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The Flip side of the Cournot - Nash competition

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Abstract.

The objective of this paper is to analyze the consequences of including environment of players in the Cournot-Nash competition. A two-persons non-cooperative game is considered. The environment is expressed in terms of “others". The function of the environment is to maintain its equilibrium. The equilibrium point of the environment is upgraded if the strategies of one or both players have a favorable impact on the environment. The equilibrium stays at its previous level otherwise. The equilibrium point of the environment is called an environmental attractor. The environmental attractor affects the strategies of the players which in turn define the reaction functions and the pay-off functions. It is shown that in the presence of an environmental attractor, players cannot reach a Nash point. In fact the reaction functions of the players act as orbits around the environmental attractor. Thus each player has a finite but compact number of strategies compatible with the environment available to them. As long as the equilibrium of the environment is maintained, any of the orbits will be acceptable choices for the players. An environmental model of a two-persons competition game is given, and the consequences are studied.

JEL: C72.

Key words: Competition; strategy; environment; environmental vectors; environmental attractors; orbits; reaction functions; payoff functions; player-environment interaction.

1 Introduction

The objective of this paper is to explore the idea that in any strategic game, players act within an environment and this environment can play a role in the outcome of the competition. The environment is expressed in terms of “others". The interaction of the environment with the players is analyzed and its' consequences in terms of finding an equilibrium point is studied. In all strategic competition models it is assumed that the players compete in a void. They take up strategies with respect to the strategies of other player(s). Neither of the players considers the impact of the environment on their strategies and the fact that the environment modifies the nature of each strategy. It is also assumed that the environment does not alter the outcome of the Nash type competition. In general the environment has one objective and that is to maintain equilibrium. The environment either rejects the strategies of both players, in which case it maintains its previous equilibrium and forces players to modify their strategies around the environmental equilibrium point, or it adopts the strategy of one of the players and in this case it updates its equilibrium point. The player whose strategy is accepted by the environment is better off. The other player has to either modify his strategy around the new equilibrium point or otherwise take the consequences of going on independently of the environment. Once one takes into account the environment of the players the concept of a Nash equilibrium point becomes obsolete. There is no winner-winner situation. In this case there will always be a player who will be better off than the other. How much a player is better off depends on how close the player's strategy is with respect to the choice of the environment. This is demonstrated in the following example. In order to clamber up the career ladder, some people slave nights and week-ends at the office. They gain a rank at the expense of their free time. But in doing so they hurt anyone else who aspires to the same goal. They too will have to give up their week-ends to keep up. In this case the strategy of one player has a direct impact on the environment. By his aspirations he has changed the equilibrium of his environment. The person with high aspirations in this example wins the game ignoring the effect of his actions on his immediate environment, his office. Though the high achiever in this example wins the competition and obtains a rank, his actions have altered his environment. His environment has accepted the new equilibrium. This new equilibrium is working longer and harder which is translated into a higher pay-off. If others follow suite and work as hard or somewhat hard they stand to gain just as our high achiever has. But if others choose to go on with their routine they forego the rewards which are higher pay-offs.

An environment can be either an economic environment, or a social environment summed up as “others", or a natural (physical) environment. In this paper only social environment is considered. To model the environment into a strategic game, the following attributes of the environment are recognized and used. Environment is impartial to both players. Environment does not choose strategies with respect to any of the players. Environment has one criterion, it's aim is to stay at equilibrium. If its equilibrium is disturbed it will do one of two things. If the change is unfavorable for environment, then environment will attempt to go back to the previous equilibrium level. If change is favorable for environment, then environment accepts this change and upgrades the previous equilibrium to a new one. Since environment automatically reverts to its previous state of equilibrium or upgrades the existing equilibrium, it affects the outcome of any strategic game by eliminating the player's equilibrium point, and choosing one player over the other as the winner of the competition.

2 Modeling of strategic games within an environment

For the purpose of modeling some assumptions are made. A two-persons non cooperative strategic game is considered. The game has several features. First, the game has rules that govern the order in which actions are taken, describe the array of allowed actions, and define how the outcome of the game is related to the actions taken. Second there are two players, each of whom is struggling consciously to do the best he can for himself. The outcome of a player depends on the actions of the other player. The player knows this, and knows that choosing the best action requires making an intelligent assessment of the actions likely to be taken by the other player. Each player possesses a complete information. Each player knows who the other player is, and all actions available to players, and all potential outcomes to both players. The joint actions of the two players determine each player's pay-off, [1],[2]. Players know that their actions alter the equilibrium of their environment. They know that this alteration is either positive or negative; and that the environment will react to change. They also know that their standing in the game either strengthens or weakens their chances of winning according to whether the environment is affected favorably or unfavorably.

Third, these players are in an environment which is affected by their game. The environment does not compete, it reacts. It is assumed that the environment is at equilibrium at the beginning of a game. This equilibrium is upgraded as a function of each player's strategy. The environment upgrades to a new equilibrium if the strategy of one of the players is favorable. The position of the player whose strategy created this new equilibrium is re-enforced in the game and the other player is now aware of this fact. The other player has to choose a strategy that is the best strategy with respect to the other player and the new environmental equilibrium point. If however the change is unfavorable, then the environment rejects the change and stays at the previous equilibrium level. The position of the player whose strategy created this negative change is weakened; the other player is aware of this. At this point the strong player has no incentive to change his strategy. To react to the other player he modifies his strategy very slightly pivoting around the new environmental equilibrium. The weaker player has two options. He accepts the new environmental equilibrium point and he picks a strategy with minimum standard deviation. Or he continues with the strategy independent of the environment hoping that this way he will incite the other player to respond in the same manner. Naturally as long as the strong player is ahead of the game his motivation to change strategy is very low. But it is possible that he gets manipulated by the weaker player and responds with a different strategy in return. In this case, if both players continue the game ignoring the environment it is possible to reach a Nash point. Then the Nash point is unstable and will change as soon as the environment reacts to the game, even if the players are not competing anymore. The environment will reverse the outcome of the Nash equilibrium since the environment aims at reaching a favorable equilibrium. Both players stand to loose in this situation. Eventually they have to readjust their strategies to the environmental equilibrium point somewhere along the path of the game.

Mathematically speaking, the equilibrium seeking action of the environment acts as an environmental attractor of a dynamic system of two-player competition. This environmental attractor will modify the behavior of the players. Thus these attractors create a new dynamic that affects the player even after the outcome of the game, when in fact the player is not competing anymore. How can environmental attractors be modeled is the topic of the next paragraph. Quantification of the environment in a two persons game is achieved first by defining every single change in the environment as a vector with components being the strategies of the two players; and second quantifying the environmental change as the change that is accepted by the majority of the environment, in other words “others". In reality for each player two situations can occur: 1) each consecutive strategy affects a large number of “others", 2) each consecutive strategy affects a small number of “others". Strategies that are small variations around the environmental attractor affect a smaller number of “others". This gives players a freedom of competing on a level that is almost independent of the environment. They can positively modify their pay-offs. Each can try to improve his pay-off slightly. This will stabilize the situation of each player. If each consecutive strategy of a player is widely different from the environmental attractor, then it impacts a higher number of “others". Players are challenged. The environment will react by forcing players to modify their respective strategies. Either the environment keeps the present attractor or moves to a new one. In both cases players face a new situation where their set of available strategies has to be revisited in order to match the force of the environmental attractor.

