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Inverse Problems For Discontinous Coefficients Sturm-Liouville Operators Depending On The Spectral Parameter Boundary Conditions

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Abstract. In this study, depending on the spectral parameter boundary conditions discontinuous coefficients Sturm-Liouville is investigated . Also Weyl function for this problem under consideration has been de.ned and uniqueness theorems for solution of inverse problem according to this function have been proved .

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1. Introduction.

In this study, we concern the problem L:

$$\ell(y) := -y'' + \left(\frac{A}{x^{\alpha}} + q_o(x)\right) y = \lambda^2 \rho(x) y, \quad 0 < x \leq \pi \tag{1.1}$$

$$(\Gamma_\alpha y)(0) - h\, y(0) = 0$$

$$\lambda\left(y'(\pi) + H_1 y(\pi)\right) = y'(\pi) + H_2 y(\pi)$$

where

$$\rho(x) = \begin{cases} 1, 0 < x \leq a \\ \beta^2, a < x \leq \pi \end{cases} \tag{1.2}$$

$A \in R^+, 1 \leq \alpha < 2$, $\beta > 0, \beta \neq 1$ are real numbers, $q_o(x)$ is real valued boundary function in $L_2(0,\pi)$, λ is a spectral parameter.

We denote that in spectral theory, the inverse problem is the usual name for any problem in which it is required to ascertain the spectral data that will determine a differential operator uniquely and a method of construction of this operator from the data. This kind of problem was first formulated and investigated by Ambartsumyan in 1929 [1]. Since 1946, various forms of the inverse problem have been considered by numerous authors G. Borg[2], N. Levinson [3], B.M. Levitan [4], etc.and now there exists an extensive literature on the [6]-[10]. Later, the inverse problems having specified singularities were considered by a number of authors [5], [19], [21]. The method of spectral mappings is an impressive device for investigating a profound class of inverse problems not only for Sturm-Liouville operators, but also for other more complicated classes of operators such as differential operators of arbitrary orders, differential operators with singularities and others.We apply the method of spectral mappings to the solution of the inverse problem for the Sturm-Liouville operator on a finite interval. In the method of spectral mappings we begin from Cauchy's integral formula for analytic functions. We apply this theorem in the complex plane of the spectral parameter for specially constructed analytic functions having singularity connected with the given spectral characteristics. This permits us to reduce the inverse problem to the so-called main equation which is a linear equation in a corresponding Banach space of sequences.

In this paper, first it is mentioned about integral representation for solution which satisfies certain initial conditions of differential equation generated by singular Sturm- Liouville operator , properties of spectral characteristics and uniqueness theorems for solution of inverse problem are discussed. After that we give a derivation of the main equation and prove its unique solvability.

We define

$(\Gamma y)(x) = y' - u(x)y$ where $u(x) = A\frac{x^{1-\alpha}}{1-\alpha}$ and let's write the expression of left hand side of equation (1.1) as follows:

(1.3)
$$\ell(y) := -[(\Gamma_\alpha y)(x)]' - u(x)(\Gamma_\alpha y)(x) - u^2(x)y(x) + q_o(x)y(x) = \lambda^2\rho(x)y(x).$$

2.Representation for the solution

We define

(2.1) $$y_1(x) = y(x)\ , y_2(x) = y'(x) - u(x)y((x)) = (\Gamma_\alpha y)(x)$$

$u(x) = A\frac{x^{1-\alpha}}{1-\alpha}$ and let's write the expression of left hand side of equation (1.1) as follows

(2.2)
$$\ell(y) := -[(\Gamma_\alpha y)(x)]' - u(x)(\Gamma_\alpha y)(x) - u^2(x)y(x) + q_o(x)y(x) = \lambda^2\rho(x)y(x).$$

Then equation (1.1) reduces to the system;

(2.3) $$\begin{cases} y_1' - y_2 = u(x)y_1 \\ y_2' + \lambda^2\rho(x)y_1 = -u(x)y_2 - u^2(x)y_1 + q_o(x)y_1 \end{cases}$$

or in matrice from

(2.4) $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda^2\rho(x) - u^2 + q_o & -u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The entries of the matrice

$$B = \begin{pmatrix} u(x) & 1 \\ -\lambda^2 - u^2(x) + q_o(x) & -u(x) \end{pmatrix}$$

are functions in $L_1[0,\pi]$.

Using the theorem in [22] Naimark (1967), there exists only one solution of system (2.3) which satisfies the same initial conditions $y_1(\xi) = \upsilon_1, y_2(\xi) = \upsilon_2$ for each $\xi \in [0,\pi]$, $\upsilon = (\upsilon_1,\upsilon_2)^T \in C^2$ especially the initial conditions $y_1(0) = 1, y_2(0) = i\lambda$.

Definition 1. *First component of solution which satisfies the initial conditions* $y_1(\xi) = \upsilon_1, y_2(\xi) = (\Gamma_\alpha y)(\xi) = \upsilon_2$ *of the system (2.3) is called as the solution the equation (1.1) which satisfies the same initial conditions.*

It was obtained in [23] by the successive approximations method that [24] the following theorem is true.First component of solution which satisfies the initial conditions $y_1(\xi) = \upsilon_1, y_2(\xi) = (\Gamma_\alpha y)(\xi) = \upsilon_2$ of the system (2.3) is called as the solution the equation (1.1) which satisfies the same initial conditions.

It was obtained in [23] by the successive approximations method that [24] the following theorem is true.

Theorem 1. *For every solutions of the problem L which satisfies the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i\lambda \end{pmatrix}$ *,the following expression is true;*
For $x > a$

$$\begin{cases} y_1(x,\lambda) = \beta^+ e^{i\lambda\mu^+(x)} + \beta^- e^{i\lambda\mu^-(x)} + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{11}(x,t)\, e^{i\lambda t} dt \\ y_2(x,\lambda) = i\lambda\beta\left(\beta^+ e^{i\lambda\mu^+(x)} - \beta^- e^{i\lambda\mu^-(x)}\right) \\ \qquad + b(x)\left[\beta^+ e^{i\lambda\mu^+(x)} + \beta^- e^{i\lambda\mu^-(x)}\right] \\ \qquad + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{21}(x,t)\, e^{i\lambda t} dt + i\lambda\beta \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{22}(x,t)\, e^{i\lambda t} dt \end{cases}$$

We see that

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)} |K_{ij}(x,t)|\, dt = \sum \int\limits_{-\mu^+(x)}^{\mu^+(x)} \left|K_{ij}^{(n)}(x,t)\right| dt \le e^{c.\sigma(x)} - 1$$

$$c = \max\left\{2\beta^+, \frac{|\beta|\,\beta^+}{2}, \frac{\beta^+}{2}\right\}$$

and

$$\sigma(x) = \int\limits_0^{\mu^+(x)} \left(\mu^+(x) - t\right)\left[|u(t)|^2 + |u(t)|\,|b(t)| + |q_o(t)|\right] dt$$

