := \Big[\Sigma^{E[s]+1,E[s]+1}(\Omega\times]0, T[), \Sigma^{E[s],E[s]}(\Omega\times]0, T[)\Big]_{1-\theta}$$

endowed with any standard norm for the interpolated space. The dual space is denoted by $\Sigma^{-s,-s}(\Omega\times]0, T[)$. From (4) we obtain easily

$$\forall\, s \geq 0 \quad H^{s,s}(\Omega\times]0, T]) \subset \Sigma^{s,s}(\Omega\times]0, T[). \tag{5}$$

By $\big[$[5], chapter 4, proposition 9.1$\big]$ $\mathcal{D}(\Omega\times]0, T[)$ is also dense in $\Sigma^{s,s}(\Omega\times]0, T[)$ if $s \geq 0$ and hence $\Sigma^{-s,-s}(\Omega\times]0, T[) \subset \mathcal{D}'(\Omega\times]0, \infty[)$. As above,

$$\Sigma^{s,s}(\Omega\times]0, T[) \subset L^2(\Omega\times]0, T[) \subset \Sigma^{-s,-s}(\Omega\times]0, T[)$$

is another Gelfand triple and $\mathcal{D}(\Omega\times]0, T[)$ is dense in $\Sigma^{-s,-s}(\Omega\times]0, T[)$.

Concerning to spaces over the boundary $\partial\Omega\times]0, T[$, if $k \in \mathbb{N} \cup \{0\}$ we define

$$J^k(\partial\Omega\times]0, T[) := \Big\{ v \;\Big|\; \|v\|_{J^k(\partial\Omega\times]0,T[)} = \Big(\sum_{j=0}^{k} \Big\| \varphi^j_{T_0 T} \frac{\partial^j v}{\partial t^j} \Big\|^2_{L^2(]0,T[,H^{k-\frac{j}{k}}(\partial\Omega))} \Big)^{\frac{1}{2}} < \infty \Big\}.$$

$J^k(\partial\Omega\times]0, T[)$ becomes a Banach space when endowed with the norm $\|.\|_{J^k(\partial\Omega\times]0,T[)}$. This definition is extended by interpolation to the case of $s \in]0, \infty[\setminus\mathbb{N}$ putting

$$J^s := [H^{E[s]+1}(\partial\Omega\times]0, T[), L^2(\partial\Omega\times]0, T[)]_{1-\frac{s}{E[s]+1}}.$$

Clearly

$$\forall\, s \geq 0 \quad H^{s+\frac{1}{2},s+\frac{1}{2}}(\partial\Omega\times]0,T[) \subset J^{s+\frac{1}{2}}(\partial\Omega\times]0,T[). \tag{6}$$

By $\big[$[5], chapter 5, section 10.3$\big]$ we see that $\mathcal{D}(\partial\Omega\times]0,T[)$ is dense in $J^s(\partial\Omega\times]0,T[)$ for every $s \geq 0$. If $J^{-s}(\partial\Omega\times]0,T[) := (J^s(\partial\Omega\times]0,T[)'$ we obtain the Gelfand triple $J^r(\partial\Omega\times]0,T[) \subset L^2(\partial\Omega\times]0,T[) \subset J^{-r}(\partial\Omega\times]0,T[)$.

To finish, given $s \geq 1$ consider also the space of distributions

$$D_{\mathcal{A}}^{-(s-1)}(\Omega\times]0,T[) := \Big\{u \in H^{-(s-1),-s}(\Omega\times]0,T[) \;\mid\; \mathcal{A}(u) \in \Sigma^{-(s+1),-(s+1)}(\Omega\times]0,T[)\Big\}$$

endowed with the norm

$$\|u\|_{D_{\mathcal{A}}^{-(s-1)}(\Omega\times]0,T[)} = \|u\|_{H^{-(s-1),-s}(\Omega\times]0,T[)} + \|\mathcal{A}(u)\|_{\Sigma^{-(s+1),-(s+1)}(\Omega\times]0,T[)}. \tag{7}$$

Then $D_{\mathcal{A}}^{-(s-1)}(\Omega\times]0,T[)$ turns out to be a Banach space. The closure of $\mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ in $D_{\mathcal{A}}^{-(s-1)}(\Omega\times]0,T[)$ will be denoted by $\mathbf{D}_{\mathcal{A}}^{-(s-1)}(\Omega\times]0,T[)$.

