SCITECH
RESEARCH ORGANISATION

Volume 13, Issue 1

Published online: January 22 2018|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Almost Oscillation Criteria for Second Order Neutral Difference Equations

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Abstract

In this paper, we consider the second order neutral difference equation of the form

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n x_{n-\sigma}^\beta = e_n, \qquad n \geq n_0,$$

where $z_n = x_n + p_n x_{n-\tau}$ and $\alpha > 0,\ \beta > 0$ are ratios of odd positive integers. Examples are provided to illustrate the results.

Keywords: Second order difference equation, Almost oscillatory, Riccati technique, Summation by parts.

1. Introduction

In this paper, we study the oscillatory behavior of second order neutral difference equation of the form

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n x_{n-\sigma}^\beta = e_n, \qquad n \geq n_1, \tag{1}$$

where $z_n = x_n + p_n x_{n-\tau}$ and $\alpha > 0,\ \beta > 0$ are ratios of odd positive integers, σ and τ are positive integers.

Throughout this paper, we assume that:

(H$_1$) $\{p_n\}$, $\{q_n\}$ and $\{e_n\}$ are positive real sequences with $0 \leq p_n \leq p \leq 1$, $q_n > 0$, $e_n \geq 0$ and $\{a_n\}$ is a positive real sequence with $\sum_{s=n_0}^{n} \frac{1}{a_s^{1/\alpha}} \to \infty$ as $n \to \infty$.

(H$_2$) α and β are ratios of odd positive integers, σ and τ are positive integers.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0$ and satisfying equation (1). A solution $\{x_n\}$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is said to be non-oscillatory.

Recently there has been an increasing interest in the study of the oscillation and non-oscillation of the second order neutral difference equations, see for example [6, 7, 10-14] and the references cited there in. In [6], we see that the oscillation criteria for second order difference equation of the form

$$\Delta(r_n \Delta(x_n + x_{n-k})^\gamma) + q_n x_{n+1}^\alpha = e_n, \quad n \in N_0 \tag{2}$$

is discussed.

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In [14], we see that the oscillation criteria for second order non-positive neutral term of the form

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n f(x_{n-\sigma}) = 0, \qquad n \geq n_0 > 0, \tag{3}$$

where $z_n = x_n - p_n x_{n-\tau}$, α is a ratio of odd positive integers.

Also Thandapani et.al [14] studied the oscillatory behavior of the equation

$$\Delta^2 (x_n + c x_{n-k})^\gamma + q_n x_{n+1}^\alpha = e_n. \tag{4}$$

This observation motivated us to study oscillation criteria for second order neutral difference equation. In section 1, we present some lemmas which will be useful to prove our main results. In section 2, we obtain new oscillation criteria of equation (1) and in section 3, we provide some examples to illustrate the main results.

Definition 1.1. *A solution $\{x_n\}$ of equation (1) is said to be almost oscillatory if either $\{x_n\}$ is oscillatory or $\{\Delta x_n\}$ is oscillatory or $x_n \to 0$ as $n \to \infty$.*

We provide two lemmas which are useful in proving the main results.

Lemma 1.1. *Set*

$$F(x) = a x^{\alpha-\gamma} + \frac{b}{x^\gamma} \quad for \quad x > 0. \tag{5}$$

If $a > 0,\ b > 0$ and $\alpha > \gamma \geq 1$, then $F(x)$ attains its minimum

$$F_{\min} = \frac{\alpha a^{\gamma/\alpha} b^{1-\gamma/\alpha}}{\gamma^{\gamma/\alpha}(\alpha-\gamma)^{1-\gamma/\alpha}}. \tag{6}$$

Lemma 1.2. *For all $x \geq y \geq 0$ and $\gamma \geq 1$, we have the following inequality*

$$x^\gamma - y^\gamma \geq (x-y)^\gamma. \tag{7}$$

2. Oscillation Results

In this section, by using the Riccati substitution we will establish new almost oscillation criteria for equation (1).

