Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 13, Issue 1
Published online: February 12 2018|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

A meshless method for reconstructing a source term in diffusion equation

Wei Liu, Baiyu Wang *

College of Computer Engineering and Applied Mathematics, Changsha University, China
Corresponding author: wangbaiyumath@163.com

Abstract

A meshless method based on the moving least squares approximation is applied to find the numerical solution of the inverse problem of diffusion equation. The problem is that reconstructing a source term using a solution specified at some internal points. Some numerical experiments are presented and discussed.

Keywords: Meshless method; moving least squares; inverse problem; diffusion equation.

MSC: 65M32; 65M70; 35R30.

1. Introduction

Inverse problem of diffusion equations appears naturally in a wide variety of physical and engineering settings, and an important class is the determination of the unknown source term from additional measurements.

There are various methods to deal with this kind of inverse problem [1-4]. In this paper, according to some ideas [5], we use the meshless method based on the moving least squares approximation.

This paper is organized as follows. In section 2, we give an outline of the moving least squares. In section 3, we solve the inverse problem using the meshless method based on the moving least squares approximation. In section 4, we give the numerical experiments and discussions.

2. The moving least squares approximation

The moving least squares approximation has been introduced by P.Lancaster and K.Salkauskas [6].

Given data values $\{u_j\}, j=1,2,\cdots,N,$ at nodes x_j, the moving least squares approximation produces a function $u^h(x)$ in a weighted square sense, it can be written as

$$u^h(x)=\sum_{i=1}^{m} p_i(x)a_i(x)=P^T(x)a(x), \tag{2.1}$$

where m is the number of terms in the basis, $p_i(x)$ are the monomial basis functions, and $a_i(x)$ are the coefficients of the basis functions.

The unknown coefficient $a(x)$ is determined by minimizing the functional $J(a)$, which is defined as

$$\begin{aligned} J(a) &= \sum_{j=1}^{n} \omega_j(x)(u^h(x)-u_j)^2 \\ &= \sum_{j=1}^{n} \omega_j(x)(\sum_{i=1}^{m} p_i(x_j)a_i(x)-u_j)^2 \\ &= (Pa-u)^T W(Pa-u) \end{aligned}$$

where $\omega_j(x)$ is the weight function with compact support at the node x_j,

$$P = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad W = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix},$$

according to the above conditions, $J(a)$ takes the derivative $a(x)$ to zero, we have

$$\frac{\partial J}{\partial a} = A(x)a(x) - B(x)u = 0,$$

that is equivalent to

$$A(x)a(x) = B(x)u, \tag{2.2}$$

where $A(x) = P^T WP, \quad B(x) = P^T W.$

From (2.2), we get

$$a(x) = A^{-1}(x)B(x)u, \tag{2.3}$$

substituting (2.3) into (2.1), we have

$$u^h(x) = P^T(x)a(x) = \phi^T(x)u = \sum_{j=1}^{n} \varphi_j(x)u_j,$$

where $\phi^T(x) = P^T(x)A^{-1}(x)B(x),$ and $\varphi_j(x)$ is called the shape function.

3. The inverse problem

The problem can be described as follows,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k(x)\frac{\partial u}{\partial x}) + f(x,t), \quad 0 < x < l, 0 < t < T, \tag{3.1}$$

where $k(x)$ is smooth enough and $k(x) > 0,$ with the initial condition

$$u(x,0) = 0, \quad 0 < x < l, \tag{3.2}$$

and the boundary conditions

$$u(0,t) = 0, u(l,t) = 0, \quad 0 \le t \le T. \tag{3.3}$$

The formulas (3.1-3.3) are the direct problem, and the inverse problem is that the functions $u(x,t)$ and $f(x,t)$ are unknown, with the additional observation of $u(x,t)$ at some internal point $x_0(0 < x_0 < l)$,

$$u(x_0,t) = E(t). \tag{3.4}$$

Assume the function $f(x,t)$ can be described as

$$f(x,t)=\eta(t)\psi(x), \tag{3.5}$$

where $\psi(x)$ is the known function, and satisfies the following restrictions:

(1) $\psi(x_0)\neq 0$,

(2) $\psi(x)$ is smooth enough,

(3) $\psi(x)=0$ on the boundary of the computational domain.

Let

$$u(x,t)=\theta(t)\psi(x)+\omega(x,t), \tag{3.6}$$

where

$$\theta(t)=\int_0^t \eta(s)ds, \tag{3.7}$$

substituting (3.6) and (3.7) into (3.1), we have

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial x}(k(x)\frac{\partial\omega}{\partial x})+\theta(t)\frac{\partial}{\partial x}(k(x)\frac{\partial\psi}{\partial x}) \qquad 0<x<l, 0<t<T, \tag{3.8}$$

from (3.6) and combining (3.4) , we get

$$\theta(t)=\frac{E(t)-\omega(x_0,t)}{\psi(x_0)}, \tag{3.9}$$

substituting (3.9) into (3.8),

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial x}(k(x)\frac{\partial\omega}{\partial x})+\frac{E(t)-\omega(x_0,t)}{\psi(x_0)}\frac{\partial}{\partial x}(k(x)\frac{\partial\psi}{\partial x}) \qquad 0<x<l, 0<t<T, \tag{3.10}$$

the initial and boundary conditions are

$$\omega(x,0)=0,\ \ 0<x<l, \tag{3.11}$$

$$\omega(0,t)=0,\quad \omega(l,t)=0,\quad 0\leq t\leq T. \tag{3.12}$$

Through the above descriptions, if we have the numerical solution $\hat{\omega}(x,t)$ of the equation (3.10), from (3.5-3.7) and (3.9), we can get the numerical solution $\hat{u}(x,t)$ and $\hat{f}(x,t)$.