2 Existence and uniqueness theorems for solutions in the case of irregular data

We begin recalling an important result in [6] which we shall need to use in the sequel. Put $H^{-1,0}(\Omega\times]0,T[) := L^2(\Omega\times]0,T[)$. Given $r \in \mathbb{N}\cup\{0\}$, let $\mathcal{W}^{r-1,r}(\Omega\times]0,T[)$ be the subspace of the functions $g \in H^{r-1,r}(\Omega\times]0,T[)$ such that $\frac{\partial^k g}{\partial t^k}(\mathbf{x},0) = 0$ in Ω for every $0 \leq k \leq r-1$ if $r > 0$ (hence if $r = 0$ there is no condition on g). Given an operator $\mathfrak{H}$ of the same type as $\mathfrak{B}$ we define $H^{r+1,r+1}_{\mathcal{A},\mathfrak{H}}(\Omega\times]0,T[)$ as the set of functions $g \in H^{r+1,r+1}(\Omega\times]0,T[)$ such that $\mathcal{A}(g) \in \mathcal{W}^{r-1,r}(\Omega\times]0,T[)$ in $\Omega\times]0,T[$,

$$g(\mathbf{x},0) = \frac{\partial g}{\partial t}(\mathbf{x},0) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \mathfrak{H}(g)(\mathbf{x},t) = 0 \quad \text{in} \quad \partial\Omega\times]0,T[.$$

Next theorem has been proved in $\big[$[6], theorem 6$\big]$:

Theorem 2 *If $r \geq 1$ the operator $\mathcal{A}$ restricted to $H^{r+1,r+1}_{\mathcal{A}\mathfrak{H}}(\Omega\times]0,T[)$ is an isomorphism onto $\mathcal{W}^{r-1,r}(\Omega\times]0,T[)$.*

Corollary 3 *Assume $F_0 \in \mathcal{D}(\Omega\times]0,T[)$, $F_1 \in \mathcal{D}(\partial\Omega\times]0,T[)$, $F_2 \in \mathcal{D}(\Omega)$ and $F_3 \in \mathcal{D}(\Omega)$. Then there exists an unique function $U(\mathbf{x},t) \in \mathcal{C}^\infty(\overline{\Omega\times]0,T[})$ such that*

$$\mathcal{A}(U) = F_0 \quad in \quad \Omega\times]0,T[, \qquad \mathfrak{B}(U)(\mathbf{x},t) = F_1 \quad in \quad \partial\Omega\times]0,T[), \tag{8}$$

$$U(\mathbf{x},0) = F_2 \qquad \frac{\partial U}{\partial t}(\mathbf{x},0) = F_3 \quad in \quad \Omega. \tag{9}$$

Proof. Let $r \in \mathbb{N}$. By theorem 2 there is $U_r \in H^{r+1,r+1}(\Omega\times]0,T[)$ verifying (8) and (9). As $H^{r+2,r+2}(\Omega\times]0,T[) \subset H^{r+1,r+1}(\Omega\times]0,T[)$ by the same theorem 2 we will have $U_r = U_{r+1}$ and by the arbitrariness of r in $\mathbb{N}$ the function $U = U_r$ is independent of r and $U \in \bigcap_{r=1}^{\infty} H^{r+1,r+1}(\Omega\times]0,T[)$ and the conclusion follows. ■

For our purposes we need to consider the following space, whose importance will be clear after next theorem 5.

Definition 4 *Let $r \geq 0$ be such that $r - \frac{1}{2} \notin \mathbb{N}\cup\{0\}$. We define $X^r(\Omega\times]0,T[)$ as the set of functions $v \in H^{r+1,r+1}(\Omega\times]0,T[)$ such that*

$$\mathfrak{V}_{\mathcal{A}}(v)(\mathbf{x},t) = 0 \quad in \ \partial\Omega\times]0,T[,$$

$$v(\mathbf{x},T) = \frac{\partial v}{\partial t}(\mathbf{x},T) = 0 \quad in \quad \Omega \quad and \quad \mathcal{A}^*(v) \in H^{r-1,r}_{0,0}(\Omega\times]0,T[),$$

provided with the norm induced by $H^{r+1,r+1}(\Omega\times]0,T[)$.