Equilibrium points in a non-cooperative competition are found based on the fixed point theory. This means that pay-off functions follow the conditions of the fixed point theory that for every strategy space (S), there is a mapping (P) of (S) such that (S) is projected into itself, ($P : S \rightarrow S$). There exists a fixed point ($s \in S$) such that ($P(s) = s$). An strategy set (S) is a compact set in, ($\mathbb{R}^n$) [3]. The introduction of the environmental equilibrium point adds complications. Complications occur in two directions. One is that the existence of the environmental attractor changes the nature of the strategy sets of the players. Second is the change in the behavior of the pay-off functions of players towards the environmental attractor. Environment eliminates the fixed point due to competition. The actual fixed point is decided by the environment. The existence of the environmental attractor changes the nature of the strategy sets. These sets will no longer be compact sets. To show this one must return to the general definition of compact sets, and the definition of a fixed point space. By definition, compact sets have the property that every collection of open sets has a finite sub-collection with the same property. A set is a compact set if and only if it is closed and bounded. This property no longer holds when an environmental attractor is introduced. The existence of this attractor means that many open sets around each set point can be considered as valid sub-sets. These sets will never be closed and bounded. This is due to the fact that at any point the environment can upgrade its equilibrium meaning that it changes the attractor thus allowing for open sets to overlap. The strategy set in the presence of the environmental attractor cannot have a fixed point property since it loses its compact property. Given (s) an element of the strategy space (S), no mapping can be found that would give (s = P(s)). The second reason is that pay-off functions in the presence of the environmental attractor become non-contractive functions. The environmental attractor forces the players to choose strategies around it. Thus pay-off functions become orbits of the environmental attractor. The idea of pay-off functions as orbits becomes clear later on when mathematical formulation of these functions is introduced.

The vector formulation of the environmental attractor ($\vec{e}$) starts here. Total number of players is (N=2). Let (S_i) be the strategy space of player (i), and (S_j) be the strategy space of player (j). Let ($S = S_i \times S_j ; i \neq j$), the strategy space of the game be the Cartesian product of the individual strategy spaces, (S_i), and (S_j). Let ($s_k^i \in S_i$), denote player (i) with strategy (k), where (($k \in K$), $K = 1, \cdots, n, K \backslash \{0\}$). (n) represents a finite set of strategies available to both players (i), and (j). let ($s_k^j \in S_j$), denote player (j) with strategy ($k' \neq k$). Before the beginning of the game, the environment is at equilibrium. In Figure 1, this equilibrium is denoted by ($\vec{e}^0_{s_0^i} = \vec{e}^0_{s_0^j}$). None of the players (i), and (j) have chosen a strategy, meaning strategy ($s_0^i = s_0^j = 0$). If both players choose the same strategy, ($s_l^i = s_l^j$), and player (i) wins the competition with strategy(l) by receiving a higher pay-off, then environment adopts the strategy of the winning player, i.e. (i). The old equilibrium point is upgraded to a new equilibrium point ($\vec{e}_{s_l^i}$). The environmental attractor for the new equilibrium point is denoted by ($\vec{e}_{s_l^i}$). Environment simply ignores the other player,(j). On the other hand, if the environment accepts the strategy choice of player (j), irrespective of a lower pay-off, then the environmental attractor shifts to (s_l^j) as is shown in Figure 1. The player whose strategy choice is adopted by the environment, is the actual winner. In Figure 1, the grid represents the field of environmental attractors. The grid is made of (n) rows of the strategies of player (i), and (n) columns of the strategies of player (j). The process of upgrading the equilibrium point continues as long as the players are engaged in a competition and as long as the environment reacts to the strategies of the winner. Each time a new environmental attractor is found it replaces the old one and the process continues from this point on.

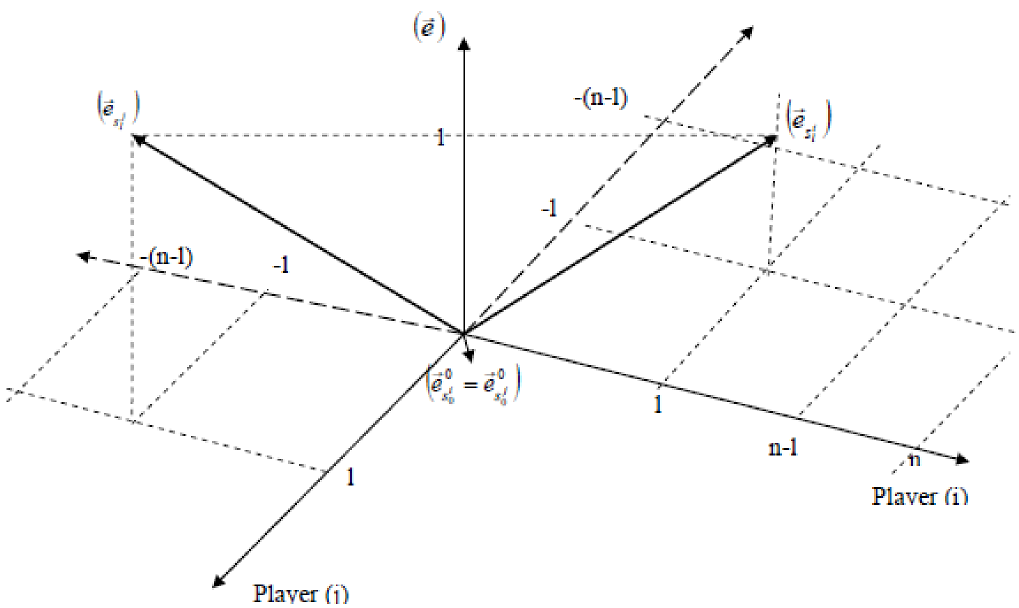

Figure 1. Position of environmental attractor when players choose the same strategy

If each player chooses a different strategy, (s_k^i), and ($s_{k'}^j$), where ($k \neq k'$), and player (i) wins the competition with strategy $(k \in (1, \cdots, n))$ in the set of given strategies, $((1, \cdots, n))$, by receiving a higher pay-off, then the environment must decide whether to adopt the strategy of the player that fits best the environment irrespective of whether player (i) wins or not. If the strategy of player (i) is adopted, then the old equilibrium point is upgraded to a new equilibrium point $(\vec{e}_{s^i_k})$, otherwise the equilibrium point of the environment does not change. It stays at the previous equilibrium point. The environmental attractor for the new equilibrium point is denoted by $(\vec{e}_{s^i_k})$ if the environment adopts the strategy of player (i). If on the other hand, it is player (j) who wins the competition with strategy (k'), by receiving a higher pay-off, and the environment adopts the strategy of the winning player, (j), (k'), then the old equilibrium point is upgraded to a new equilibrium point. The environmental attractor for the new equilibrium point is denoted by $(\vec{e}_{s^j_{k'}})$. This situation is shown in Figure 2, with the strategy of player (i), being (k=n-l), and the strategy of player (j), being $(k' = l)$. Figure 3, depicts the trajectory of the environmental attractor during (3) consecutive competition runs. In Figure 3, $(R(h), h = 1, 2, 3)$ indicates rounds of competition. After the first round of competition R(1), the environmental attractor is at equilibrium point $(\vec{e}^{\,1}_{s^i_l})$. The second round of competition, R(2), shifts the environmental attractor to equilibrium point $(\vec{e}^{\,2}_{s^i_{n-l}})$. The equilibrium point shifts from $(\vec{e}^{\,1}_{s^i_l})$ to $(\vec{e}^{\,2}_{s^i_{n-l}})$. The third round of competition R(3), shifts the environmental attractor to equilibrium point $(\vec{e}^{\,3}_{s^j_l})$. This process goes on as long as players compete and the environment accepts the strategy of one of the players.

