SCITECH
RESEARCH ORGANISATION

Volume 13, Issue 2
Published online: February 24, 2018|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Dynamic investigation of ellipses inscribed in a rectangle

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Abstract

Described is a dynamic investigation of an ellipse inscribed in a rectangle, with a view to properties conserved while making changes. We find mathematically the locus from which an ellipse is viewed at an angle of 90°, as well as the locus of the foci of an ellipse rotated in a rectangle that inscribes it. Would also study the case where a given ellipse is inscribed by a parallelogram, the lengths of whose sides can be changed.

Keywords:

Loci; Dynamic investigation; Geometric properties; Conservation and change.

Introduction

Research in education seeks methods for improving the quality of teaching and learning, and therefore it also focuses on integration of technology in teaching. The technological tool allows one to construct and represent mathematical objects in a dynamic manner, while providing the user with feedback during the solution of problems [1-2]. Learning that combines the use of dynamic software allows the students to discover mathematical phenomena, mathematical models, various representations, generalizations and relations between graphical representations, with reference to mathematical concepts [3]. One of the difficulties that exists with those who study the subject of loci is understanding the dynamic nature of the locus, and the dynamic interrelation between different points in the given drawing. When studying using a technological tool the students can describe better mathematical concepts and relations, when compared to teaching that does not use a technological tool. The students gain a better understanding of these abstract concepts, and they gain access to mathematical ideas on a high level [4].

As part of activities for getting acquainted with the large contribution of the usage of computer technology in teaching mathematics, students were engaged in discovery of properties and calculation of the parameters of ellipses inscribed in a given rectangle.

Most of the properties were discovered using Geogebra, and subsequently the calculated parameters permitted an accurate construction of the ellipses.

We believe that the activities can be expanded by asking questions such as: what happens if the ellipse is inscribed in a rectangle, in a rhombus or in other geometrical shapes? This activity was tried successfully among students of the higher grades in high school (who study mathematics on an advanced level), as well as in an advanced investigation course taught the pre-service teachers of mathematics in colleges of education. Of course, the visual result obtained on the computer screen is not sufficient, and one must require the students to formally prove the property as required in mathematics. At the first stage in activity was held of getting acquainted with the properties of the ellipse as a prelude to the investigation of an ellipse inscribed in a circle. The proofs presented in the paper are those gathered from the students after instructions and clues. The methodical discussion dealt with the geometrical properties of an ellipse, and the properties conserved during a change.

Presentation of the task

Ellipse inscribed in a rectangle

Given is a rectangle whose side lengths are 2a, 2b, which inscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, as shown in Figure 1.

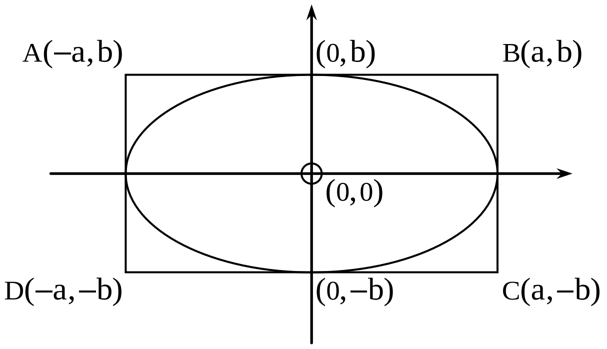

Figure 1

Claim 1

All the rectangles in which this ellipse is inscribed have their vertices on the circle whose equation is: $x^2+y^2=a^2+b^2$.

Proof:

in fact we have to prove that the locus all the points from which the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is viewed at a right angle is the circle $x^2+y^2=a^2+b^2$ (figure 2).

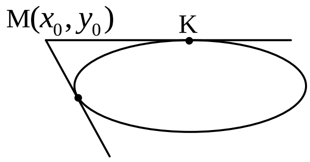

Figure 2

Method A

Let M be an arbitrary point (x_0,y_0). It is given that the tangents to the ellipse that issue from M are perpendicular.

We denote the equations of the tangents that issue from this point by:

(1) $m_1x_0+n_1=y_0$ (the tangent MK), where K is a point on the ellipse.

(2) $m_2x_0+n_2=y_0$.

It is well known that the tangency condition of the straight line $y=mx+n$ with the given ellipse is: $m^2a^2+b^2=n^2$. In other words:

(3) $m_1{}^2a^2+b^2=n_1{}^2$

and

(4) $m_2^2a^2 + b^2 = n_2^2$

From the relations (1) and (2) we obtain x_0 , y_0 through m_1 , m_2 , n_1 , n_2, and from the relations (3) and (4) we obtain a^2 , b^2 through m_1 , m_2 , n_1 , n_2 using the fact that $m_1m_2 = -1$ and obtaining $x_0^2 + y_0^2 = a^2 + b^2$.