$(i,j = 1,2)$ Where, $b(x) = -\frac{1}{2}\int\limits_0^x \left[u^2(s) - q_0(s)\right] e^{-\frac{1}{2}\int\limits_s^x u(t)dt} ds$ and we get

$$K_{11}(x,\mu^+(x)) = \frac{\beta^+}{2\beta}u(x) + \frac{1}{2}\beta b(a)\, e^{-\int\limits_a^x u(t)dt} + \frac{\beta^+}{2\beta}b(a) - \frac{\beta^-}{2\beta}u(a) + \frac{\beta^-}{2}b(a),$$

$$K_{21}(x,\mu^+(x)) = \frac{\beta^+}{2\beta}(b'(x) - b'(a))$$

$$-\frac{1}{2}\int\limits_a^x \left[u^2(s) - q_o(s)\right] K_{11}(s,\mu^+(s))\, ds - \frac{1}{2}\int\limits_a^x u(s)\, K_{21}(s,\mu^+(s))\, ds,$$

$K_{22}(x,\mu^{+}(x)) = \frac{\beta^{+}}{2\beta}(2b(x)-u(x)) + \frac{\beta}{2}b(a)e^{-\int_a^x u(t)dt} - \frac{\beta^{+}}{2\beta}b(a) + \frac{\beta^{-}}{2\beta}u(a) + \frac{\beta^{-}}{2}b(a),$

$$K_{22}(x,-\mu^{-}(x)+0) - K_{22}(x,-\mu^{-}(x)-0) = \frac{A}{1-\alpha}e^{\int_a^x u(t)dt}$$

$$K_{22}(x,\mu^{-}(x)+0) - K_{22}(x,\mu^{-}(x)-0) = \frac{\beta^{-}A}{2\beta^2(1-\alpha)}\int_a^x \left[u(t)b(t)+u^2(t)-q_o(t)\right]e^{\frac{1}{\beta}\int_t^x u(\xi)d\xi}dt$$

and $\frac{\partial K_{ij}(x,.)}{\partial x}, \frac{\partial K_{ij}(x,.)}{\partial t} \in L_2(0,\pi),\ i,j=1,2\ ,\ \beta^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\beta}\right), \mu^{+}(x) = \beta x - \beta a + a, \mu^{-}(x) = -\beta x + \beta a + a$

Note For $0 < x \leq a$, there are in [16]

3. Properties of the spectrum

In this section, properties of the spectrum of problem L have been given.

Let us denote problem L as L_0 in the case of $A=0$ and $q_0(x) \equiv 0$

When $A=0$ and $q_0(x) \equiv 0$, it is easily shown that solution $\varphi_0(x,\lambda)$ satisfying the initial conditions $\varphi_0(0,\lambda) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$ is shown as $\varphi_0(x,\lambda) = \frac{y_0(x,\lambda) - \overline{y_0(x,\lambda)}}{2i}$

for $x < a$

$$\begin{cases} \varphi_{01}(x,\lambda) = \sin\lambda x \\ \varphi_{02}(x,\lambda) = \lambda\cos\lambda x \end{cases}$$

for $x > a$,

$$\begin{cases} \varphi_{01}(x,\lambda) = \beta^{+}\sin\lambda\mu^{+}(x) + \beta^{-}\sin\lambda\mu^{-}(x) \\ \varphi_{02}(x,\lambda) = \lambda\beta(\beta^{+}\cos\lambda\mu^{+}(x) - \beta^{-}\cos\lambda\mu^{-}(x)) \end{cases}$$

Let $\Delta_0(\lambda)$ be a characteristic function of problem L_0 . Then characteristic equation of problem L_0 is the form.

$$\Delta_0(\lambda) = \beta^{+}\sin\lambda\mu^{+}(\pi) + \beta^{-}\sin\lambda\mu^{-}(\pi)$$

Roots λ_n^0 of this equation are eigenvalues of problem L_0.

Lemma1: Let $y,z \in D(L_0^*)$. Then

$$(L_0^* y, z) = \int_0^{\pi}\rho(x)\ell(y)\overline{z}dx = (y, L_0^* z) + [y,\overline{z}]\left(|_0^{a-0} + |_{a+0}^{\pi}\right)$$

where

$[y,\overline{z}]\left(|_0^{a-0} + |_{a+0}^{\pi}\right) = \left[(\Gamma_\alpha\overline{z})(x)\,y(x) - (\Gamma_\alpha y)(x)\,\overline{z(x)}\right]\left(|_0^{a-0} + |_{a+0}^{\pi}\right).$

Lemma2:

$\inf_{n\neq m}\left|\lambda_n^0 - \lambda_m^0\right| = c > 0$, i.e., roots of characteristic equation $\Delta_0(\lambda) = 0$ are separated.

Proof:

Let us assume that sequence $\{\lambda_n^0\}$ has two subsequences $\left\{\lambda_{n_p}^0\right\}$ and $\left\{\tilde{\lambda}_{n_p}^0\right\}$ such that

$$\lambda_{n_p}^0 \neq \tilde{\lambda}_{n_p}^0, \lambda_{n_p}^0 \to \infty, \tilde{\lambda}_{n_p}^0 \to \infty \quad (p \to \infty)$$

and

$$\lim_{p\to\infty}\left|\lambda^0_{n_p}-\widetilde{\lambda}^0_{n_p}\right|=0$$

If we use orthogonality of eigenfunctions $\varphi_0\left(x,\lambda^0_{n_p}\right)$ and $\varphi_0\left(x,\widetilde{\lambda}^0_{n_p}\right)$ of problem L_0 in space $L_2(0,\pi)$,

$$0=\int_0^\pi \rho(x)\varphi_0\left(x,\lambda^0_{n_p}\right)\varphi_0\left(x,\widetilde{\lambda}^0_{n_p}\right)dx$$
$$=\int_0^\pi \rho(x)\varphi_0\left(x,\lambda^0_{n_p}\right)\left[\varphi_0(x,\widetilde{\lambda}^0_{n_p})-\varphi_0\left(x,\lambda^0_{n_p}\right)\right]dx+\int_0^\pi \rho(x)(\varphi_0\left(x,\lambda^0_{n_p}\right))^2dx$$
$$\geq\int_0^\pi \rho(x)\varphi_0\left(x,\lambda^0_{n_p}\right)\left[\varphi_0(x,\widetilde{\lambda}^0_{n_p})-\varphi_0\left(x,\lambda^0_{n_p}\right)\right]dx+\int_0^a \rho(x)(\varphi_0\left(x,\lambda^0_{n_p}\right))^2dx$$
$$=\int_0^\pi \rho(x)\varphi_0\left(x,\lambda^0_{n_p}\right)\left[\varphi_0(x,\widetilde{\lambda}^0_{n_p})-\varphi_0\left(x,\lambda^0_{n_p}\right)\right]dx+\int_0^a \sin^2\lambda^0_{n_p}x dx$$
$$=\int_0^\pi \rho(x)\varphi_0\left(x,\lambda^0_{n_p}\right)\left[\varphi_0(x,\widetilde{\lambda}^0_{n_p})-\varphi_0\left(x,\lambda^0_{n_p}\right)\right]dx+\frac{a}{2}-\frac{\sin 2\lambda^0_{n_p}a}{2\lambda^0_{n_p}}$$

From representing of function $\varphi_0(x,\lambda)$, we get that

$$\lim_{p\to\infty}\left|\varphi_0\left(x,\widetilde{\lambda}^0_{n_p}\right)-\varphi_0\left(x,\lambda^0_{n_p}\right)\right|=0$$

i.e., as $p\to\infty$, $\left|\varphi_0\left(x,\widetilde{\lambda}^0_{n_p}\right)-\varphi_0\left(x,\lambda^0_{n_p}\right)\right|=0$ uniformly converges to zero with respect to x in the interval $[0,\pi]$. For this reason, if we pass through the limit as $p\to\infty$ then inequality $0\geq\frac{a}{2}$ is obtained.