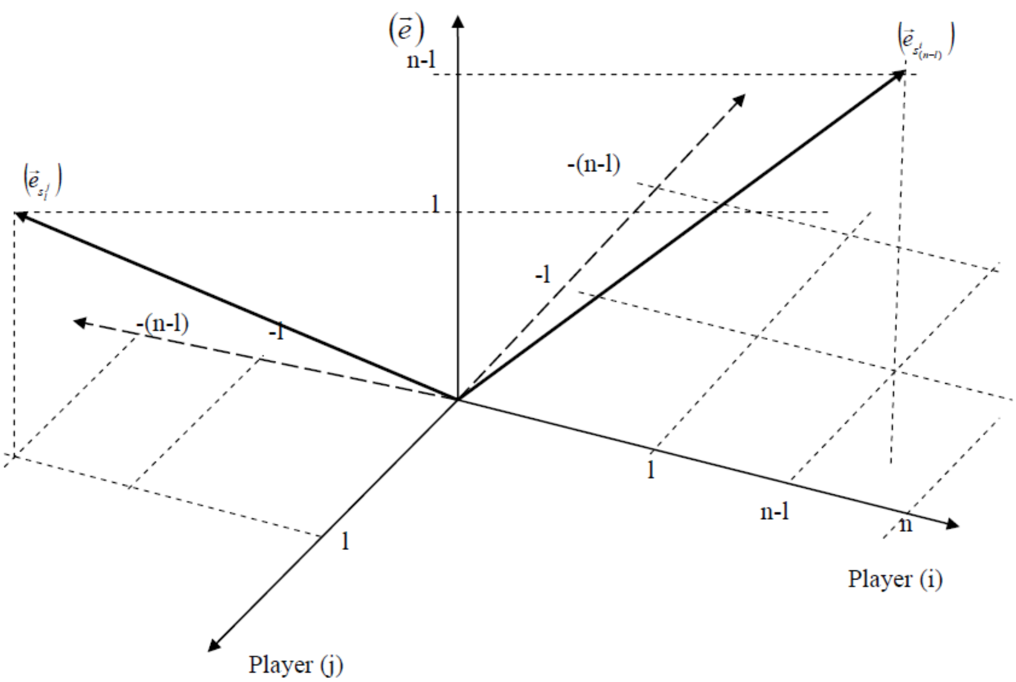

Figure 2. Position of environmental attractor when players choose different strategies

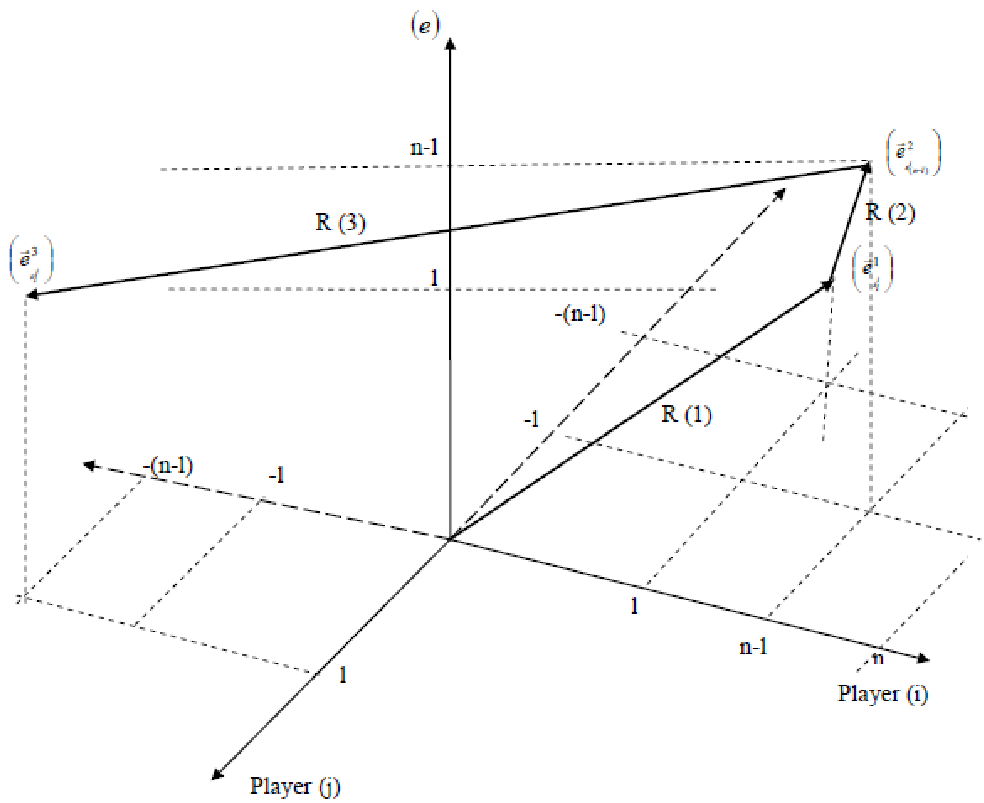

Figure 3. Trajectory of environmental attractor after (3) rounds of competition with different strategies