Method B

$$\begin{cases} mx_0 + n = y_0 \\ m^2a^2 + b^2 = n^2 \end{cases}$$

We isolate n from the first equation and substitute it in the second equation:

$$m^2a^2 + b^2 = y_0^2 - 2mx_0y_0 + m^2x_0^2$$

We arrange the second equation as a quadratic equation with the unknown m:

$$m^2(a^2 - x_0^2) + 2mx_0y_0 - (y_0^2 - b^2) = 0$$

Based on the perpendicularity condition $m_1m_2 = -1$, and using the Vieta Theorem, we have:

$$\frac{b^2 - y_0^2}{a^2 - x_0^2} = -1 \quad \Rightarrow \quad x_0^2 + y_0^2 = a^2 + b^2$$

Conclusion: All the ellipses inscribed in the rectangle ABCD (Figure 1) have the property that if d_1 is the large axis and d_2 is the length of the small axis, then: $a^2 + b^2 = \left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2$.

The presentation of both methods of solution highlights the beauty of mathematics, whereby a solution can be obtained for a particular problem by different methods using mathematical tools from the same field or by combining tools from different fields [5 -8].

An applet was prepared which shows the circle that is the locus of all the rectangles that describe a given ellipse. Using the rotation circle, the rectangle inscribing the ellipse can be rotated. When the focus F_1 is dragged along the x axis, and the point B – along the y-axis, the parameters of the ellipse change, together with the radius of the circle on which lie the vertices of the rectangles that inscribe the ellipse.

Link 1: https://www.geogebra.org/m/cNAD6RCJ

An ellipse and the circle which lie the vertices of the rectangles that inscribe it.

Claim 2

Given is an ellipse whose center is O (the point of intersection of the axes of the ellipse) and l_1 and l_2 are two parallel tangents points of intersection with the ellipse are M and N (Figure 3).

Then the straight line that connects the points M and N passes through the center of the ellipse O.

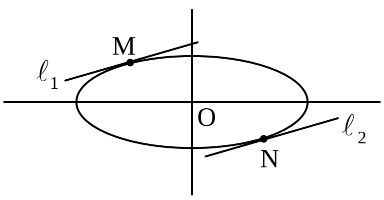

Figure 3

Proof

If the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the slopes of the tangents are m, then their equations are: $y=mx+n_1$ and $y=mx+n_2$. From the tangency conditions there holds:

$$m^2a^2+b^2=n_1{}^2$$
$$m^2a^2+b^2=n_2{}^2$$

Therefore $n_1=-n_2\equiv n>0$, in other words the equations of the tangents are:

$$\ell_1\ :\ y=mx+n$$
$$\ell_2\ :\ y=mx-n$$

l_1 has a single point of intersection with the ellipse $x^2b^2+y^2a^2=a^2+b^2$,

in other words the equation $x^2b^2+(mx+n)^2a^2-a^2b^2=0$ has the following single solution:

$$x^2(b^2+m^2a^2)+2mnxa^2+n^2-a^2b^2=0$$

$$x_1=\frac{-2mna^2}{b^2+m^2a^2}$$

Also $x_2=\frac{2mna^2}{b^2+m^2a^2}$ for ℓ_2.

Therefore, $\frac{x_1+x_2}{2}=0$ and it immediately

follows that: $\frac{y_1+y_2}{2}=0$, therefore MN passes

through O and O is the middle of MN.

An applet was prepared which shows that the chord MN that connects the tangency points to tangents that are parallel to the ellipse – passes through the center of the ellipse. When the point M is dragged along the curve of the ellipse, the location tangency point N of the parallel tangent changes, and the chord MN passes through the center of the ellipse.

Link 2: https://www.geogebra.org/m/ZmfTZCQE

The chord that connects the tangency points of two tangents that are parallel to the ellipse passes through the center of the ellipse.

Note: known also is the property that the midpoint of parallel chords in an ellipse lies on a straight line that intersects the ellipse at M and N, and that the slope of all the straight lines equals the slope of the tangents at the points N and M, as shown in figure 4.