Theorem 2.1. *Assume that there exists a sequence $\{\rho_n\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[\rho_s Q_s^* - \frac{(\Delta\rho_s)^2}{4\rho_s} a_{s-\sigma}\right] = \infty, \tag{8}$$

and

$$\sum_{s=n_0}^{n-1} \sum_{u=n_0}^{s-1} (M q_u \pm e_u)^{\frac{1}{\alpha}} = \infty, \tag{9}$$

where

$$Q_n = \frac{\beta q_n^{\alpha/\beta} e_n^{1-\alpha/\beta} (1-p_n)^\alpha}{\alpha^{\alpha/\beta}(\beta-\alpha)^{1-\alpha/\beta}}, \tag{10}$$

and

$$Q_n^* = \min\{Q_n,\ d^{\beta-\alpha}(1-P_n)q_n - d^{-\alpha} e_n\},$$

$M > 0$ and $d > 0$. Then every solution of equation (1) is almost oscillatory.

Proof. Suppose that sequence $\{x_n\}$ is not almost oscillatory solution of equation (1). There exists a positive solution $\{x_n\}$ of equation (1) such that $x_{n-\tau} > 0$ and $x_n > 0$ for all $n \geq n_1 \geq n_0$. Then by definition of almost oscillatory there are two possible cases arise. $\Delta x_n > 0$ eventually positive (or) $\Delta x_n < 0$ eventually negative

Case Ia: Assume that $\Delta x_n > 0$ for all $n \geq n_1$. Thus $\Delta z_n > 0$ for all $n \geq n_1$, we have $x_n \geq (1-p_n) z_n$. Then from equation (1) and (H_2), we have

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n(1-p_n)^\beta z_{n-\sigma}^\beta \leq e_n$$

$$\Delta(a_n(\Delta z_n)^\alpha) \leq -q_n(1-p_n)^\beta z_{n-\sigma}^\beta + e_n. \tag{11}$$

From the above inequality, we have

$$a_{n-\sigma}(\Delta z_{n-\sigma})^{\alpha} \geq a_{n+1}(\Delta z_{n+1})^{\alpha}, \quad n \geq n_1. \tag{12}$$

Define

$$w_n = \frac{\rho_n a_n(\Delta z_n)^{\alpha}}{z_{n-\sigma}^{\alpha}}, \quad n \geq n_1. \tag{13}$$

Then $w_n > 0$,

$$\begin{aligned}
\Delta w_n &= \frac{\rho_n \Delta(a_n(\Delta z_n)^{\alpha})}{z_{n-\sigma}^{\alpha}} + a_{n+1}(\Delta z_{n+1})^{\alpha} \Delta\left(\frac{\rho_n}{z_{n-\sigma}^{\alpha}}\right), \\
&= \frac{\rho_n \Delta(a_n(\Delta z_n)^{\alpha})}{z_{n-\sigma}^{\alpha}} + a_{n+1}(\Delta z_{n+1})^{\alpha}\left[\Delta\left(\frac{1}{z_{n-\sigma}^{\alpha}}\right)\rho_n + \frac{\Delta \rho_n}{z_{n+1-\sigma}^{\alpha}}\right], \\
&= \frac{\rho_n \Delta(a_n(\Delta z_n)^{\alpha})}{z_{n-\sigma}^{\alpha}} + \frac{a_{n+1}\Delta\rho_n(\Delta z_{n+1})^{\alpha}}{z_{n+1-\sigma}^{\alpha}} - \frac{\rho_n a_{n+1}(\Delta z_{n+1})^{\alpha}\Delta(z_{n-\sigma}^{\alpha})}{z_{n-\sigma}^{\alpha} z_{n+1-\sigma}^{\alpha}} \\
&= \frac{\rho_n \Delta(a_n(\Delta z_n)^{\alpha})}{z_{n-\sigma}^{\alpha}} + \frac{\Delta\rho_n w_{n+1}}{\rho_{n+1}} - \frac{\rho_n w_{n+1}}{\rho_{n+1}} \frac{\Delta z_{n-\sigma}^{\alpha}}{z_{n-\sigma}^{\alpha}}.
\end{aligned} \tag{14}$$

It follows from $\Delta z_n > 0$, we have $z_{n+1-\sigma} > z_{n-\sigma}$, and $z_{n+1-\sigma}^{\alpha} \geq z_{n-\sigma}^{\alpha}$, we obtain

$$\frac{1}{z_{n+1-\sigma}^{\alpha}} < \frac{1}{z_{n-\sigma}^{\alpha}}. \tag{15}$$