Shift in environmental attractor occurs if the environment accepts the strategy of one of the players. Let the environment consist of (M) number of people not involved in the Cournot-Nash competition. The environment is designated by "others". Each environmental attractor represents a subset of (M) who has accepted the strategy of a player. Let's designate this subset by $(L^r_{s^i_k} \in M; k = 1, \cdots, n; i = 1, \cdots, I)$. (r) is the environmental attractor number,(k) is a strategy, and (i) is the number of players, (in this case I=2). Each new environmental attractor,$(\vec{e}^{\,r}_{s^i_k})$ has a corresponding subset $(L^r_{s^i_k})$. Each time there is a shift in environmental attractor, this implies that the new environmental subset is larger than the previous one, $(L^r_{s^i_k} > L^r_{s^i_{k-1}})$. This procedure is demonstrated in Figure 3. In Figure 3, the initial environmental attractor is at $(\vec{e}^{\,0}_{s^i_0} = \vec{e}^{\,0}_{s^j_0})$, where $(\vec{e}^{\,0}_{s^i_0} = a^i_{s^i_0} \times \vec{s^i_0} - b^j_{s^j_0} \times \vec{s^j_0} + c^i_{e^i_0} \times \vec{e^i_0})$ or $(\vec{e}^{\,0}_{s^j_0} = b^j_{s^j_0} \times \vec{s^j_0} - a^i_{s^i_0} \times \vec{s^i_0} + c^i_{e^j_0} \times \vec{e^j_0})$. $(\vec{s^i}, \vec{s^j}, \vec{e^\bullet})$ are unit vectors representing player(i) strategy choice axis, player (j) strategy choice axis, and the environment strategy choice axis,$(\vec{e^\bullet})$ where $(\bullet)$ represents either (i), or (j). $(a^i_{s^i_0})$, $(b^j_{s^j_0})$, and $(c^\bullet_{e^\bullet_0})$, are the coefficients. The coefficients are the quantitative values of the strategy choices of the players. The strategy that is not adopted by the environment, is represented as a negative entity. The environmental attractor is at point zero since players have chosen not to compete $(s^i_0 = s^j_0 = e^\bullet_0 = 0)$. In the first round of competition, R(1), the environmental attractor shifts to a new point($\vec{e}^{1}_{s^i_l}$) once the players compete with strategies (s^i_l, s^j_{n-l}) and a subset of (M), $(L^1_{s^i_l} \in M)$, $(L^1_{s^i_l} > L^1_{s^j_{n-l}})$, and $(L^1_{s^i_l} \bigcap L^1_{s^j_{n-l}} = 0)$ adopts the strategy of player (i). The environmental attractor of the first round R(1), is formulated as $(\vec{e}^{1}_{s^i_l} = a^i_{s^i_l} \times \vec{s^i} - b^j_{s^j_{n-l}} \times \vec{s^j} + c^i_{e^i_l} \times \vec{e^i}$). In the second round of competition, R(2), the players compete with strategies (s^i_{n-l}, s^j_l). Again the environment adopts the strategy of player (i), and the environmental attractor shifts to point $(\vec{e}^{2}_{s^i_{n-l}} = a^i_{s^i_{n-l}} \times \vec{s^i} - b^j_{s^j_l} \times \vec{s^j} + c^i_{e^i_{n-l}} \times \vec{e^i})$. The subset of (M), that accepts the strategy of player (i) in R(2), $(L^2_{s^i_l} > L^1_{s^i_l})$, is greater than the subset in R(1). In the third round, R(3), the environmental attractor shifts to $(\vec{e}^{3}_{s^j_{n-l}} = -a^i_{s^i_{n-l}} \times \vec{s^i} + b^j_{s^j_l} \times \vec{s^j} + c^j_{e^j_l} \times \vec{e^j})$. Players compete with strategies (s^i_l, s^j_{n-l}). This time the majority of the environment adopts the strategy of player (j), and thus $(L^3_{s^j_{n-l}} > L^2_{s^i_l})$. The environmental attractors shift as long as players compete and the majority takes sides.

3 Analysing environmental attractor

In the previous section two elements are identified as important factors when environmental attractors are introduced in a two-persons competition. Element one is the strategy sets of players and the environment. The strategy sets of players (i), and (j) and the environment (e), $(S_i \subset S)$, $(S_j \subset S)$, $(E \subset S)$ are subsets of a general compact and convex space (S). It is defined that there exists a mapping (P) from (S_i) into (S_i). Any element $(s \in S_i)$ has a mapping (P) such that $(s = P(s))$, in which case (s) is considered to be a fixed point of (S_i). Similarly, there exists a mapping (P') from (S_j) into (S_j). Any element $(s' \in S_j)$ has a mapping (P') such that $(s' = P'(s'))$, in which case (s') is considered to be a fixed point of (S_j). Let $(E = (e_1, e_1, \cdots, e_M) : E\backslash\{S_i\}; E\backslash\{S_j\})$ be the set of strategy choices for the environment. (M) is the number of strategies in the set (E). These strategy choices do not include the strategy choice set of players (i), and (j). Set (E) is a compact convex set such that $((E \cap S_i) = 0)$, and $((E \cap S_j) = 0)$. Once players (i), and (j) are engaged in a competition, then, the strategy set of the environment (E) has to expand to include (S_i), and (S_j). In order to simplify the arguments that follow, let the two players have the same strategy set, $(S_i = S_j \subset S)$. Let's denote this equality set by (Ł). This simplification is possible since the case $(S_i \neq S_j)$, where the two strategy sets of players are not equal, is just an extension of the equality case. Let's denote the expanded set (E), by $(E' = E \cup Ł)$. Must show that the expanded set (E'), has a fixed point in the set (Ł). The following theorem proves the existence of a fixed point for the expanded set (E').

Given that players (i), and (j) are engaged in a competition, the strategy sets of the two players have fixed points. Let (s) be the fixed point of the set (Ł). Let (f) be a map $(f : Ł \to Ł)$ such that $(f(s) = s)$. The strategy set (E) of the environment or the "others" has a fixed point, since by induction "others" is an extension of one player, and the strategy set of a player is a fixed point set, then the strategy set of the environment is a fixed point set. Let the fixed point of the strategy set of the environment (E), be denoted by point (e). There exists a map $(g : E \to E)$ such that $(g(e) = e)$. The existence of fixed points for the sets (Ł), and (E), does not follow that the expanded set (E') is a fixed point set. To illustrate that the set (E') is a fixed point set, the following Theorem is introduced.

Theorem 3.1. *If sets (Ł), and (E), are fixed point sets, then the expanded set (E') is a fixed point set.*

Proof. By definition, the strategy set of the environment can be expanded to include the strategy set of the players, $(E' = E \cup Ł)$. Since both sets (Ł), and (E) are retractions of the set (E'), and they both are fixed point sets, then the set (E'), is a fixed point set. □

Theorem 3.2. *The fixed point of the set (E') is the point (s), the fixed point of the set (Ł).*

Proof. Let $(B_\epsilon(e))$ be a closed ball with center (e), and radius $(\epsilon = d(e - s))$, where $(d(e - s))$ is the distance from point (e) to point (s) on the boundary $(s \in \delta B_\epsilon)$ of $(B_\epsilon(e))$. Let (f) be a map $(f : Ł \to Ł)$, and g be a map $(g : E \to E)$, then by the Banach contraction principal, [8], $(d(f(s) - g(e)) \leqslant \alpha \times d(e - s))$, with $(\alpha = 1)$. Now, taking the limit of both sides as $(\lim_{(e \to s)})$, $(\lim_{(e \to s)} d(f(s) - g(e)) \leqslant \lim_{(e \to s)} d(e - s))$, shows that there exists a map d : $(d : B_\epsilon(e) \to E')$ that takes (E') to the boundary (δB_ϵ) of the closed ball $(B_\epsilon(e))$. By the Theorem of nonlinear alternative, [9] there exists a unique fixed point, (s) on the boundary of the ball $(B_\epsilon(e))$ such that $(s = f(s))$. □

Based on Theorem 3.1 the set (E') will always have a fixed point if it is made up of exactly (2) fixed point sets. By Theorem 3.2 the fixed point of the set (E') is found in the set (Ł). The conclusion is that the fixed point of the environment strategy set (E') is in the set (Ł) of the strategy sets of the players in the Nash equilibrium.