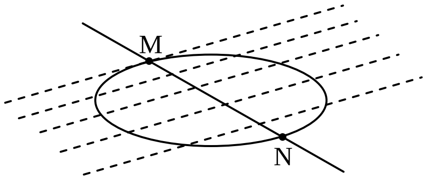

Figure 4

An applet was prepared showing ellipse with the tangents at the point M, and three chords that are parallel to this tangent. The middles of these chords lie on the straight line that passes through the point of tangency M, through the center of the ellipse, and whose continuation intersects the ellipse at the point N. the point M can be dragged along the ellipse, thereby changing the inclination angle of the tangent at the point, as well as those of the parallel chords.

Link 3: https://www.geogebra.org/m/n4mr8GER

A tangent to an ellipse and chords with their midpoints parallel to it.

Conclusion:

If a parallelogram inscribes an ellipse and the points of intersection are M, N, P, Q (Figure 5), then MNPQ is a parallelogram.

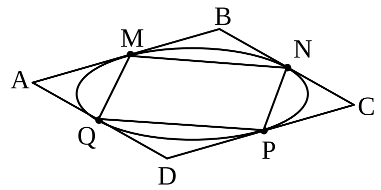

Figure 5

Proof

This is an immediate result of Claim 2, since QN and MP bisect each other.

Claim 3

In the terms of Figure 5 there holds: the sides of the parallelogram MNPQ are parallel to the diagonals of the parallelogram ABCD (in other words $\frac{\mathrm{AM}}{\mathrm{MB}} = \frac{\mathrm{AQ}}{\mathrm{QD}}$).

Proof

Based on the above conclusion, if $\mathrm{M}(x_1, y_1)$ then $\mathrm{P}(-x_1, -y_1)$ and if $\mathrm{N}(x_0, y_0)$ then $\mathrm{Q}(-x_0, -y_0)$. The tangents MN and PQ intersect at the point B, and using the equations of the tangents we obtain $\mathrm{B}\left(\frac{a^2(y_1 - y_0)}{y_1 x_0 - x_1 y_0}, \frac{-b^2(x_1 - x_0)}{y_1 x_0 - x_1 y_0}\right)$.

D is symmetric to B with respect to the origin.

A simple check shows that the slope of NP equals the slope of BD.

An applet was prepared in which a parallelogram inscribes a given ellipse (with its axes. The points of tangency were connected to form an inner quadrilateral inscribed in the parallelogram. We denote the angles of the inner quadrilateral: α, β, γ, δ.

The point M, which is one of the vertices of the parallelogram, can be dragged to change the lengths of its sides. For every location of the point M the size of the angle of the quadrilateral inscribed in the ellipse appears on the screen, and since the alternate angles are equal, the inner quadrilateral is a parallelogram.

Link 4: https://www.geogebra.org/m/Ygmn8EUR

A parallelogram that inscribes an ellipse and an inner parallelogram inscribed in it.

Particular case

If ABCD is a rectangle, the parallelograms MNPQ are light ray paths between mirrors (the sides of ABCD), and the perimeter of each of them is fixed

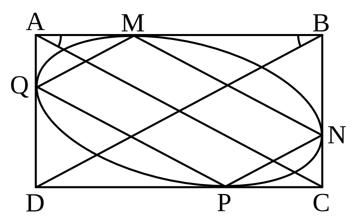

Figure 6

and equal to 2AC (Figure 6).

Proof

From Claim 3, $MN \parallel AC$ and $MQ \parallel BD$, and therefore, $\angle AMQ = \angle BMN = \angle B_1 = \angle A_1$. If $\frac{AM}{AB} = \alpha$, then $\frac{MB}{AB} = 1-\alpha$, therefore $\frac{MN}{AC} + \frac{MQ}{AC} = 1$, and therefore the perimeter of the parallelogram QMNP is fixed.

Note: The path MNPQ is also called the billiard ball path, and it is the shortest path for a ball to travel hitting the four walls.

An applet was prepared, which shows a rectangular billiard table in which a red ellipse is inscribed. The four points of tangency between the ellipse and the table form a parallelogram inscribed in the ellipse and also in the rectangle, and outlining the path of the ball.

Link 5: https://www.geogebra.org/m/HDnEPM5r

A billiard table for illustrating the ellipse inscribed in a rectangle.

Finding the parameters a_1, b_1 and c_1 of an ellipse inscribed in a given circle

In the rectangle ABCD (Figure 7) we draw through its center two perpendicular straight lines d_1 and d_2, which shall form the axes of an ellipse inscribed in the rectangle. We denote $OMB = \alpha$. Clearly, in this case MN is tangent to the ellipse inscribed in the rectangle.