From inequalities (12), (14) and (15), we obtain

$$\Delta w_n \leq \frac{\rho_n \Delta(a_n(\Delta z_n)^{\alpha})}{z_{n-\sigma}^{\alpha}} + \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \frac{\rho_n}{\rho_{n+1}^2}\frac{w_{n+1}^2}{a_{n-\sigma}}. \tag{16}$$

In the view of (11), (13) and (16), we obtain

$$\Delta w_n \leq -\rho_n\left[q_n(1-P_n)^{\beta} z_{n-\sigma}^{\beta-\alpha} - \frac{e_n}{z_{n-\sigma}^{\alpha}}\right] + \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \frac{\rho_n}{\rho_{n+1}^2}\frac{w_{n+1}^2}{a_{n-\sigma}}. \tag{17}$$

Set

$$G(x) = q_n(1-\rho_n)^{\beta} x^{\beta-x} - \frac{e_n}{x^{\alpha}}.$$

It is easy to verify that function G is increasing for positive constant. Since x is increasing, there is a constant $d > 0$ such that $x \geq d > 0$ and

$$G(x) \geq q_n(1-\rho_n)^{\beta} d^{\beta-\alpha} - d^{-\alpha} e_n = Q_n^*. \tag{18}$$

From inequality (17) and (18), we obtain

$$\Delta w_n \leq -\rho_n Q_n^* + \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \frac{\rho_n}{\rho_{n+1}^2}\frac{w_{n+1}^2}{a_{n-\sigma}}. \tag{19}$$

Using completing the square, we have

$$\begin{aligned}
\Delta w_n &\leq -\rho_n Q_n^* + \frac{(\Delta\rho_n)^2 a_{n-\sigma}}{4\rho_n} \\
&\leq -\left[\rho_n Q_n^* - \frac{(\Delta\rho_n)^2 a_{n-\sigma}}{4\rho_n}\right].
\end{aligned}$$

Summing both side the above inequality from n_1 to $n-1$, we obtain

$$w_n - w_{n_1} < \sum_{s=n_1}^{n-1}\left[\rho_s Q_s^* - \frac{(\Delta\rho_s)^2 a_{s-\sigma}}{4\rho_s}\right]. \tag{20}$$

Noting that $w_n \geq 0$ for $n \in N$, we have $w_{n_1} > w_{n_1} - w_n$. Hence

$$\sum_{s=n_1}^{n-1}\left[\rho_s Q_s^* - \frac{(\Delta\rho_s)^2 a_{s-\sigma}}{4\rho_s}\right] \leq w_{n_1} - w_n \leq w_{n_1}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1}\left[\rho_s Q_s^* - \frac{(\Delta\rho_s)^2 a_{s-\sigma}}{4\rho_s}\right] \leq w_{n_1}. \tag{21}$$

which is contradiction to (8).

Case Ib: Assume that $\Delta x_n < 0$ for all $n \geq n_1$, then $\Delta z_n < 0$ for all $n \geq n_1$. From $x_n > 0$ and $\Delta x_n < 0$, we obtain

$$\lim_{n\to\infty} x_n = l > 0. \tag{22}$$

Then there exists $n_2 \in N$ such that $x_{n-\sigma}^{\beta} \geq l^{\beta}$ for $n \geq n_2$. Therefore, we have

$$\Delta(a_n(\Delta z_n)^{\alpha}) \leq -q_n l^{\beta} + e_n. \tag{23}$$

Set $l^{\beta} = M$. Summing the last inequality from n_2 to $n-1$, we get

$$a_n(\Delta z_n)^\alpha < a_n(\Delta z_n)^\alpha - a_{n_2}(\Delta z_{n_2})^\alpha \le -\sum_{s=n_2}^{n-1}[Mq_s - e_s]$$

and

$$\Delta z_n \le -\left[\sum_{s=n_2}^{n-1} Mq_s - e_s\right]a_n^{-1/\alpha}. \tag{24}$$

Again summing the above inequality from n_2 to $n-1$, we obtain

$$z_n \le z_{n_2} - \left[\sum_{s=n_2}^{n-1}\left[\sum_{u=n_2}^{s-1}[Mq_u - e_u]a_s^{-1/\alpha}\right]\right]. \tag{25}$$

Letting $n \to \infty$, from condition (9) implies that z_n is negative for all $n \ge n_2$. This contradiction ended the proof of this case.