4 Reaction functions and environmental attractors

The payoff functions of players (i,j), are denoted by $(g_{s_k^i} \in \Re)$, and $(g_{s_k^j} \in \Re)$, where $(\Re)$ is the space of real numbers. $(g_{s_k^i})$ takes on a new characteristic in addition to the characteristics that are attributed to it in the standard Cournot-Nash game such as continuity, boundedness, and concavity. The new characteristic is the strategy dynamics. Strategies of players interact with each other and create a dynamic. This dynamic dictates what diametrically opposed players can gain and loose. The payoff function, $(g_{s_k^i}(p))$ is the function of strategy dynamics of players $((i, j); i \neq j)$. The payoff functions in the presence of the environmental attractor become orbits, [3]. The reason for the shape of the payoff functions is the behavior of the reaction functions which become dynamic due to the interaction of the strategies of the two players. The payoff functions rather than being the formulation of gains, they reflect the interplay of strategies.To define the payoff function one must define the interaction of the these strategies. These strategies in turn are influenced by environmental attractors. To construct and formulate strategy interactions one can attach a desirability (V) level to each choice. Logically each time a player picks a strategy this strategy has to have the highest level of desirability for that player. Desirability is a function of the strategies of the players. Each player picks the strategy that maximizes his gains, while simultaneously reducing the strategy choices of his opponent and forcing him to pick a less desirable strategy, and thus incur a loss. This is shown in Figure 4. At each round of competition, there exists one strategy with the highest desirability level. Once a player selects this strategy, the other player is forced to pick a strategy in either region (1), or region (2). If the opponent picks the same exact strategy, it is known in advance that he is incurring losses. Therefore, if for example, player (i) picks strategy (l), among (n) available strategies, then player (j) has to pick strategies either within the set $(1, \cdots, (l-1))$, or the $((l+1), \cdots, n)$. Since the two sets are symmetric with respect to desirability, let's assume that player (j) picks a strategy from set $((l+1), \cdots, n)$. In the second round of competition, the two players must choose from the set of residual strategies. The residual strategy set does not include the strategy that is adopted by the environment. At this point there are (n-1) strategies available to players (i), and (n-2) strategies available to player (j). This is shown in Figure 5.

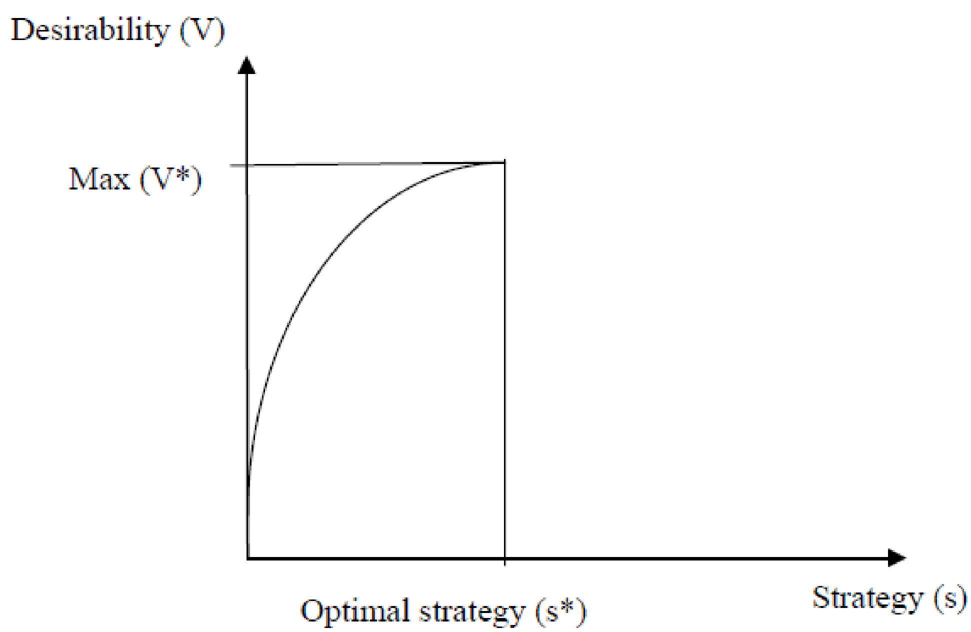

Figure 4. Strategies and their desirability levels

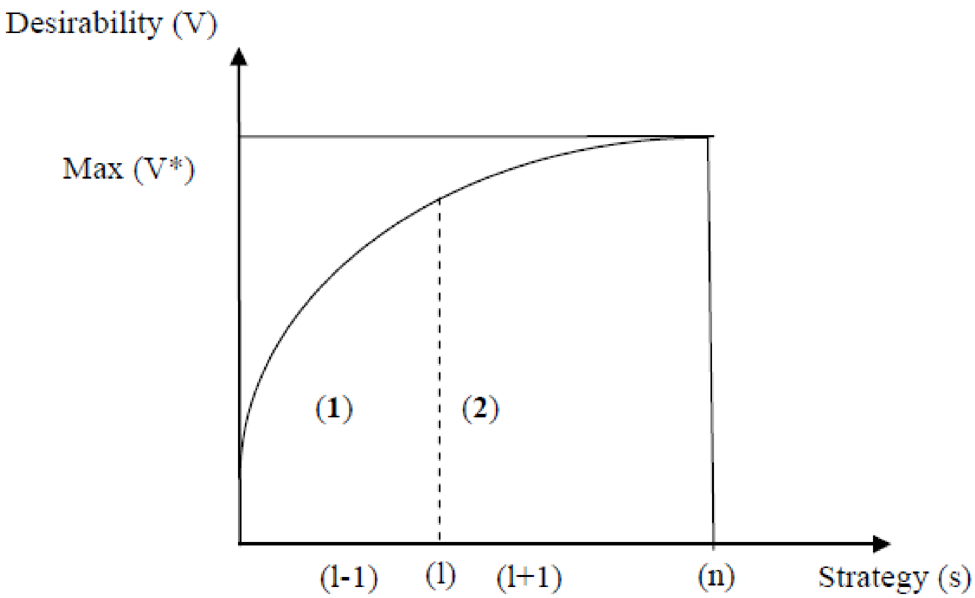

Figure 5. Strategy regions for player (j)

Added to this desirability factor is the interplay of the strategies of the players. This is formulated in the reaction functions (φ_x), where $(x = s_k^i)$ and (φ_y), where $(y = s_k^j)$ of players (i), and (j) respectively. The best formulation for the reaction functions is a polynomial formulation. The polynomial formulation of the strategy interaction for player (i) is given as:

$$\varphi_x = x \times (x^2 + y^2 + \lambda) \tag{4.1}$$

$(V = \lambda)$ is a constant the choice of which depends on the desirability level of the strategy choice of player (i). The interpretation of equation (4.1) is as follows: The strategy choice of player (i) is given by variable (x). The strategy choice of the other player, (j) is given by variable (y). Player (i) is confident about his strategy choice, which is shown by the power of (x) and the value of (λ). But player (i) takes into account the importance of the strategy choice of the other player, the term (y^2) and acknowledges the interaction of his strategy choice with the strategy choice of the other player, the term $(x \times y^2)$. A numerical example of such a function can be given in Figure 6. It is assumed that both players (i) and (j) have 100 strategy choices. The desirability range for both players is between (0) and (10).