This contradiction gives to proof of Lemma2.

Denote $\Delta(\lambda)=\langle\psi(x,\lambda),\varphi(x,\lambda)\rangle$, where $\langle y(x),z(x)\rangle := y(x)(\Gamma_\alpha z)(x)-(\Gamma_\alpha y)(x)z(x)$. According to Liouville formula, $\langle\psi(x,\lambda),\varphi(x,\lambda)\rangle$ is not depend on x.

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be of the equations of (1.1) to have the solution with initial conditions,

$\varphi(0,\lambda)=1$, $(\Gamma_\alpha\varphi)(0,\lambda)=h$, $\psi(\pi,\lambda)=\lambda-1$, $\psi(\pi,\lambda)=H_2-H_1\lambda$.

Clearly, for each x, functions $\langle\psi(x,\lambda),\varphi(x,\lambda)\rangle$ are entire in λ and $\Delta(\lambda)=V(\varphi)=-U(\psi)$

Lemma3: Eigenvalues of the problem L are simple that is $\dot{\Delta}(\lambda_n)\neq 0$

Proof: Since $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ are solutions of equation (1.1), it is obtained that

$$\begin{aligned}-\psi''(x,\lambda)+[u'(x)+q_o(x)]\psi(x,\lambda) &= \lambda^2\rho(x)\psi(x,\lambda)\\ -\varphi''(x,\lambda_n)+[u'(x)+q_o(x)]\varphi(x,\lambda_n) &= \lambda_n^2\rho(x)\varphi(x,\lambda_n)\end{aligned}$$

If first equation is multiplied by $\varphi(x,\lambda_n)$, second equation is multiplied by $\psi(x,\lambda)$ and subtracting them side by side and finally integrating over the interval $[0,\pi]$, then the following equality is obtained:

$$\frac{d}{dx}\langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle=\left(\lambda^2-\lambda_n^2\right)\rho(x)\psi(x,\lambda)\varphi(x,\lambda_n)$$

$$\left(\lambda^2-\lambda_n^2\right)\int_0^{\pi}\rho(x)\,\psi(x,\lambda)\,\varphi(x,\lambda_n)\,dx=\langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle\left[|_0^{a-0}+|_{a+0}^{\pi}\right]$$

$\alpha_n=\int_0^{\pi}\rho(x)\,\varphi^2(x,\lambda_n)\,dx$ is considered , it is obtained that

$\int_0^{\pi}\rho(x)\,\psi(x,\lambda)\,\varphi(x,\lambda_n)\,dx=-\dot{\Delta}(\lambda_n)$ as $\lambda\to\lambda_n$.

For all $x\in[0,\pi]$, we get from existing of constants γ_n which satisfy the equality $\psi(x,\lambda_n)=\gamma_n\varphi(x,\lambda_n)$ that
$\alpha_n\gamma_n=-\dot{\Delta}(\lambda_n)$.

It is obvious that $\dot{\Delta}(\lambda_n)\neq 0$. So lemma is proved .

Lemma4: The eigenvalues λ_n of problem L have the following asymptotic behaviour :

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{\delta_n}{\lambda_n^0}. \tag{3.1}$$

where $\delta_n\in\ell_2$ and d_n is a bounded sequence.

Proof: Denote

$$\begin{aligned}\Gamma_n &= \left\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\delta,\delta>0,n=0,1,2,...\right\}\\ G_\delta &= \left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta,\delta>0,n=0,1,2,...\right\}\end{aligned}$$

where δ is sufficiently small positive number . $z=x+iy$ including benefiting from $|\sin z|=\sqrt{\sin^2 x+\sinh^2 y}$ equality , for $\lambda\in\overline{G_\delta}$ there $|\Delta_0(\lambda)|=\left|\beta^+\sin\lambda\mu^+(\pi)+\beta^-\sin\lambda\mu^-(\pi)\right|$
$\geq\frac{\beta^+}{2}C_\delta e^{|\operatorname{Im}\lambda|\mu^+(\pi)}$
so that it is $C_\delta>0$.

On the other hand , including

$$\varphi_1(x,\lambda)=\beta^+\cos\lambda\mu^+(x)+\beta^-\cos\mu^-(x)+\int_{-\mu^+(x)}^{\mu^+(x)}K_{11}(x,t)\cos\lambda t dt$$

and

$$\varphi_2(x,\lambda)=-\lambda\beta\left(\beta^+\sin\lambda\mu^+(x)-\beta^-\sin\lambda\mu^-(x)\right)+b(x)\left(\beta^+\cos\lambda\mu^+(x)+\beta^-\cos\mu^-(x)\right)$$
$$+\int_{-\mu^+(x)}^{\mu^+(x)}K_{21}(x,t)\cos\lambda t dt-\lambda\beta\int_{-\mu^+(x)}^{\mu^+(x)}K_{22}(x,t)\sin\lambda t dt,$$

characteristic function of the problem L is obtained as

$$\Delta(\lambda)=(\lambda-1)\,\varphi_2(\pi,\lambda)+(\lambda H_1-H_2)\,\varphi_1(\pi,\lambda)$$
$$=\Delta_0(\lambda)+\int_{-\mu^+(\pi)}^{\mu^+(\pi)}K_{22}(\pi,t)\sin\lambda t dt$$
$$-\frac{1}{\lambda\beta}(\beta\beta^+\sin\lambda\mu^+(\pi)-\beta\beta^-\sin\lambda\mu^-(\pi)+(b(\pi)+H_1)\left(\beta^+\cos\lambda\mu^+(\pi)+\beta^-\cos\lambda\mu^-(\pi)\right)$$

$$+\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}\left[K_{21}(\pi,t)+H_1K_{11}(\pi,t)\right]\cos\lambda t dt\}$$
$$+\frac{1}{\lambda^2\beta}\{(b(\pi)+H_2)\left(\beta^{+}\cos\lambda\mu^{+}(\pi)+\beta^{-}\cos\lambda\mu^{-}(\pi)\right)$$
$$+\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}\left[K_{21}(\pi,t)+H_2K_{11}(\pi,t)\right]\cos\lambda t dt\}=0.$$

Also , $K_{22}(x,\cdot)\in L_1(-\mu^{+}(x),\mu^{+}(x))$,

$$\lim_{|\lambda|\to+\infty}\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}K_{22}(\pi,t)\sin\lambda t dt=0$$

can be written , we get $|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{C_\delta}{2}e^{|\operatorname{Im}\lambda|\mu^{+}(\pi)}$

Thus,

$$|\Delta_0(\lambda)|>C_\delta e^{|\operatorname{Im}\lambda|\mu^{+}(\pi)}>\frac{C_\delta}{2}e^{|\operatorname{Im}\lambda|\mu^{+}(\pi)}>|\Delta(\lambda)-\Delta_0(\lambda)|$$

such that n is sufficiently large natural number.

It follows from that for sufficiently large values of n , functions $\Delta_0(\lambda)$ and $[\Delta(\lambda)-\Delta_0(\lambda)]+\Delta_0(\lambda)=\Delta(\lambda)$ have the same number of zeros counting multiplicities inside contour Γ_n according to Rouche's theorem . That is , they have the theorem $(n+1)$ number of zeros : $\lambda_0,\lambda_1,...,\lambda_n$. Analogously , it is shown by Rouche's theorem that for sufficiently large values of n , function $\Delta(\lambda)$ has a unique of zero inside each circle $\left|\lambda-\lambda_n^0\right|<\delta$.