Next, we use the meshless method based on the moving least squares approximation solving the problem (3.10-3.12).

The approximate function $\hat{\omega}(x,t)$ of $\omega(x,t)$ can be represented as

$$\hat{\omega}(x,t)=\sum_{j=1}^{n}\lambda_j(t)\varphi_j(x),$$

where $\varphi_j(x)$ is the shape function described in section 2 and $\lambda_j(t)(1\leq j\leq n)$ are the unknown functions must be founded.

Then

$$\frac{\partial\hat{\omega}}{\partial t}=\sum_{j=1}^{n}\lambda_j'(t)\varphi_j(x), \qquad \frac{\partial\hat{\omega}}{\partial x}=\sum_{j=1}^{n}\lambda_j(t)\varphi_j'(x), \qquad \frac{\partial^2\hat{\omega}}{\partial x^2}=\sum_{j=1}^{n}\lambda_j(t)\varphi_j''(x),$$

so the equation (3.10) can be rewritten as

$$\sum_{j=1}^{n}\lambda_j'(t)\varphi_j(x)=\frac{\partial}{\partial x}(k(x)\sum_{j=1}^{n}\lambda_j(t)\varphi_j'(x))+\frac{E(t)-\sum_{j=1}^{n}\lambda_j(t)\varphi_j(x_0)}{\psi(x_0)}\frac{\partial}{\partial x}(k(x)\frac{\partial\psi}{\partial x}),$$

for the numerical computation, we apply one step forward difference formula to time domain t, then

$$\sum_{j=1}^{n}\frac{\lambda_j(t_{s+1})-\lambda_j(t_s)}{\Delta t}\varphi_j(x)=\frac{\partial}{\partial x}(k(x)\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j'(x))+\frac{E(t_s)-\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j(x_0)}{\psi(x_0)}\frac{\partial}{\partial x}(k(x)\frac{\partial\psi}{\partial x}),$$

that is equivalent to

$$\sum_{j=1}^{n}\lambda_j(t_{s+1})\varphi_j(x)=\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j(x)+\Delta t[\frac{\partial}{\partial x}(k(x)\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j'(x))+\frac{E(t_s)-\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j(x_0)}{\psi(x_0)}\frac{\partial}{\partial x}(k(x)\frac{\partial\psi}{\partial x})], \tag{3.13}$$

by substituting each x_k for x,

$$\sum_{j=1}^{n}\lambda_j(t_{s+1})\varphi_j(x_k)=\hat{\omega}(x_k,t_{s+1}),\qquad \sum_{j=1}^{n}\lambda_j(t_s)\varphi_j(x_k)=\hat{\omega}(x_k,t_s),$$

so for (3.13),

$$\hat{\omega}(x_k,t_{s+1})=\hat{\omega}(x_k,t_s)+\Delta t[\frac{\partial}{\partial x}(k(x_k)\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j'(x_k))+\frac{E(t_s)-\sum_{j=1}^{n}\lambda_j(t_s)\varphi_j(x_0)}{\psi(x_0)}\frac{\partial}{\partial x}(k(x_k)\frac{\partial\psi}{\partial x})], \tag{3.14}$$

from (3.14) and the conditions (3.11)-(3.12), we can obtain $\lambda_j(t), j=1,2,\cdots,n,$ and the numerical solution $\hat{\omega}(x,t)$, then we get the numerical solution

$$\hat{f}(x_k,t_s)=\hat{\eta}(t_s)\psi(x_k),\quad \hat{u}(x_k,t_s)=\hat{\theta}(t_s)\psi(x_k)+\hat{\omega}(x_k,t_s).$$

4. Numerical experiments and discussions

Consider the problem (3.1-3.5), with the conditions

$$k(x)=1,\ E(t)=t,$$

and we let $l=1, T=1, x_0=0.5.$

The exact solutions are

$$u(x,t)=t\sin(\pi x),\ f(x,t)=(1+\pi^2 t)\sin(\pi x),$$

with $\psi(x)=\sin(\pi x).$

Firstly, we plot the error functions $f(x,t)-\hat{f}(x,t)$ and $u(x,t)-\hat{u}(x,t)$ in Figure 1.