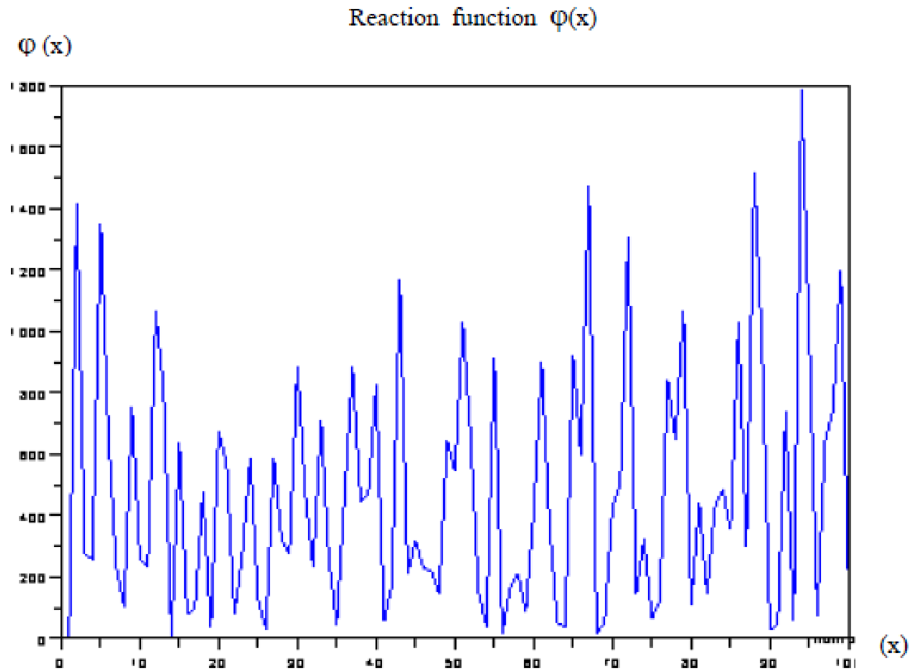

Figure 6. Graphical representation of polynomial function (φ_x)

Values of the polynomial (φ_x) give a quantitative meaning to the reaction function of players (i). In Figure 6, the x-axis represents the number of strategies of player (i), which is chosen to be equal to (100). When player (i) picks strategy (x=95), the reaction function ($\varphi_x = 1900$) units is at maximum. The strategy dynamics of player (j) can be represented by a polynomial function similar to player (i) denoted by (φ_y).

$$\varphi_y = y \times (x^2 + y^2 + \mu) \tag{4.2}$$

Equation (4.2) has the same interpretation as equation (4.1). In equation (4.2) player (j) has confidence in his choice of strategy, variable (y), which is re-enforced. Still the strategy choice of player (i) affects player (j), and this is represented by variable (x). ($V = \mu$) is the desirability level of the strategy choice of player (j). The numerical example of the reaction function of player (j) is given in Figure 7. The x-axis represents the strategy choices of player (j). When player (j) picks strategy (y=85), the reaction function ($\varphi_y = 1800$) units is at maximum.

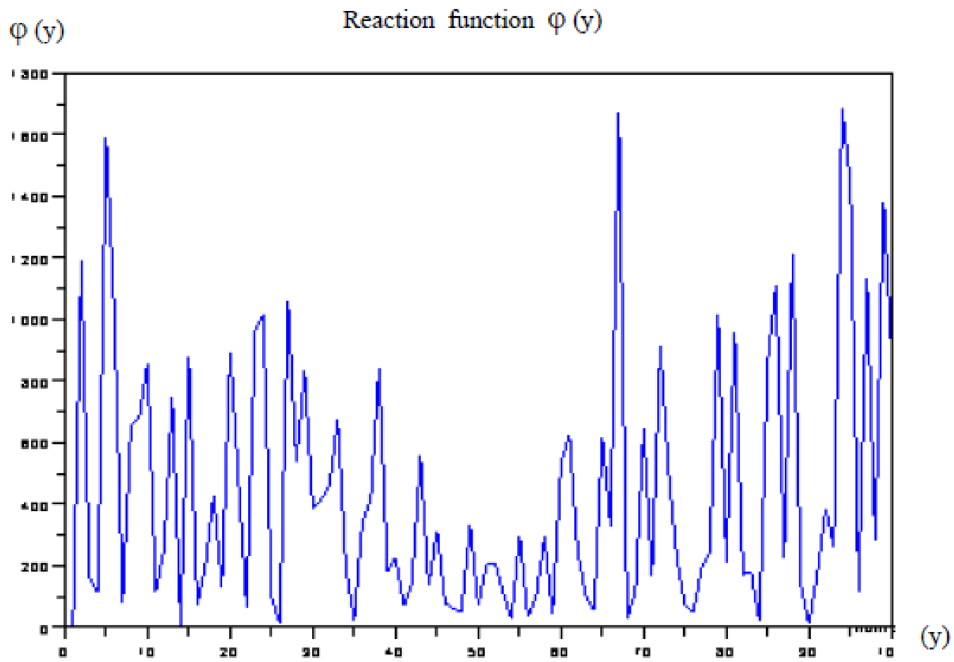

Figure 7. Graphical representation of polynomial function (φ_y)

Environmental attractor has a significant influence on the strategy choice dynamics of the players. The reaction function of the environment is denoted by a polynomial equation $(\varphi_{x,y})$ of degree (4) as:

$$\varphi_{x,y} = (x^2 + y^2)^2(\pm)x^2(\pm)y^2(\pm)\upsilon \tag{4.3}$$

Equation (4.3) can be interpreted as the magnitude of the environmental vector which is fortified and prioritized by adding the strategy choice of one player as an acceptable environmental variable and subtracting the strategy choice of the other player as the rejected variable. Variable (υ) represents either the desirability level of the strategy choice of player (i), (λ), or of player (j), (μ). Figures 8, and 9, give numerical examples of polynomial $(\varphi_{x,y})$. In Figure 8, the environmental attractor accepts the choice strategy of player (i), which is (x = 95). The value of the corresponding environmental attractor $(\varphi_{x,y} \cong 34000)$ is maximum at (34000) units. In Figure 9, environmental attractor accepts the choice strategy of player (j), which is (y = 85). The value of the corresponding environmental attractor $(\varphi_{x,y} \cong 24000)$ is maximum at (24000) units.

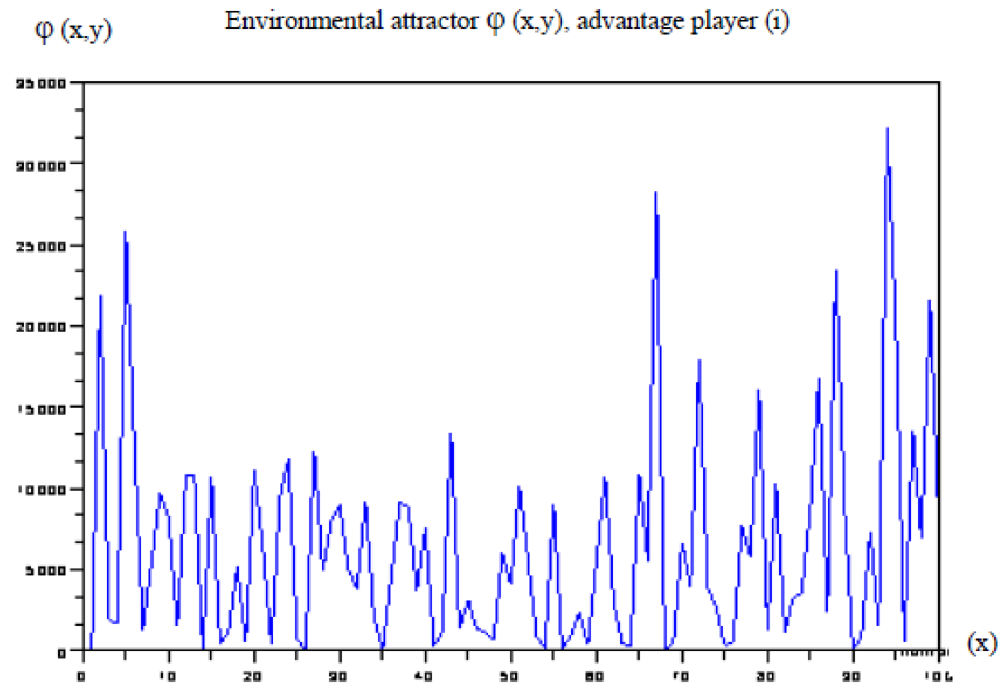

Figure 8. Graphical representation of polynomial $(\varphi_{x,y})$, advantage player (i)