Finally, we assume that $\{x_n\}$ is an eventually negative solution of equation (1). There exists $n_3 \in N$ such that $x_n < 0$ for all $n \ge n_3$. We use the transformation $y_n = -x_n$ in equation (1). Then we have $\{y_n\}$ is an eventually positive solution of the equation

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n y_{n-\sigma}^\beta = -e_n \tag{26}$$

where $z_n = y_n + p_n y_{n-\tau} > 0$.

We have two possible cases arise. $\Delta y_n > 0$ eventually positive (or) $\Delta y_n < 0$ eventually negative

Case IIa: Assume that $\Delta y_n > 0$ for all $n \ge n_3$. Thus $\Delta z_n > 0$ for all $n \ge n_3$.

Define

$$w_n = \frac{\rho_n a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha}, \quad n \ge n_3. \tag{27}$$

Thus $w_n > 0$ and from (14) and (26), we have

$$\Delta w_n \le -\rho_n\left[q_n(1-P_n)^\beta z_{n-\sigma}^{\beta-\alpha} + \frac{e_n}{z_{n-\sigma}^\alpha}\right] + \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \frac{\rho_n w_{n+1}}{\rho_{n+1}}\frac{\Delta(z_{n-\sigma}^\alpha)}{z_{n-\sigma}^\alpha}. \tag{28}$$

Putting $a = q_n(1-P_n)^\beta$, $b = -e_n$ and $x = z_{n-\sigma}$, we have

$$F(u) = q_n(1-P_n)^\beta u^{\beta-\alpha} - \frac{e_n}{u^\alpha}.$$

Using Lemma 1.1, we have

$$F(u) \ge \frac{\beta q_n^{\alpha/\beta}(1-p_n)^\alpha e_n^{1-\alpha/\beta}}{\alpha^{\alpha/\beta}(\beta-\alpha)^{1-\alpha/\beta}}, \tag{29}$$

and also (18) are hold. Then the rest of the proof similar to that of the above and hence is omitted.

Case IIb: Assume that $\Delta y_n < 0$ for all $n \ge n_3$. Hence $\{y_n\}$ is a positive solution and the proof is similar to that of Case Ib and hence omitted. The proof is now complete. ■

Corollary 2.1 *Assume that all the conditions of Theorem 2.1 hold except the condition (8) is replaced by*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\rho_s Q_s^* = \infty, \tag{30}$$

and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\frac{(\Delta\rho_s)^2 a_{s-\sigma}}{4\rho_s} < \infty. \tag{31}$$

Then every solution of equation (1) is almost oscillatory.

Before starting the next theorem, we define functions $h, H: N_0 \times N_0 \to \mathbb{R}$ such that

(i). $H_{m,n} = 0$ for $m \ge n \ge 0$, $H_{m,n} > 0$ for $m > n > 0$.

(ii). $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \le 0$ for $m \ge n \ge 0$ and there exists a positive real sequence $\{\rho_n\}$ such that

$$\Delta_2 H_{m,n} + \frac{\Delta\rho_s}{\rho_{s+1}}H_{m,s} = -h(m,s)\sqrt{H(m,s)} \quad for \quad m > s > 0.$$

Theorem 2.2 *Assume that condition (9) holds. If there exists a positive real sequence $\{\rho_n\}$ such that*

$$\limsup_{n\to\infty}\frac{1}{H_{m,n}}\sum_{s=n_0}^{n-1}\left[H_{m,s}\rho_s Q_s^* - \frac{1}{4}\frac{\rho_{s+1}}{\rho_s}a_{s-\sigma}h^2(m,s)\right] = \infty, \tag{32}$$

then every solution of equation (1) is almost oscillatory.

Proof. Proceeding as in Theorem 2.1, we have two cases to consider.