Sincle δ is sufficiently small number , representing of $\lambda_n=\lambda_n^0+\varepsilon_n$ is acquired where $\lim\limits_{n\to\infty}\varepsilon_n=0$.

Since numbers λ_n are zeros of characteristic function $\Delta(\lambda)$,

$$\Delta(\lambda_n)=\Delta\left(\lambda_n^0+\varepsilon_n\right)=\Delta_0\left(\lambda_n^0+\varepsilon_n\right)+\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}K_{22}(\pi,t)\sin(\lambda_n^0+\varepsilon_n)tdt$$
$$-\frac{1}{(\lambda_n^0+\varepsilon_n)\beta}\{\beta\beta^{+}\sin(\lambda_n^0+\varepsilon_n)\mu^{+}(\pi)-\beta\beta^{-}\sin(\lambda_n^0+\varepsilon_n)\mu^{-}(\pi)$$
$$+(b(\pi)+H_1)\left(\beta^{+}\cos(\lambda_n^0+\varepsilon_n)\mu^{+}(\pi)+\beta^{-}\cos(\lambda_n^0+\varepsilon_n)\mu^{-}(\pi)\right)$$
$$+\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}\left[K_{21}(\pi,t)+H_1K_{11}(\pi,t)\right]\cos(\lambda_n^0+\varepsilon_n)tdt\}$$
$$+\frac{1}{(\lambda_n^0+\varepsilon_n)^2\beta}\{(b(\pi)+H_2)\left(\beta^{+}\cos(\lambda_n^0+\varepsilon_n)\mu^{+}(\pi)+\beta^{-}\cos(\lambda_n^0+\varepsilon_n)\mu^{-}(\pi)\right)$$
$$+\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}\left[K_{21}(\pi,t)+H_2K_{11}(\pi,t)\right]\cos(\lambda_n^0+\varepsilon_n)tdt\}$$
$$=0\ .$$

On the other hand ,

$\Delta_0\left(\lambda_n^0+\varepsilon_n\right)=(\dot{\Delta}_0\left(\lambda_n^0\right)+o(1))\varepsilon_n.$

In that case the equality

$$\left(\dot{\Delta}_0\left(\lambda_n^0\right)+o\left(1\right)\right)\varepsilon_n+\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}K_{22}\left(\pi,t\right)\sin(\lambda_n^0+\varepsilon_n)tdt$$

$$-\frac{1}{(\lambda_n^0+\varepsilon_n)\beta}\{\beta\beta^+\sin(\lambda_n^0+\varepsilon_n)\mu^+\left(\pi\right)-\beta\beta^-\sin(\lambda_n^0+\varepsilon_n)\mu^-(\pi)$$

$$+\left(b\left(\pi\right)+H_1\right)\left(\beta^+\cos(\lambda_n^0+\varepsilon_n)\mu^+\left(\pi\right)+\beta^-\cos(\lambda_n^0+\varepsilon_n)\mu^-\left(\pi\right)\right)$$

$$+\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}\left[K_{21}\left(\pi,t\right)+H_1K_{11}\left(\pi,t\right)\right]\cos(\lambda_n^0+\varepsilon_n)tdt\}$$

$$+\frac{1}{(\lambda_n^0+\varepsilon_n)^2\beta}\{(b\left(\pi\right)+H_2)\left(\beta^+\cos(\lambda_n^0+\varepsilon_n)\mu^+(\pi)+\beta^-\cos(\lambda_n^0+\varepsilon_n)\mu^-\left(\pi\right)\right)$$

$$+\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}\left[K_{21}\left(\pi,t\right)+H_2K_{11}\left(\pi,t\right)\right]\cos(\lambda_n^0+\varepsilon_n)tdt\}=0$$

is obtained .

Since function $\Delta_0\left(\lambda_n\right)$ is type of [25] , the number $\gamma_\delta>0$ exists such that for all n , $\left|\dot{\Delta}_0\left(\lambda_n^0\right)\right|\geqslant\gamma_\delta$.

If the study [26] (see also [27]) is used , then we get that $\lambda_n^0=\dfrac{\pi n}{\mu^+\left(\pi\right)}+\Psi_1\left(n\right)$

If we use the expression of ε_n , then

$$\varepsilon_n\approx-\frac{\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}K_{22}\left(\pi,t\right)\sin\lambda_n^0tdt}{\dot{\Delta}_0\left(\lambda_n^0\right)+\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}tK_{22}\left(\pi,t\right)\cos\lambda_n^0tdt}$$

$$+\frac{1}{\lambda_n^0\dot{\Delta}_0\left(\lambda_n^0\right)}\{\beta^+\sin\lambda_n^0\mu^+\left(\pi\right)-\beta^-\sin\lambda_n^0\mu^-(\pi)$$

$$+\frac{b(\pi)+H_1}{\beta}\left(\beta^+\cos\lambda_n^0\mu^+(\pi)+\beta^-\cos\lambda_n^0\mu^-\left(\pi\right)\right)$$

$$+\frac{1}{\beta}\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}\left[K_{21}\left(\pi,t\right)+H_1K_{11}\left(\pi,t\right)\right]\cos\lambda_n^0tdt\}.$$

Since

$$\left\{\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)}K_{22}\left(\pi,t\right)\sin\lambda_n^0tdt\right\}_{n\geq1}\in\ell_2.$$

They are obtained that

$$\delta_n = -\frac{\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)} K_{22}(\pi,t)\sin\lambda_n^0 t dt}{\dot{\Delta}_0(\lambda_n^0) + \int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)} tK_{22}(\pi,t)\cos\lambda_n^0 t dt}, \ (\delta_n) \in \ell_2,$$

$$d_n = \frac{1}{\dot{\Delta}_0(\lambda_n^0)}\{\beta^+ \sin\lambda_n^0\mu^+(\pi) - \beta^- \sin\lambda_n^0\mu^-(\pi) + \frac{b(\pi)+H_1}{\beta}\left(\beta^+\cos\lambda_n^0\mu^+(\pi) + \beta^-\cos\lambda_n^0\mu^-(\pi)\right)$$

$$+\frac{1}{\beta}\int\limits_{-\mu^+(\pi)}^{\mu^+(\pi)} [K_{21}(\pi,t) + H_1K_{11}(\pi,t)]\cos\lambda_n^0 t dt\}$$

(d_n) is a bounded sequence .
Thus, for the eiqenvalues λ_n of the problem L , asymptotic formula (3.1) is true .
Therefore Lemma4 is proved .