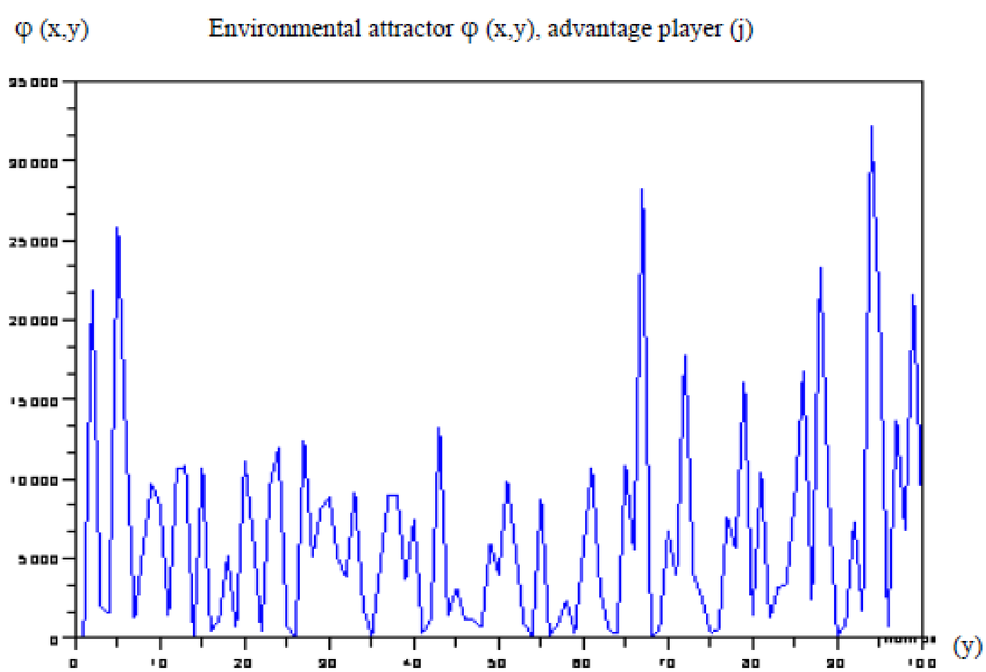

Figure 9. Graphical representation of polynomial $(\varphi_{x,y})$, advantage player (j)

5 Formulation of desirability

The most important factor about choosing a strategy, is the degree of its effectiveness. Given that players (i), and (j) both have the same strategy set $(s_1, \cdots, s_n)$, then the desirability of any strategy for player (i) is whether this strategy can maximize his pay-off function, minimize the pay-off function of the opponent, and limit the strategy choices of player (j). This means that at any round of competition, if player (i) can pick a strategy among the set of available strategies, $(s_1, \cdots, s_n)$, then player (j) can pick among (n-1) residual strategies. Let the strategy set of player (i) be denoted by (S), and the residual strategy set of player (j) be denoted by $(S' \in S)$. Let $(s^r \in S)$ be the strategy chosen by player (i) during the round (r) of competition, and $(s'^r \in S')$ be the strategy chosen by player (j) from the set of residual strategy set (S'). Let $(V(s^r, s'^r))$ represent the desirability function. The objective of player (i) is to maximize the desirability of the chosen strategy (s^r), while minimizing the desirability of the strategy chosen by player (j), (s'^r), $(\underset{s^r}{Max} V(s^r, s'^r) = \underset{s^r}{Max}\underset{s'^r}{Min} V(s^r, s'^r))$. Given that there is a finite choice of strategies available during a round of competition, it is appropriate to formulate the desirability function as, $(V(s^r, s'^r) = w \times s^r \times e^{(-a \times s'^r)})$. It is assumed that the desirability function $(V(s^r, s'^r))$ is differentiable. Before going on to finding the solution of the desirability function, must introduce some background materials. Let the set of strategies be an ordered set based on the perceived importance of each strategy. The ordered set of strategies is denoted by $(S^0 = (s_1^0 < s_2^0 < \cdots < s_n^0)$. The residual ordered set available to the competing player is denoted by $(S'^0 \in S^0)$. At any round of competition, once one player chooses from the ordered strategy set (S^0), then there are (n-1) residual strategy choices left for the competing player to choose from [10]. The following theorem and Lemma are used to find the maximum-minimum solution to the desirability function.

Theorem 5.1. *Given an ordered set $(S^0 = (s_1^0 < s_2^0 < \cdots < s_n^0)$, and given $((s_l^0)$, the strategy choice, then the directional derivative of $(V(s_l^0))$, $(D_{s_j^0} V(s_l^0))$ from either direction $(j < l)$ or $(j > l)$ exists and is given by $(D_{s_j^0} V(s_l^0) = \frac{(V(s_j^0) - V(s_l^0))}{d_j})$ where $(d_j = |s_j^0 - s_l^0| = \sqrt{(s_j^0 - s_l^0)^2})$ if $(j < l)$, and $(d_j = |s_j^0 - s_l^0| = \sqrt{(s_j^0 - s_l^0)^2})$ if $(j > l)$. Then $(\lim_{s_j^0 \to s_l^0} \sum_{j=1}^{l-1} D_{s_j^0} V(s_l^0) = s_l^0)$, for $(j < l)$.*

Proof. Given (s_l^0), the function $(V(s_l^0))$ is maximum at this point. By definition the derivative is given as $(D_{s_j^0} V(s_l^0) = \frac{(V(s_j^0) - V(s_l^0))}{d_j})$. Since the set $(S^0 = (s_1^0 < s_2^0 <$ $cdots < s_n^0)$ is ordered then $((V(s_1^0) < V(s_2^0) < \cdots < V(s_n^0))$. Therefore $((D_{s_j^0} V(s_1^0) < D_{s_j^0} V(s_2^0) < \cdots < D_{s_j^0} V(s_l^0))$ for $(j < l)$ and $((D_{s_j^0} V(s_n^0) > D_{s_j^0} V(s_{n-1}^0) > \cdots > D_{s_j^0} V(s_l^0))$ for $(j > l)$. The left derivatives of the point $((s_l^0)$ are positive and in increasing order, $(D_{s_j^0} V(s_l^0) > 0)$, and the right derivatives of the point (s_l^0) are negative and in decreasing order, $(D_{s_j^0} V(s_l^0) < 0)$. Maximization requires that only the left side of the point (s_l^0) should be considered. This results in the limit of the derivatives approaching the point (s_l^0), $(\lim_{s_j^0 \to s_l^0} \sum_{j=1}^{l-1} D_{s_j^0} V(s_l^0) = s_l^0)$ for $(j < l)$. □

Lemma 5.2. *Suppose that in the ordered set $(S^0 = (s_1^0 < s_2^0 < \cdots < s_n^0)$, there is one strategy choice that maximizes the desirability function $(V(s^r, s'^r))$. Then there* *exists* $(\varrho > 0)$ *such that the directional derivative* $(D_{s_j^0} V(s_l^0) = \varrho)$ *if* $(s_j^0 \in [s_1^0, s_{l-1}^0])$.