Case I: Assume that $\Delta x_n > 0$ for all $n \ge n_1$. Define w_n by (13), then $w_n > 0$ and satisfies

$$\Delta w_n \leq -\rho_n Q_n^* + \frac{\Delta\rho_n w_{n+1}}{\rho_{n+1}} - \frac{\rho_n w_{n+1}^2}{\rho_{n+1}^2 a_{n-\sigma}}. \tag{33}$$

Multiply both side by $H_{m,n}$ and summing from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} H_{m,s}\Delta w_s \leq -\sum_{s=n_1}^{n-1} H_{m,s}\rho_s Q_s^* + \sum_{s=n_1}^{n-1} H_{m,s}\Delta\rho_s \frac{w_{s+1}}{\rho_{s+1}} - \sum_{s=n_1}^{n-1} H_{m,s}\frac{\rho_s w_{s+1}^2}{\rho_{s+1}^2 a_{s-\sigma}}$$

$$\sum_{s=n_1}^{n-1} H_{m,s}\rho_s Q_s^* \leq -\sum_{s=n_1}^{n-1} H_{m,s}\Delta w_s + \sum_{s=n_1}^{n-1} H_{m,s}\Delta\rho_s \frac{w_{s+1}}{\rho_{s+1}} - \sum_{s=n_1}^{n-1} H_{m,s}\frac{\rho_s w_{s+1}^2}{\rho_{s+1}^2 a_{s-\sigma}}. \tag{34}$$

By using summation by parts, we obtain

$$\sum_{s=n_1}^{n-1} H_{m,s}\rho_s Q_s^* \leq H_{m,n_1} w_{n_1} + \sum_{s=n_1}^{n-1}\left[\left[\Delta_2 H_{m,s} + H_{m,s}\frac{\Delta\rho_s}{\rho_{s+1}}\right] w_{s+1} - H_{m,s}\frac{\rho_s w_{s+1}^2}{\rho_{s+1}^2 a_{s-\sigma}}\right]. \tag{35}$$

Using completing the square in the last inequality, we obtain

$$\sum_{s=n_1}^{n-1}\left[H_{m,s}\rho_s Q_s^* - \frac{\rho_{s+1}}{4\rho_s} a_{n-\sigma} h^2(m,s)\right] \leq H_{m,n} w_{n_1},$$

$$\frac{1}{H_{m,n}}\sum_{s=n_1}^{n-1}\left[H_{m,s}\rho_s Q_s^* - \frac{\rho_{s+1}}{4\rho_s} a_{s-\sigma} h^2(m,s)\right] \leq w_{n_1},$$

which contradicts the assumption (32).

Next we consider the case when $x_n < 0$ for all $n \geq n_1$. We use the transformation $y_n = -x_n$ then y_n is a positive solution of equation

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n^\beta y_{n-\sigma} = -e_n \tag{36}$$

when $z_n = y_n + P_n y_{n-\tau}$. Define w_n by (13) and (29) holds. The remaining of the proof is similar to that of first case of Theorem 2.1 and hence omitted. The proof of the case II is similar to that of second case of Theorem 2.1. The proof is now complete. ■

Corollary 2.2 *Assume that all the conditions of Theorem 2.2 hold except the condition (32) is replace by*

$$\limsup_{n\to\infty} \frac{1}{H_{m,n}}\sum_{s=n_1}^{n-1} H_{m,s}\rho_s Q_s^* = \infty,$$

and

$$\limsup_{n\to\infty} \frac{1}{H_{m,n}}\sum_{s=n_1}^{n-1} \frac{\rho_{s+1}a_{s-\sigma}h^2(m,s)}{4\rho_s} < \infty.$$

Then every solution of equation (1) is almost oscillatory.

3. Examples

In this section, we provide three examples.

Example 3.1. *Let us consider the second order neutral difference equation of the form*

$$\Delta\left(2 + (-1)^n \Delta(x_n + \tfrac{1}{2}x_{n-1})^3\right) + 2x_{n-3} = 2(-1)^{n-4}, \quad n \geq 4. \tag{37}$$

Here $a_n = 2 + (-1)^n$, $p_n = \frac{1}{2}$, $q_n = 2$, $\tau = 1$, $\sigma = 3$, $\alpha = 3$, $\beta = 1$ and $e_n = 2(-1)^{n-4}$. All the conditions of Theorem 2.1 are satisfied. Hence every solutions of equation (37) is almost oscillatory. In fact one such solution is $x_n = (-1)^n$. Here $\{x_n\}$ is oscillatory.