Theorem 5 *Let $r \geq 1$ be such that $r - \frac{1}{2} \notin \mathbb{N}$. Then $\mathcal{A}^*$ is an isomorphism from $H_{0,0}^{r-1,r}(\Omega\times]0,T[)$ onto $X^r(\Omega\times]0,T[)$.*

Proof. Assume first $r \in \mathbb{N}$. Clearly $H_{0,0}^{r-1,r}(\Omega\times]0,T[) \subset \mathcal{W}^{r-1,r}(\Omega\times]0,T[)$ and the variable change $t = T - t'$ transform the operator $\mathcal{A}^*$ defined in $\Omega\times]0,T[$ in the operator $\mathcal{A}$ defined in $\Omega\times]0,T[$. Then $\mathcal{A}^* : H_{0,0}^{r-1,r}(\Omega\times]0,T[) \longrightarrow X^r(\Omega\times]0,T[)$ is a bijective map as a consequence of theorem 2 applied to the operator $\mathcal{A}$ with respect to the temporal variable t'. On the other hand it is clear that $\mathcal{A}^*$ is continuous from $X^r(\Omega\times]0,T[)$ into $H_{0,0}^{r-1,r-1}(\Omega\times]0,T[)$. Let $\{v_m\}_{m=1}^{\infty} \subset X^r(\Omega\times]0,T[)$ be a sequence such that $\lim_{m\to\infty} v_m = v$ in $H^{r+1,r+1}(\Omega\times]0,T[)$ and $\lim_{m\to\infty} \mathcal{A}^*(v_m) = \overline{v}$ in $H^{r-1,r}(\Omega\times]0,T[)$. Then $\lim_{n\to\infty} \mathcal{A}^*(v_m) = \overline{v}$ in $H_{0,0}^{r-1,r-1}(\Omega\times]0,T[)$. Since $\lim_{m\to\infty} \mathcal{A}^*(v_m) = \mathcal{A}^*(v)$ in $H^{r-1,r-1}(\Omega\times]0,T[)$ we obtain $\mathcal{A}^*(v) = \overline{v} \in H^{r-1,r}_{0,0}(\Omega\times]0,T[)$. Then $\mathcal{A}^*$ has closed graph and by the closed graph theorem it is continuous and by the open map theorem $\mathcal{A}^*$ becomes an isomorphism indeed.

The result for arbitrary $r \geq 1$ follows easily by complex interpolation. ■

Following theorem is the main result of this section.

Theorem 6 ***Existence and uniqueness of the solutions in the case of irregular data.*** *Let $r \geq 1$ verifying the conditions*

$$r + \frac{1}{2} - \left(\frac{i+1}{2}\right)\left(\frac{r+1}{r+2}\right) \notin \mathbb{Z}, \quad i = 0, 1 \tag{10}$$

and

$$\forall\, 0 \leq k \leq r \quad r + \frac{3}{2} - \left(\frac{r+2}{r+1}\right)\left(k + \frac{1}{2}\right) \notin \mathbb{Z}. \tag{11}$$

Assume that

$$F_0 \in \Sigma^{-(r+1),-(r+1)}(\Omega\times]0,T[), \quad F_1 \in J^{-(r+\frac{1}{2}-o_{\mathfrak{B}})}(\partial\Omega\times]0,T[), \tag{12}$$

$$F_2 \in \Sigma^{-(r-\frac{1}{2})}(\Omega) \quad \text{and} \quad F_3 \in \Sigma^{-(r+\frac{1}{2})}(\Omega). \tag{13}$$

There exist a unique distribution $U(\mathbf{x},t) \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega\times]0,T[)$ such that

$$\mathcal{A}(U) = F_0 \quad \text{in} \quad \Omega\times]0,T[, \qquad \mathfrak{B}(U) = F_1 \quad \text{in} \quad \partial\Omega\times]0,T[) \tag{14}$$

and

$$U(\mathbf{x},0) = F_2 \quad \text{and} \quad \frac{\partial U}{\partial t}(\mathbf{x},0) = F_3 \quad \text{in} \quad \Omega \tag{15}$$

where the traces are computed in $H^{-(r+\frac{1}{2}-o_{\mathfrak{B}}),-(r+\frac{1}{2}-o_{\mathfrak{B}})\frac{r+2}{r+1}}(\partial\Omega\times]0,T[)$, $H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$ and $H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega)$ respectively.