Proof. From Theorem 5.1, for any point $(s_1^0 < s_j^0 < s_l^0)$, the directional derivative $(D_{s_j^0} V(s_l^0) > 0)$ is positive, and at the limit $(\lim_{s_j^0 \to s_l^0} D_{s_j^0} V(s_l^0) = s_l^0)$. In general, this can be extended to any multiples of (s_l^0), which is denoted by (ϱ). Therefore, Theorem 5.1, can be generalized to $(D_{s_j^0} V(s_l^0) = \varrho)$. □

By Lemma 5.2, the directional derivative of $(V(s^r, s'^r) = w \times s^r \times e^{(-a \times s'^r)})$, is $(D_{s^r} V(s^r, s'^r) = \varrho(s^r))$ for any choice of $(s^r > 0)$, where $(\varrho(s^r))$ is a fixed value that corresponds to the strategy choice (s^r).

6 Payoff functions as orbits of the environmental attractor

The payoff functions of players (i), and (j) can be formulated as the interaction of the strategy dynamics of the two players. The payoff function of player (i), given the strategy choice of player (j) is $(g_{x,y})$, and the payoff function of player (j) given the strategy choice of player (i), is given as $(g_{y,x})$. The payoff functions are formulated as in equations (6.1), and (6.2).

$$g_{x,y} = v_x \times (\varphi_x \times \varphi_{x,y}) - v_y \times \varphi_y \tag{6.1}$$

$$g_{y,x} = v_y \times (\varphi_y \times \varphi_{x,y}) - v_x \times \varphi_x \tag{6.2}$$

Both (v_x) and (v_y) are monetary rewards associated with strategy choices of players. The Nash equilibrium point is where the derivatives of player's payoff functions with respect to the strategy choices of the players are equal. This is shown in equation (6.3).

$$\frac{\partial g_{x,y}}{\partial x} = \frac{\partial g_{y,x}}{\partial y} \tag{6.3}$$

The derivative of the player with strategy choice (x), $(\frac{\partial g_{x,y}}{\partial x} = f(v_x, \lambda))$ is a function of the monetary reward, (v_x), and the magnitude of desirability, (λ). Similarly, the derivative of the player with strategy choice (y), $(\frac{\partial g_{y,x}}{\partial y} = f(v_y, \mu))$ is a function of the monetary reward, (v_y), and the magnitude of desirability, (μ). Thus the payoff functions of the players become orbits around the strategy choice attractor. Figure 10, shows payoff functions as orbits around the environmental attractor (0). The environmental attractor (0) occurs when players (i), and (j), do not choose strategies or the environment chooses not to adopt any of the strategy choices of players (i), and (j) at the start of the competition procedure. A range (0,0.35) around the environmental attractor (0) is considered.

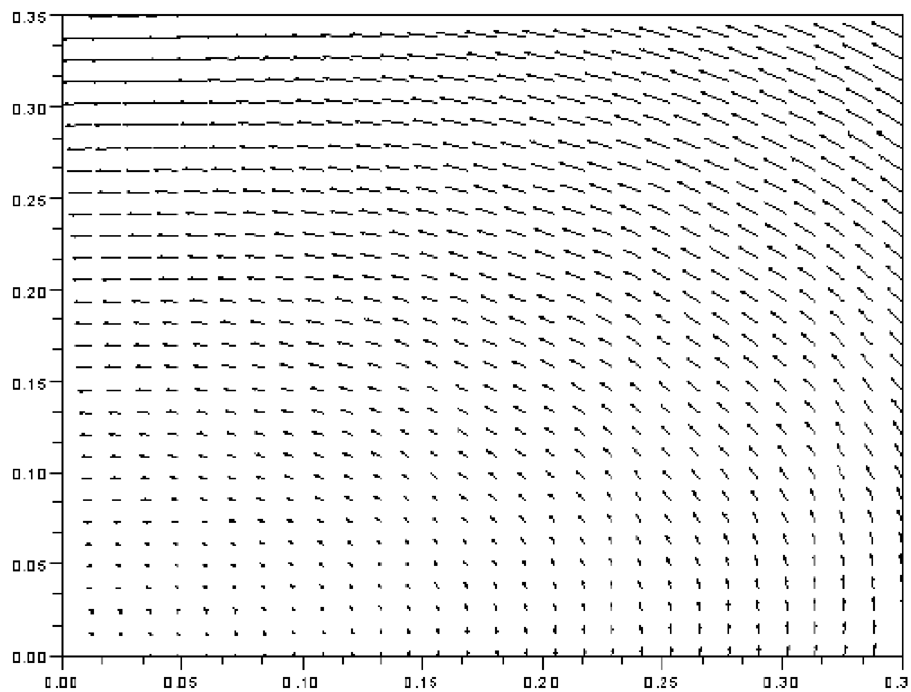

Figure 10. Payoff functions as orbits around the environmental attractor (0)

Payoff functions do not intersect at a fixed point. How can an equilibrium point be defined, [4],[5],[6],[7]. The most evident answer is that there is no equilibrium point. A player adopts an orbit that best fits his objectives and best corresponds to the environmental attractor. The other player reacts to this and in his turn adopts an orbit that best responds to the environmental attractor and the other player. The orbits of the two players cannot intersect and thus prevent any possibility of an equilibrium solution. The non-existence of an equilibrium point is not a sign of chaos. Though no equilibrium point can be found, there is still many possibilities for the two players to choose among orbits available to them and find one that is most optimal for them.

7 Conclusion

In this paper a special case of strategic competition is presented. The idea of competition within an environment that reacts is new. It presents new aspects of a strategic competition that have not been considered up to now. The environmental interference is different from cooperative or n-persons games. Environment does not compete. It has one permanent objective and that is to preserve the equilibrium of the environment. If this equilibrium is disturbed then environment reacts to restore it back to either its previous state or if change is perceived as positive for the environment, then it will upgrade the previous state to a new state. Environment does not take sides;

it adopts a strategy of one of the players if it is an improvement compared to its previous state. Though this act re-enforces the position of the player whose strategy is chosen by the environment, it does not in any way lead to the player winning the game. Therefore, environment does not cooperate or ally itself with any of the players.

New ideas are introduced in this paper. The environmental attractor represents the impact of the environment on the players. Players can not choose strategies to counteract the environment. The attractor forces the players to modify their strategies with respect to the environmental attractor. The reaction functions of the players behave differently from what is assumed in the fixed point theory. The reaction functions become orbits around the environmental attractor. This concept is new. Is there an equilibrium point in such a circumstance? The answer in this paper is no. There is no equilibrium point in the traditional sense. Each player chooses an appropriate orbit that fits best the objectives that he pursues given the environmental attractor.

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