Example 3.2. *Let us consider the second order neutral difference equation of the form*

$$\Delta(2 - (-1)^n \Delta(x_n + x_{n-2})) + x_{n-2}^3 = 14 + 29(-1)^{n-3}, \quad n \geq 3. \tag{38}$$

Here $a_n = 2 - (-1)^n$, $p_n = 1$, $q_n = 1$, $\tau = 2$, $\sigma = 2$, $\alpha = 1$, $\beta = 3$ and $e_n = 14 + 29(-1)^{n-3}$. All the conditions of Theorem 2.1 are satisfied. Hence every solutions of equation (38) is almost oscillatory. In fact one such solution is $x_n = 2 + (-1)^{n+1}$. Here $\{x_n\}$ is non-oscillatory and Δx_n is oscillatory.

Example 3.3. *Let us consider the second order neutral difference equation of the form*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-2})\right) + n^3 x_{n-1}^3 = \frac{(n-1)n(n+1)(n+2)-(6n+9)}{(n-1)n(n+1)(n+2)}, \quad n \geq 3. \tag{39}$$

Here $a_n = \frac{1}{n}$, $p_n = 2$, $q_n = n^3$, $\tau = 2$, $\sigma = 1$, $\alpha = 1$, $\beta = 3$ and $e_n = \frac{(n-1)n(n+1)(n+2)-(6n+9)}{(n-1)n(n+1)(n+2)}$. All the conditions of Theorem 2.1 are satisfied. Hence every solutions of equation (39) is almost oscillatory. In fact one such solution is $x_n = \frac{1}{n+1}$. Here $\{x_n\}$ is tends to zero as $n \to \infty$.

References

[1] R.P. Agarwal, Difference Equations and Inequalities, *Second Edition, Marcel Dekker, New York,* 2000.

[2] R.P. Agarwal, M. Bohner, S.R. Grace and D.'O. Regan, Discrete Oscillation Theory, *Hindawi Publ. Corp., New York*, 2005.

[3] R.P. Agarwal, M.M.S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order delay difference equations, *Appl. Math. Lett.,* **10**(2)(1997), 103-109.

[4] J. Cheng, Existence of nonoscillatory solution of second order linear difference equations, *Appl. Math. Lett.,* **20**(2007), 892-899.

[5] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, *Clarendon Press, Oxford,* 1991.

[6] R. Jankowski and E. Schmeidel, Almost oscillation criterca for second order neutral difference equations with quasi-differences, *International Journal of Difference Equations,* **9**(1)(2014), 77-86.

[7] J. Jiang, Oscillation criteria for second order quasilinear neutral delay difference equations, *Appl. Math. Comput.*, **125**(2002), 287-293.

[8] W.G. Kelley and A.C. Peterson, Difference Equations: An Introduction with Applications, *Acad. Press, New York,* 1991.

[9] G. Ladas, Ch.G. Philos and Y.G. Sticas, Sharp condition for the oscillation of delay difference equations, *J. Appl. Math. Simul.,* **2**(2)(1989), 101-112.

[10] H.J. Li and C.C. Yeh, Oscillation criteria for second order neutral delay difference equations, *Comput. Math. Appl.*, **36**(1999), 123-132.

[11] E. Thandapani, J.R. Graef and P.W. Spikes, On the oscillation of second order quasilinear difference equations, *Nonlinear World,* **3**(1996), 545-565.

[12] E. Thandapani and K. Mahalingam, Oscillation and nonoscillation of second order neutral difference equations, *Czechoslovak Math. J.,* **53**(128)(2003), 935-947.

[13] E. Thandapani and S. Selvarangam, Oscillation theorems for second order nonlinear neutral difference equations, *J. Math. Comput. Sci.,* **2**(4)(2012), 866-879.

[14] E. Thandapani, S. Selvarangam, R. Rama and M. Madhan, Improved oscillation criteria for second order nonlinear delay difference equations with nonpositive neutral term, *Fasciculi Mathematici*, 2016.

[15] E. Thandapani, E. Vijaya and M. Gyori, New oscillation criteria for forced super-linear neutral type difference equations, *Fasc. Math.,* (Pre-Print).