Proof. By definition of $X^r(\Omega\times]0,T[)$ and the theorem of traces $\big[$ [5], chapter 4, theorem 2.1$\big]$ in $H^{r+1,r+1}(\Omega\times]0,T[)$ the maps

$$u \in X^r(\Omega\times]0,T[) \longrightarrow u(\mathbf{x},0) \in H^{r+\frac{1}{2}}(\Omega),$$

$$u \in X^r(\Omega\times]0,T[) \longrightarrow \frac{\partial u}{\partial t}(\mathbf{x},0) \in H^{r-\frac{1}{2}}(\Omega)$$

and

$$u \in X^r(\Omega\times]0,T[) \longrightarrow \mathfrak{R}_{\mathcal{A}}(u)\big|_{\partial\Omega\times]0,T[}(\mathbf{x},t) \in H^{r+\frac{1}{2}-o_{\mathfrak{B}},r+\frac{1}{2}-o_{\mathfrak{B}}}(\partial\Omega\times]0,T[)$$

are well defined and continuous. Then by (5), (6), the continuity of trace mappings and the definition of the topology in $X^r(\Omega\times]0,T[)$ the linear form L defined for $v \in X^r(\Omega\times]0,T[)$ by

$$L(v) = \langle F_0, v\rangle + \langle F_1, \mathfrak{R}_{\mathcal{A}}(v(\mathbf{x},t))\rangle + \Big\langle F_2, \Big(\lambda\, v(\mathbf{x},0) - \eta \frac{\partial v}{\partial t}(\mathbf{x},0)\Big)\Big\rangle + \eta\Big\langle F_3, v(\mathbf{x},0)\Big\rangle$$

is continuous. Since $\mathcal{A}^*$ is an isomorphism from $X^r(\Omega\times]0,T[)$ onto $H_{0,0}^{r-1,r}(\Omega\times]0,T[)$ (theorem 5) there is an unique $U := ((\mathcal{A}^*)')^{-1}(L) \in H^{-(r-1),-r}(\Omega\times]0,T[)$ such that

$$\forall\, v \in X^r(\Omega\times]0,T[) \quad \langle U, \mathcal{A}^*(v)\rangle = \langle L, v\rangle \tag{16}$$

holds. As $\mathcal{D}(\Omega\times]0,T[) \subset X^r(\Omega\times]0,T[)$, by definition of L we obtain

$$\forall\, v \in \mathcal{D}(\Omega\times]0,T[) \quad \langle F_0, v\rangle = \langle L, v\rangle = \langle U, \mathcal{A}^*(v)\rangle = \langle \mathcal{A}(U), v\rangle. \tag{17}$$

Since $\mathcal{D}(\Omega\times]0,T[)$ is dense in $H_{0,0}^{r-1,r}(\Omega\times]0,T[)$ and in $\Sigma^{r+1,r+1}(\Omega\times]0,T[)$ we obtain

$$\mathcal{A}(U) = F_0 \tag{18}$$

and U becomes a solution of the given equation.

Let us see that $U \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega\times]0,T[)$. By density of $\mathcal{D}(\Omega\times]0,T[)$, $\mathcal{D}(\partial\Omega\times]0,T[)$ and $\mathcal{D}(\Omega)$ in the corresponding next spaces, we find sequences $\{f_k\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega\times]0,T[)$, $\{g_k\}_{k=1}^{\infty} \subset \mathcal{D}(\partial\Omega\times]0,T[)$, $\{U_{2k}\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega)$ and $\{U_{3k}\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega)$ such that

$$\lim_{k\to\infty} f_k = F_0 \quad \text{in } \Sigma^{-(r+1),-(r+1)}(\Omega\times]0,T[) \tag{19}$$

$$\lim_{k\to\infty} g_k = F_1 \quad \text{in } J^{-\left(r+\frac{1}{2}-o_{\mathfrak{B}}\right)}(\partial\Omega\times]0,T[) \tag{20}$$

$$\lim_{k\to\infty} U_{2k} = F_2 \quad \text{in } \Sigma^{-(r-\frac{1}{2})}(\Omega) \quad \text{and} \quad \lim_{k\to\infty} U_{3k} = F_3 \quad \text{in } \Sigma^{-(r+\frac{1}{2})}(\Omega). \tag{21}$$