4. Weyl Solution and Properties of The Weyl Function

Let $\Phi(x,\lambda) = \begin{pmatrix} \Phi_1(x,\lambda) \\ \Phi_2(x,\lambda) \end{pmatrix}$ be solution of (1.1) under the conditions $\Phi_1(0,\lambda) = 1$ and $\Phi_1(\pi,\lambda) = 0$. The function koşullarını $\Phi(x,\lambda)$ is called the Weyl solution . We shall assume that $S(x,\lambda), \varphi(x,\lambda)$ and $\Psi(x,\lambda)$ are solutions of equation (1.1) that satisfy the initial conditions

$$\Psi(\pi,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } S(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is clear that the functions $\Psi(x,\lambda)$ and $S(x,\lambda)$ are entire in λ . Then the function $\Psi(x,\lambda)$ can be represented as follows :

$$\Psi(x,\lambda) = \Psi_1(0,\lambda)\varphi(x,\lambda) - \Delta(\lambda)S(x,\lambda)$$

or

$$-\frac{\Psi(x,\lambda)}{\Delta(\lambda)} = -\frac{\Psi_1(0,\lambda)}{\Delta(\lambda)}\varphi(x,\lambda) + S(x,\lambda)$$

Denote

$$M(\lambda) = -\frac{\Psi_1(0,\lambda)}{\Delta(\lambda)}$$

It is clear that

$$\Phi(x,\lambda) = -\frac{\Psi(x,\lambda)}{\Delta(\lambda)} = M(\lambda)\varphi(x,\lambda) + S(x,\lambda) \tag{4.1}$$

The function $\Phi(x,\lambda)$ is called the Weyl solution and the function $M(\lambda)$ is called the Weyl function for the boundary value problem L. The Weyl solution and Weyl function are meromorphic functions with respect to $\{\lambda_n\}_{n\geq 0}$ having poles in the spectrum of the problem L .

Theorem1: The following representation holds ;

$$M(\lambda) = \frac{1}{\alpha_0(\lambda-\lambda_0)} + \sum_{n=1}^{\infty}\left\{\frac{1}{\alpha_n(\lambda-\lambda_n)} + \frac{1}{\alpha_n^0\lambda_n^0}\right\}.$$

Proof:

Let's write a representation solution of $\Psi(x,\lambda)$ as representation solution of $\varphi(x,\lambda)$:

for $x>a$

$$\begin{cases} \Psi_1(x,\lambda)=\cos\lambda(\pi-x)+\int\limits_0^{\pi+x}\widetilde{N}_{11}(x,t)\cos\lambda t dt \\ \Psi_2(x,\lambda)=-\lambda\sin\lambda(\pi-x)+b(x)\cos\lambda(\pi-x)+\int\limits_0^{\pi+x}\widetilde{N}_{21}(x,t)\cos\lambda t dt \\ -\lambda\int\limits_0^{\pi+x}\widetilde{N}_{22}(x,t)\sin\lambda t dt \end{cases}$$

for $x<a$

$$\begin{cases} \Psi_1(x,\lambda)=\beta^+\cos\lambda(\mu^+(\pi)-\mu^+(x))+\beta^-\cos\lambda(\mu^-(\pi)-\mu^-(x)) \\ +\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{11}(x,t)\cos\lambda t dt \\ \Psi_2(x,\lambda)=-\lambda\beta(\beta^+\sin\lambda(\mu^+(\pi)-\mu^+(x))-\beta^-\sin\lambda(\mu^-(\pi)-\mu^-(x)) \\ +b(x)(\beta^+\cos\lambda(\mu^+(\pi)-\mu^+(x))+\beta^-\cos\lambda(\mu^-(\pi)-\mu^-(x))+\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{21}(x,t)\cos\lambda t dt \\ -\lambda\beta\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{22}(x,t)\sin\lambda t dt \end{cases}$$

where $\widetilde{N}_{ij}(x,t)=N_{ij}(x,t)-N_{ij}(x,-t), i,j=1,2$. In the case of $A=0$ ve $q_o(x)=0$ we write solution with $\Psi_{01}(x,\lambda)$ and $\Psi_{02}(x,\lambda)$, so we have

$$\begin{cases} \Psi_1(x,\lambda)=\Psi_{01}(x,\lambda)+f_1 \\ \Psi_2(x,\lambda)=\Psi_{02}(x,\lambda)+f_2 \end{cases}$$

where

$$f_1=\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{11}(x,t)\cos\lambda t dt \text{ ve}$$

$$f_2=b(x)(\beta^+\cos\lambda(\mu^+(\pi)-\mu^+(x))+\beta^-\cos\lambda(\mu^-(\pi)-\mu^-(x))$$
$$+\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{21}(x,t)\cos\lambda t dt-\lambda\beta\int\limits_0^{\mu^+(\pi)+\mu^+(x)}\widetilde{N}_{22}(x,t)\sin\lambda t dt.$$

On the other hand , we can write

$$M(\lambda)-M_0(\lambda)=\frac{\Psi_1(0,\lambda)}{\Psi_2(0,\lambda)}-\frac{\Psi_{01}(0,\lambda)}{\Psi_{02}(0,\lambda)}=-\frac{f_1}{\Delta(\lambda)}+\frac{f_2}{\Delta(\lambda)}M_0(\lambda).$$

Since $\lim\limits_{|\lambda|\to\infty} e^{-|\operatorname{Im}\lambda\mu^{+}(\pi)|}|f_i(\lambda)|=0$ and $|\Delta(\lambda)|>C_\delta e^{|\operatorname{Im}\lambda|\mu^{+}(\pi)}$ for $\lambda\in G_\delta$, we get that ,

$$\lim_{|\lambda|\to\infty}\sup_{\lambda\in G_\delta}|M(\lambda)-M_0(\lambda)|=0 \tag{4.2}$$

Weyl function $M(\lambda)$ is meromorphic with respect to poles λ_n . We calculate that

$$\operatorname*{Re}_{\lambda=\lambda_n} s\,M(\lambda) = \frac{\Psi_1(0,\lambda_n)}{\dot{\Delta}(\lambda_n)}=\frac{1}{\dot{\Delta}(\lambda_n)\varphi_1(\pi,\lambda_n)}=-\frac{1}{\alpha_n} \qquad (4.3)$$

$$\operatorname*{Re}_{\lambda=\lambda_n^0} s\,M_0(\lambda) = \frac{\Psi_{01}\left(0,\lambda_n^0\right)}{\dot{\Delta}\left(\lambda_n^0\right)}=\frac{1}{\dot{\Delta}_0\left(\lambda_n^0\right)\varphi_{01}\left(\pi,\lambda_n^0\right)}=-\frac{1}{\alpha_n^0} \qquad (4.3)$$

Consider the contour integral

$$I_n(x)=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)-M_0(\mu)}{\lambda-\mu}d\mu\ ,\ \lambda\in int\Gamma_n$$

By virtue of (4.2) , we have $\lim\limits_{n\to\infty} I_n(x)=0$. On the other hand , the residue theorem and (4.3) yield

$$I_n(x)=-M(\lambda)+M_0(\lambda)+\sum_{\lambda_n\in int\Gamma_n}\frac{1}{\alpha_n(\lambda-\lambda_n)}-\sum_{\lambda_n^0\in int\Gamma_n}\frac{1}{\alpha_n^0\left(\lambda_n^0-\lambda\right)}$$

and theorem is proved .

Form a $\widetilde{L}$ problem by taking $\widetilde{q_0}(x)$, $\widetilde{A}$, $\widetilde{\lambda}$ in L problem:

Theorem 2: If $M(\lambda)=\widetilde{M}(\lambda)$ then $L=\widetilde{L}$. Thus the specification of the Weyl function uniquely determines the operator.