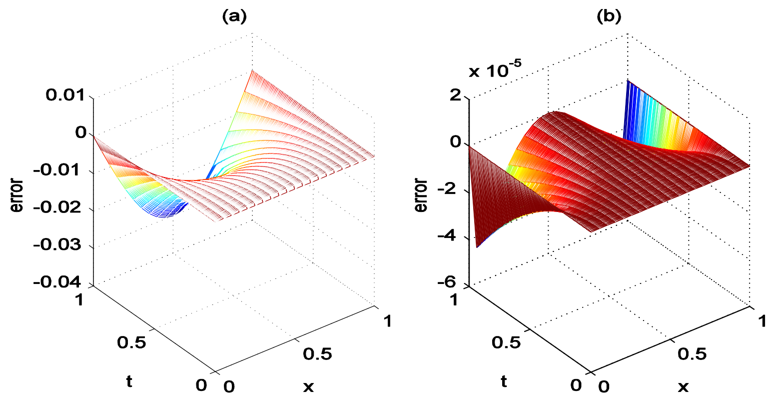

Fig 1: (a) $f(x,t)-\hat{f}(x,t)$,**(b)** $u(x,t)-\hat{u}(x,t)$ with $\Delta x=0.025, \Delta t=0.0001$.

From Figure 1, we see that the approximation effect is good.

Secondly, in order to test the sensitivity of this method on overspecified data $u(x_0,t)=E(t)$, we give small perturbation on $E(t)$, and the artificial error is introduced into the additional specification data by defining function

$$E_\gamma(t)=E(t)(1+\gamma),$$

the result are shown in Figure 2.

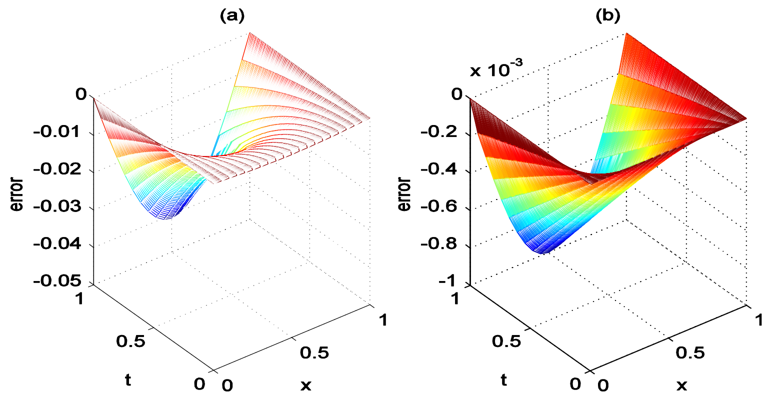

Fig 2: (a) $f(x,t)-\hat{f}(x,t)$,**(b)** $u(x,t)-\hat{u}(x,t)$ with $\gamma=0.001$.

From Figure 2, we see that the approximation effect is worse, moreover, there is no obvious oscillation in error graph.

At last, in order to further illustrate the accuracy, stability and effectiveness of the method, we define the error of function $f(x,t)$ as follows

$$E(f)=\sqrt{\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(f(x_i,t_j)-\hat{f}(x_i,t_j))^2}{MN}},$$

where $f(x_i,t_j)$ and $\hat{f}(x_i,t_j)$ are the exact and numerical solution at the node x_i,t_j, respectively, M and N are the number of nodes about x and t, the definition of $E(u)$ is same to $E(f)$. The results are given in Table 1.

Table 1. The errors for different cases

Δx	γ	$E(u)$	$E(f)$
0.05	0	7.7985×10^{-5}	3.7323×10^{-2}
0.025	0	1.0462×10^{-5}	9.1843×10^{-3}
0.025	0.001	4.0759×10^{-4}	1.3592×10^{-2}
0.025	0.01	4.0368×10^{-3}	5.4752×10^{-2}

From Table 1, we get that when Δt is fixed, with the increase of the number of nodes, the error decreases, when there is noise data, the error decreases with the decrease of the noise parameter. So our method is stable and effective.

5. Conclusion

In this paper, we use the meshless method based on the moving least squares approximation to solve the inverse problem of reconstructing a source term in diffusion equation. From the figures and the table, we can see that this method is accurate and efficient.

Acknowledgements

This work was supported by Scientific Research Fund of Scientific and Technological Project of Changsha City, (Grant No. ZD1601077, K1705078).

References

[1] J.R.CANNON and P.DUCHATEAU, Structural identification of an unknown source term in a heat equation, Inverse Probl., 1998,14:535-551.

[2] A.A.BURYKIN and A.M.DENISOV, Determination of the unknown sources in the heat-conduction equation, Comput.Math.Model., 1997,8:309-313.

[3] A. FARCAS and D. LESNIC, The boundary-element method for the determination of a heat source dependent on one variable, J. Eng. Math., 2006,54:375-388.

[4] A.G.FATULLAYEV, E.CAN, Numerical procedures for determining unknown source parameters in parabolic equations, Math. Comput. Simul., 2000,54:159-167.

[5] A.A.SAMARSKII, P.N.VABISHCHEVICH, Numerical methods for solving inverse problems of mathematical physics, Walter de Gruyter, Berlin, New Yory.2007.

[6] P.LANCASTER, K.SALKAUSKAS, Surface generated by moving least squares methods, Math.Comput., 1981,37:141-158.