Remark that under these conditions, since $r+\frac{1}{2}-o_{\mathfrak{B}} < \left(r+\frac{1}{2}-o_{\mathfrak{B}}\right)\frac{r+2}{r+1}$ it follows from (6) and (20) that

$$\lim_{k\to\infty} g_k = F_1 \quad \text{in } H^{-\left(r+\frac{1}{2}-o_{\mathfrak{B}}\right),-\left(r+\frac{1}{2}-o_{\mathfrak{B}}\right)\frac{r+2}{r+1}}(\partial\Omega\times]0,T[). \tag{22}$$

Analogously, since $r-\frac{1}{2} < r+1-\frac{3}{2}\frac{r+1}{r+2}$ and $r+\frac{1}{2} < r+1-\frac{1}{2}\frac{r+1}{r+2}$, by (4) and (21) one has

$$\lim_{k\to\infty} U_{2k} = F_2 \text{ in } H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \quad \text{and} \quad \lim_{k\to\infty} U_{3k} = F_3 \text{ in } H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega). \tag{23}$$

We consider for $k \in \mathbb{N}$ the linear form L_k sending every $v \in X^r(\Omega\times]0,T[)$ to

$$L_k(v) = \Big\langle f_k, v\Big\rangle + \Big\langle g_k, \mathfrak{R}_{\mathcal{A}}(v(\mathbf{x},t))\Big\rangle + \Big\langle U_{2k}, \lambda\, v(\mathbf{x},0) - \eta\frac{\partial v}{\partial t}(\mathbf{x},0)\Big\rangle + \eta\,\langle U_{3k}, v(\mathbf{x},0)\rangle\,.$$

As above, by (5), (12), (13), (20) and (21) $L_k \in (X^r(\Omega\times]0,T[))'$ and there is an *unique* $U_k := ((\mathcal{A}^*)')^{-1}(L_k) \in H^{-(r-1),-r}(\Omega\times]0,T[)$ such that

$$\forall\, v \in X^r(\Omega\times[0,T[) \quad L_k(v) = \langle U_k, \mathcal{A}^*(v)\rangle. \tag{24}$$

On the other hand, by corollary 3, the mixed boundary value problem, $k \in \mathbb{N}$

$$\mathcal{A}(\overline{U}_k) = f_k \quad \text{in} \quad \Omega\times]0,T[, \qquad \mathfrak{B}(\overline{U}_k) = g_k \quad \text{in} \quad \partial\Omega\times]0,T[, \tag{25}$$

$$\overline{U}_k(\mathbf{x},0) = U_{2k}(\mathbf{x}), \quad \frac{\partial\overline{U}_k}{\partial t}(\mathbf{x},0) = U_{3k}(\mathbf{x}) \quad \text{in } \Omega \tag{26}$$

has an *unique* solution $\overline{U}_k \in \mathcal{C}^\infty(\overline{\Omega\times]0,T[})$.

By the continuity of the involved trace mappings and by (19), (20) and (21) we see that $\lim_{k\to\infty} L_k = L$ in $\left(X^r(\Omega\times]0,T[)\right)'$, and $((\mathcal{A}^*)')^{-1}$ being an isomorphism, we obtain

$$\lim_{k\to\infty} U_k = U \quad \text{in} \quad H^{-(r-1),-r}(\Omega\times]0,T[). \tag{27}$$

and by Green's formula (3) and (25), (26) and the definition of $X^r(\Omega\times]0,T[)$ we obtain

$$\forall\, v \in X^r(\Omega\times]0,T[) \qquad \langle\overline{U}_k, \mathcal{A}^*(v)\rangle = \langle\mathcal{A}(\overline{U}_k), v\rangle = L_k(v).$$

which, by the uniqueness in (24) implies

$$\forall\, k \in \mathbb{N} \qquad U_k = \overline{U}_k \tag{28}$$

and hence, from (25) we obtain $\mathcal{A}(U_k) = \mathcal{A}(\overline{U}_k) = f_k$ for every $k \in \mathbb{N}$. By (18) and (19) we have $\mathcal{A}(U) = \lim_{k\to\infty} \mathcal{A}(U_k)$ in $\Sigma^{-(r+1),-(r+1)}(\Omega\times]0,T[)$ that, with (27) gives $\lim_{k\to\infty} U_k = U$ in $D_{\mathcal{A}}^{-(r-1)}(\Omega\times]0,T[)$, i.e. $U \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega\times]0,T[)$. It follows from (27) and (28) $U = \lim_{k\to\infty} U_k = \lim_{k\to\infty} \overline{U}_k$ in $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega\times]0,T[)$ and by the theorem of continuity of the trace mappings [[7], theorem 4] and (22) we obtain