Proof:

Let us define the matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula.

$$P(x,\lambda)\begin{pmatrix}\widetilde{\varphi}_1(x,\lambda) & \widetilde{\Phi}_1(x,\lambda)\\ \widetilde{\varphi}_2(x,\lambda) & \widetilde{\Phi}_2(x,\lambda)\end{pmatrix}=\begin{pmatrix}\varphi_1(x,\lambda) & \Phi_1(x,\lambda)\\ \varphi_2(x,\lambda) & \Phi_2(x,\lambda)\end{pmatrix}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

Using (4.1) and (4.4) we calculate

$$\begin{cases}P_{11}(x,\lambda)=\varphi_1(x,\lambda)\widetilde{\Phi}_2(x,\lambda)-\Phi_1(x,\lambda)\widetilde{\varphi}_2(x,\lambda)\\ P_{12}(x,\lambda)=\Phi_1(x,\lambda)\widetilde{\varphi}_1(x,\lambda)-\varphi_1(x,\lambda)\widetilde{\Phi}_1(x,\lambda)\\ P_{21}(x,\lambda)=\varphi_2(x,\lambda)\widetilde{\Phi}_2(x,\lambda)-\Phi_2(x,\lambda)\widetilde{\varphi}_2(x,\lambda)\\ P_{22}(x,\lambda)=\Phi_2(x,\lambda)\widetilde{\varphi}_1(x,\lambda)-\varphi_2(x,\lambda)\widetilde{\Phi}_1(x,\lambda)\end{cases} \tag{4.5}$$

It follows from (4.5) , (4.1), and $\Delta(\lambda)=\langle\Psi(x,\lambda),\varphi(x,\lambda)\rangle$.

$$P_{11}(x,\lambda)=1+\frac{1}{\Delta(\lambda)}\left[\Psi_1(x,\lambda)\left(\widetilde{\varphi}_2(x,\lambda)-\varphi_2(x,\lambda)\right)-\varphi_1(x,\lambda)\left(\widetilde{\Psi}_2(x,\lambda)-\Psi_2(x,\lambda)\right)\right]$$

$$P_{12}(x,\lambda)=\frac{1}{\Delta(\lambda)}\left[\varphi_1(x,\lambda)\widetilde{\Psi}_1(x,\lambda)-\Psi_1(x,\lambda)\widetilde{\varphi}_1(x,\lambda)\right]$$

$$P_{21}(x,\lambda)=\frac{1}{\Delta(\lambda)}\left[\Psi_2(x,\lambda)\widetilde{\varphi}_2(x,\lambda)-\varphi_1(x,\lambda)\widetilde{\Psi}_2(x,\lambda)\right]$$

$$P_{22}(x,\lambda)=1+\frac{1}{\Delta(\lambda)}\left[\varphi_2(x,\lambda)\left(\widetilde{\Psi}_1(x,\lambda)-\widetilde{\Psi}_2(x,\lambda)\right)-\Psi_2(x,\lambda)\left(\widetilde{\varphi}_1(x,\lambda)-\widetilde{\varphi}_2(x,\lambda)\right)\right]$$

For $\lambda\in G_\delta$,$|\Delta(\lambda)|>C_\delta e^{|\operatorname{Im}\lambda|\mu^+(\pi)}$ from the Lebesgue lemma ,

$$(4.6)\qquad\left\{\begin{array}{l}\lim\limits_{\substack{\lambda\to\infty\\ \lambda\in G_\delta}}\max\limits_{0\le x\le\pi}|P_{11}(x,\lambda)-1|=\lim\limits_{\substack{\lambda\to\infty\\ \lambda\in G_\delta}}\max\limits_{0\le x\le\pi}|P_{22}(x,\lambda)-1|\\ =\lim\limits_{\substack{\lambda\to\infty\\ \lambda\in G_\delta}}\max\limits_{0\le x\le\pi}|P_{12}(x,\lambda)|=\lim\limits_{\substack{\lambda\to\infty\\ \lambda\in G_\delta}}\max\limits_{0\le x\le\pi}|P_{21}(x,\lambda)|=0\end{array}\right.$$

According to (4.1) , (4.5) and (4.6) we have

$$\begin{aligned}P_{11}(x,\lambda)&=\varphi_1(x,\lambda)\widetilde{S}_2(x,\lambda)-S_1(x,\lambda)\widetilde{\varphi}_2(x,\lambda)+\left(\widetilde{M}(\lambda)-M(\lambda)\right)\varphi_1(x,\lambda)\widetilde{\varphi}_2(x,\lambda)\\ P_{12}(x,\lambda)&=\widetilde{\varphi}_1(x,\lambda)\widetilde{S}_1(x,\lambda)-\widetilde{S}_1(x,\lambda)\varphi_1(x,\lambda)+\left(M(\lambda)-\widetilde{M}(\lambda)\right)\varphi_1(x,\lambda)\widetilde{\varphi}_1(x,\lambda)\\ P_{21}(x,\lambda)&=\varphi_2(x,\lambda)\widetilde{S}_2(x,\lambda)-S_2(x,\lambda)\widetilde{\varphi}_2(x,\lambda)+\left(\widetilde{M}(\lambda)-M(\lambda)\right)\varphi_2(x,\lambda)\widetilde{\varphi}_2(x,\lambda)\\ P_{22}(x,\lambda)&=\widetilde{\varphi}_1(x,\lambda)S_2(x,\lambda)-\widetilde{S}_1(x,\lambda)\varphi_2(x,\lambda)+\left(M(\lambda)-\widetilde{M}(\lambda)\right)\varphi_2(x,\lambda)\widetilde{\varphi}_1(x,\lambda)\end{aligned}$$

Thus if $M(\lambda)=\widetilde{M}(\lambda)$ then the functions $P_{jk}(x,\lambda)$ are entire in λ for each fixed x . Together with (4.6) we get that

$P_{11}(x,\lambda)\equiv 1, P_{12}(x,\lambda)\equiv 0, P_{21}(x,\lambda)\equiv 0, P_{22}(x,\lambda)\equiv 1$

We get

$$\varphi_1(x,\lambda)\equiv\widetilde{\varphi}_1(x,\lambda),\varphi_2(x,\lambda)\equiv\widetilde{\varphi}_2(x,\lambda),\Phi_1(x,\lambda)\equiv\widetilde{\Phi}_1(x,\lambda),\Phi_2(x,\lambda)\equiv\widetilde{\Phi}_2(x,\lambda)$$

for all x and λ . Consequently $L=\widetilde{L}$ dir .

5. Inverse Problem With Two Spectrum

L and $\widetilde{L}$, respectively , $\{\lambda_n\}$ and $\left\{\widetilde{\lambda}_n\right\}$ to get the problem of eigenvalues.

Lemma.1: If $\lambda_n=\widetilde{\lambda}_n$, then $\mu^+(\pi)=\widetilde{\mu}^+(\pi)$.