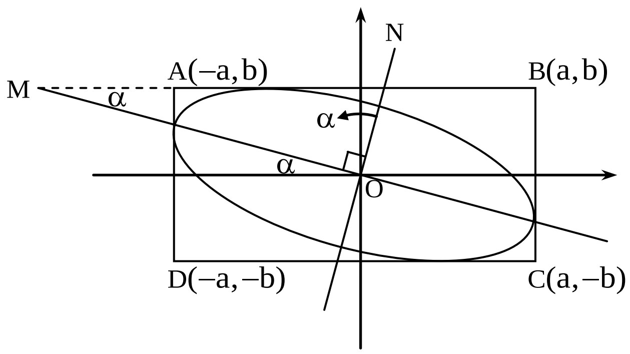

Figure 7

We denote by a_1 and b_1 the parameters of the inscribed ellipse.

From the conclusion of Claim 1, there holds:

(3) $a_1^{\,2} + b_1^{\,2} = a^2 + b^2 = OA^2$

and from the tangency condition of MN and the inscribed ellipse, there holds:

(4) $tg^2\alpha \cdot a_1^{\,2} + b_1^{\,2} = ON^2$

We subtract relation (4) from relation (3) and obtain:

$$a_1^{\,2}(1 - tg^2\alpha) = OA^2 - ON^2$$

$$a_1^{\,2} = \frac{(OA^2 - ON^2)}{\cos 2\alpha} \cdot \cos^2\alpha$$

$$a_1^{\,2} = \frac{(a^2+b^2)\cos^2\alpha - \frac{b^2}{\cos^2\alpha} \cdot \cos^2\alpha}{\cos 2\alpha}$$

$$a_1^{\,2} = \frac{a^2\cos^2\alpha - b^2\sin^2\alpha}{\cos 2\alpha}$$

$$b_1^{\,2} = a^2 + b^2 - a_1^{\,2} = \frac{(a^2+b^2)(\cos^2\alpha - \sin^2\alpha) - a^2\cos^2\alpha + b^2\sin^2\alpha}{\cos 2\alpha}$$

$$b_1^{\,2} = \frac{b^2\cos^2\alpha - a^2\sin^2\alpha}{\cos 2\alpha}$$

$$c_1^{\,2} = a_1^{\,2} - b_1^{\,2} = \frac{a^2 - b^2}{\cos 2\alpha}$$

Claim

The locus of the foci of all the ellipses inscribed in a rectangle whose sides are 2a, 2b (a>b) is a rectangular hyperbola: $x^2 - y^2 = a^2 - b^2$.

Proof

In the previous section we calculated $c_1^{\,2}$. The coordinates of the point whose focus is at a distance of c_1 from the origin are:

$$x_{c_1}^{\,2} = \frac{a^2 - b^2}{\cos 2\alpha}\cdot\cos^2\alpha$$

$$y_{c_1}^{\,2} = \frac{a^2 - b^2}{\cos 2\alpha}\cdot\sin^2\alpha$$

Hence: $x_{c_1}^{\,2} - y_{c_1}^{\,2} = \dfrac{(a^2-b^2)(\cos^2\alpha - \sin^2\alpha)}{\cos 2\alpha} = a^2 - b^2$

Figure 8 shows the locus of the foci of the ellipse, when it is rotated in the inscribing rectangle

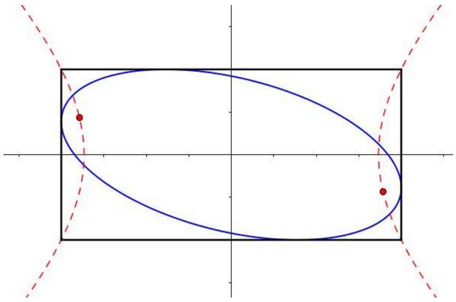

Figure 8

An applet was prepared to illustrate the locus of foci of the ellipse inscribed in a rectangle. In this applet there are two bars that allow the dimensions of the rectangle in which the ellipse is inscribed to be changed, as well as a rotation circle allowing the ellipse to be rotated and the locus of the foci of the ellipse to be presented at any stage. The locus is a hyperbola passing through the vertices of the inscribing rectangle.

Link 6: https://www.geogebra.org/m/HmHcS5DK

Presentation of the locus of the foci of an ellipse rotated inside a rectangle.

Summary

The present paper presents geometrical properties, and in particular – properties that “move” inside a given rectangle.

For students who study analytical geometry on an advanced level, this subject is an opportunity to delve into and expand their knowledge base and skills.

In addition, by using dynamic geometric software (GeoGebra), the students discover hypotheses by themselves, try to prove them and experience the feeling of "doing mathematics".

Everything stated with regards to the students is also true with regard to pre-service teachers and in-service teachers, both as part of their training and as part of further studies.

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