$$\mathfrak{B}(U) = \lim_{k\to\infty} \mathfrak{B}(\overline{U}_k) = \lim_{k\to\infty} g_k = F_1 \quad \text{in } H^{-(r+\frac{1}{2}-o_{\mathfrak{B}}),-(r+\frac{1}{2}-o_{\mathfrak{B}})\frac{r+2}{r+1}}(\partial\Omega\times]0,T[).$$

Analogously, by [[7], theorem 2] and (23) we obtain $U(\mathbf{x},0) = \lim_{k\to\infty} U_{2k}(\mathbf{x}) = F_2(\mathbf{x})$ in $H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$ and $\frac{\partial U}{\partial t}(\mathbf{x},0) = \lim_{k\to\infty} U_{3k}(\mathbf{x}) = F_3(\mathbf{x})$ in $H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega)$. ■

Corollary 7 *Let* $(\mathbf{x}_0, t_0) \in \Omega\times]0,T[$. *There is* $G \in \bigcap_{\varepsilon>0} \mathbf{D}_{\mathcal{A}}^{-(\frac{n}{2}+\varepsilon-2)}(\Omega\times]0,T[)$ *such that*

$$\mathcal{A}(G) = \delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0) \quad \textit{in } \Omega\times]0,T[,$$

$$G(\mathbf{x},0) = \frac{\partial G}{\partial t}(\mathbf{x},0) = 0 \quad \textit{in } \Omega$$

and

$$\mathfrak{B}(G)(\mathbf{x},t) = 0 \qquad \textit{in } \partial\Omega\times]0,T[.$$

Moreover, for every $\varepsilon > 0$ *there exists a sequence* $\{\varphi_k\}_{k=1}^{\infty} \subset \mathcal{C}^{\infty}(\overline{\Omega\times]0,T[})$ *such that for every* $k \in \mathbb{N}$ *one has* $\{\mathcal{A}(\varphi_k)\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega\times]0,T[)$,

$$\varphi_k(\mathbf{x},0) = \frac{\partial \varphi_k}{\partial t}(\mathbf{x},0) = 0 \quad \textit{in } \Omega, \tag{29}$$

$$\mathfrak{B}(\varphi_k)(\mathbf{x},t) = 0 \quad \textit{in } \partial\Omega\times]0,T[\tag{30}$$

and moreover,

$$\lim_{k\to\infty} \varphi_k = G \quad \textit{in } \mathbf{D}_{\mathcal{A}}^{-(\frac{n}{2}+\varepsilon-2)}(\Omega\times]0,T[)$$

and

$$\lim_{k\to\infty} \mathcal{A}(\varphi_k) = \delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0) \quad \textit{in } \Sigma^{-(\frac{n}{2}+\varepsilon),-(\frac{n}{2}+\varepsilon)}(\Omega\times]0,T[).$$

Proof. By [[8], proposition 3] for every $\varepsilon > 0, \varepsilon \notin \mathbb{Q}$ one has

$$\delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0) \in \Sigma^{-(\frac{n}{2}+\varepsilon),-(\frac{1}{2}+\varepsilon)}(\Omega\times]0,T[) \subset \Sigma^{-(\frac{n}{2}+\varepsilon),-(\frac{n}{2}+\varepsilon)}(\Omega\times]0,T[)$$

and $r+1 := \frac{n}{2}+\varepsilon$ verifies (10) and (11). The corollary follows directly from theorem 6 and the observation that, in the actual case, the sequences $\{g_k\}_{k=1}^{\infty}, \{U_{2k}\}_{k=1}^{\infty}$ and $\{U_{3k}\}_{k=1}^{\infty}$ used in the proof of theorem 6 can be taken equal to 0 for all $k \in \mathbb{N}$ and hence every function $\overline{U}_k$ verifies the initial and boundary conditions (29) and (30). ■

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