İspat: By using the behaviour of λ_n' s and $\lambda_n=\widetilde{\lambda}_n$ We have

$$\lambda_n^o+\delta_n+\left(\frac{d_n}{\lambda_n^o}\right)=\widetilde{\lambda_n^o}+\widetilde{\delta}_n+\left(\frac{\widetilde{d}_n}{\widetilde{\lambda_n^o}}\right)$$

From here is

$$\frac{\pi}{\mu^+(\pi)}n+\psi_1(n)+\delta_n+\left(\frac{d_n}{\lambda_n^o}\right)=\frac{\pi}{\widetilde{\mu}^+(\pi)}n+\widetilde{\psi_1}(n)+\widetilde{\delta}_n+\left(\frac{\widetilde{d}_n}{\widetilde{\lambda_n^o}}\right)$$

By dividing both sides of the last equation by n

$$\frac{\pi}{\mu^+(\pi)}+\frac{\psi_1(n)}{n}+\frac{\delta_n}{n}+\left(\frac{d_n}{n\lambda_n^o}\right)=\frac{\pi}{\widetilde{\mu}^+(\pi)}+\frac{\widetilde{\psi_1}(n)}{n}+\left(\frac{\widetilde{\delta}_n}{n}\right)+\left(\frac{\widetilde{d}_n}{n\widetilde{\lambda_n^o}}\right)$$

$$\lambda_n^o+\delta_n+\left(\frac{d_n}{\lambda_n^o}\right)=\widetilde{\lambda_n^o}+\widetilde{\delta}_n+\left(\frac{\widetilde{d}_n}{\widetilde{\lambda_n^o}}\right)$$

$\sup\limits_n|\psi_1(n)|<+\infty$, $\sup\limits_n\left|\widetilde{\psi_1}(n)\right|<+\infty$ when $n\to\infty$, then we have

$\mu^+(\pi)=\widetilde{\mu}^+(\pi)$

Lemma.2: a and β are determined uniquely by the sequence $\{\lambda_n\}$, if $\lambda_n=\widetilde{\lambda}_n$,then $\beta=\widetilde{\beta}$ and $a=\widetilde{a}$ $(n=1,2,3,...)$

Proof:

$\Delta(\lambda)-\Delta_o(\lambda)=O\left(e^{|\operatorname{Im}\lambda|\mu^+(\pi)}\right)$

$\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_o(\lambda)=O\left(e^{|\mathrm{Im}\,\lambda|\widetilde{\mu}^+(\pi)}\right)$

can be written in the expressions of $\Delta(\lambda),\Delta_o(\lambda),\widetilde{\Delta}(\lambda),\widetilde{\Delta}_o(\lambda)$ functions.

$\lambda_n=\widetilde{\lambda}_n$ and $\Delta(\lambda),\widetilde{\Delta}(\lambda)$ functions , λ according to the first order entire functions in accordance with Hadamard factorization theorem , for $\lambda\in\mathbb{C}$

$$\Delta(\lambda)\equiv C\widetilde{\Delta}(\lambda) \tag{5.1}$$

The equality from (5.1), for $\forall\lambda\in\mathbb{C}$

$$\Delta_o(\lambda)-C\widetilde{\Delta}_o(\lambda)=C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_o(\lambda)\right]-[\Delta(\lambda)-\Delta_o(\lambda)] \tag{5.2}$$

can be written.

If we write instead of $\Delta_o(\lambda)$ and $\widetilde{\Delta}_o(\lambda)$ from (5.2) , we have

$$C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_o(\lambda)\right]-[\Delta(\lambda)-\Delta_o(\lambda)]=\left[\beta^+\sin\lambda\mu^+(\pi)+\beta^-\sin\lambda\mu^-(\pi)\right]$$
$$-C\left[\widetilde{\beta}^+\sin\lambda\widetilde{\mu}^+(\pi)+\widetilde{\beta}^-\sin\lambda\widetilde{\mu}^-(\pi)\right]$$

Later ,

If both sides of the equation (5.2) is multiplied by $\left(\sin\lambda\mu^+(\pi)+\sin\lambda\widetilde{\mu}^+(\pi)\right)$ and finally integrating over the interval $(0,T)$, then the following equality is obtained .

$$\int_0^T\left\{C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_o(\lambda)\right]-[\Delta(\lambda)-\Delta_o(\lambda)]\right\}\left(\sin\lambda\mu^+(\pi)+\sin\lambda\widetilde{\mu}^+(\pi)\right)d\lambda$$

$$=\int_0^T\{\left[\beta^+\sin\lambda\mu^+(\pi)+\beta^-\sin\lambda\mu^-(\pi)\right]$$
$$-C\left[\widetilde{\beta}^+\sin\lambda\widetilde{\mu}^+(\pi)+\widetilde{\beta}^-\sin\lambda\widetilde{\mu}^-(\pi)\right]\}\left(\sin\lambda\mu^+(\pi)+\sin\lambda\widetilde{\mu}^+(\pi)\right)d\lambda$$
$$=\int_0^T\left[\beta^+\sin\lambda\mu^+(\pi)+\beta^-\sin\lambda\mu^-(\pi)\right]\left(\sin\lambda\mu^+(\pi)+\sin\lambda\widetilde{\mu}^+(\pi)\right)d\lambda$$
$$-C\int_0^T\left[\widetilde{\beta}^+\sin\lambda\widetilde{\mu}^+(\pi)+\widetilde{\beta}^-\sin\lambda\widetilde{\mu}^-(\pi)\right]\left(\sin\lambda\mu^+(\pi)+\sin\lambda\widetilde{\mu}^+(\pi)\right)d\lambda$$
$$=\int_0^T\beta^+\sin^2\lambda\mu^+(\pi)d\lambda+\int_0^T\beta^+\sin\lambda\mu^+(\pi)\sin\lambda\widetilde{\mu}^+(\pi)\,d\lambda$$
$$+\int_0^T\beta^-\sin\lambda\mu^-(\pi)\sin\lambda\mu^+(\pi)\,d\lambda+\int_0^T\beta^-\sin\lambda\mu^-(\pi)\sin\lambda\widetilde{\mu}^+(\pi)\,d\lambda$$
$$-C\int_0^T\widetilde{\beta}^+\sin\lambda\widetilde{\mu}^+(\pi)\sin\lambda\mu^+(\pi)\,d\lambda-C\int_0^T\widetilde{\beta}^+\sin^2\lambda\widetilde{\mu}^+(\pi)\,d\lambda-$$
$$-C\int_0^T\widetilde{\beta}^-\sin\lambda\widetilde{\mu}^-(\pi)\sin\lambda\mu^+(\pi)\,d\lambda-C\int_0^T\widetilde{\beta}^-\sin\lambda\widetilde{\mu}^-(\pi)\sin\lambda\widetilde{\mu}^+(\pi)\,d\lambda$$
$$=\frac{\beta^+}{2}T-\frac{C\widetilde{\beta}^+}{2}T-\frac{\beta^+}{4\mu^+(\pi)}\sin 2T\mu^+(\pi)+\frac{\beta^+\sin T(\mu^+(\pi)-\widetilde{\mu}^+(\pi))}{2(\mu^+(\pi)-\widetilde{\mu}^+(\pi))}-\frac{\beta^+\sin T(\mu^+(\pi)+\widetilde{\mu}^+(\pi))}{2(\mu^+(\pi)+\widetilde{\mu}^+(\pi))}$$
$$+\frac{\beta^-\sin T(\mu^+(\pi)-\mu^-(\pi))}{2(\mu^+(\pi)-\mu^-(\pi))}-\frac{\beta^-\sin T(\mu^+(\pi)-\mu^-(\pi))}{2(\mu^+(\pi)-\mu^-(\pi))}.+\frac{\beta^-\sin T(\widetilde{\mu}^+(\pi)-\mu^-(\pi))}{2(\widetilde{\mu}^+(\pi)-\mu^-(\pi))}-$$

$$-\frac{\beta^{-}\sin T(\widetilde{\mu}^{+}(\pi)+\mu^{-}(\pi))}{2(\widetilde{\mu}^{+}(\pi)+\mu^{-}(\pi))}-C[\frac{\widetilde{\beta}^{+}\sin T(\widetilde{\mu}^{+}(\pi)-\mu^{+}(\pi))}{2(\widetilde{\mu}^{+}(\pi)-\mu^{+}(\pi))}-\frac{\widetilde{\beta}^{+}\sin T(\widetilde{\mu}^{+}(\pi)+\mu^{+}(\pi))}{2(\widetilde{\mu}^{+}(\pi)+\mu^{+}(\pi))}$$
$$-\frac{\widetilde{\beta}^{+}\sin 2T\widetilde{\mu}^{+}(\pi)}{4\widetilde{\mu}^{+}(\pi)}+\frac{\widetilde{\beta}^{-}\sin T(\mu^{+}(\pi)-\widetilde{\mu}^{-}(\pi))}{2(\mu^{+}(\pi)-\widetilde{\mu}^{-}(\pi))}-\frac{\widetilde{\beta}^{-}\sin T(\widetilde{\mu}^{-}(\pi)+\mu^{+}(\pi))}{2(\widetilde{\mu}^{-}(\pi)+\mu^{+}(\pi))}$$
$$+\frac{\widetilde{\beta}^{-}\sin T(\widetilde{\mu}^{+}(\pi)-\widetilde{\mu}^{-}(\pi))}{2(\widetilde{\mu}^{+}(\pi)-\widetilde{\mu}^{-}(\pi))}-\frac{\widetilde{\beta}^{-}\sin T(\widetilde{\mu}^{+}(\pi)+\widetilde{\mu}^{-}(\pi))}{2(\widetilde{\mu}^{+}(\pi)+\widetilde{\mu}^{-}(\pi))}].$$

Let's divide both sides by T ; we get

$$O(\frac{1}{T})=\frac{\beta^{+}}{2}-\frac{C\widetilde{\beta}^{+}}{2}-O(\frac{1}{T})-CO(\frac{1}{T})$$

Taking the limits as $T\to\infty$;

$$\beta^{+}=C\widetilde{\beta}^{+}$$

In other case ;

$$C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_{o}(\lambda)\right]-\left[\Delta(\lambda)-\Delta_{o}(\lambda)\right]=$$

$$\left[\beta^{+}\sin\lambda\mu^{+}(\pi)+\beta^{-}\sin\lambda\mu^{-}(\pi)\right]-C\left[\widetilde{\beta}^{+}\sin\lambda\widetilde{\mu}^{+}(\pi)+\widetilde{\beta}^{-}\sin\lambda\widetilde{\mu}^{-}(\pi)\right].$$

If both sides of the equation (5.2) is multiplied by $\left(\sin\lambda\mu^{-}(\pi)+\sin\lambda\widetilde{\mu}^{-}(\pi)\right)$ and finally integrating over the interval $(0,T)$, then the following equality is obtained ,

$$\int_{0}^{T}\left\{C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_{o}(\lambda)\right]-\left[\Delta(\lambda)-\Delta_{o}(\lambda)\right]\right\}\left(\sin\lambda\mu^{-}(\pi)+\sin\lambda\widetilde{\mu}^{-}(\pi)\right)d\lambda$$
$$=\int_{0}^{T}\{\left[\beta^{+}\sin\lambda\mu^{+}(\pi)+\beta^{-}\sin\lambda\mu^{-}(\pi)\right]$$
$$-C\left[\widetilde{\beta}^{+}\sin\lambda\widetilde{\mu}^{+}(\pi)+\widetilde{\beta}^{-}\sin\lambda\widetilde{\mu}^{-}(\pi)\right]\}\left(\sin\lambda\mu^{-}(\pi)+\sin\lambda\widetilde{\mu}^{+}(\pi)\right)d\lambda$$
$$=\int_{0}^{T}\frac{\beta^{+}}{2}\left[\cos\lambda(\mu^{+}(\pi)-\mu^{-}(\pi))-\cos\lambda(\mu^{+}(\pi)+\mu^{-}(\pi)\right]d\lambda$$
$$+\int_{0}^{T}\frac{\beta^{+}}{2}\left[\cos\lambda(\mu^{+}(\pi)-\widetilde{\mu}^{-}(\pi))-\cos\lambda(\mu^{+}(\pi)-\widetilde{\mu}^{-}(\pi))\right]d\lambda$$
$$+\int_{0}^{T}\beta^{-}\left[\frac{1-\cos 2\lambda\mu^{-}(\pi)}{2}\right]d\lambda+\int_{0}^{T}\frac{\beta^{-}}{2}\left[\cos\lambda(\mu^{-}(\pi)-\widetilde{\mu}^{-}(\pi))-\cos\lambda(\widetilde{\mu}^{-}(\pi)+\mu^{-}(\pi))\right]d\lambda$$
$$-C[\int_{0}^{T}\frac{\beta^{+}}{2}\left[\cos\lambda(\widetilde{\mu}^{+}(\pi)-\mu^{-}(\pi))-\cos\lambda(\widetilde{\mu}^{+}(\pi)+\mu^{-}(\pi))\right]d\lambda$$
$$+\int_{0}^{T}\frac{\widetilde{\beta}^{+}}{2}\left[\cos\lambda((\widetilde{\mu}^{+}(\pi)-\widetilde{\mu}^{-}(\pi))-\cos\lambda((\widetilde{\mu}^{+}(\pi)+\widetilde{\mu}^{-}(\pi))\right]d\lambda$$
$$+\int_{0}^{T}\frac{\widetilde{\beta}^{-}}{2}\left[\cos\lambda(\widetilde{\mu}^{-}(\pi)-\mu^{-}(\pi))-\cos\lambda(\widetilde{\mu}^{-}(\pi)+\mu^{-}(\pi))\right]d\lambda+\int_{0}^{T}\widetilde{\beta}^{-}\left(\frac{1-\cos 2\lambda\widetilde{\mu}^{-}(\pi)}{2}\right)d\lambda]$$

Let's divide both sides by T ;

$$O(\frac{1}{T})=\frac{\beta^{-}}{2}-\frac{C\widetilde{\beta}^{-}}{2}+O(\frac{1}{T})-CO(\frac{1}{T})$$

Taking the limits as $T\to\infty$; $\beta^{-}=C\widetilde{\beta}^{-}$.

$\beta^{+}=C\widetilde{\beta}^{+}$ and $\beta^{-}=C\widetilde{\beta}^{-}$ hand side collect ; $C=1$ is , so , $\beta^{+}=\widetilde{\beta}^{+}$ and $\beta^{-}=\widetilde{\beta}^{-}$ since $\beta=\widetilde{\beta}$.

$\mu^{+}(\pi)=\widetilde{\mu}^{+}(\pi)$ and $\beta=\widetilde{\beta}$ main equation is obtained that $a=\widetilde{a}$.

$L_0(A)=-y''+\frac{A}{x^{\alpha}}y$ and $\widetilde{L_0}(\widetilde{A})=-y''+\frac{\widetilde{A}}{x^{\alpha}}y$; $\lambda_n^0(A)$, including the eigenvalues of the problem $L_0(A)=-y''+\frac{A}{x^{\alpha}}y$,

Lemma.3: If $\lambda_n^0=\widetilde{\lambda_n^0}$, $H_1=\widetilde{H_1}$ then $A=\widetilde{A}$.

Proof: Using

$$\lambda_n^0=\frac{\pi n}{\mu^{+}(\pi)}+\frac{d_n^0}{n}+\frac{\delta_n^0}{n} \text{ and}$$

$$d_n^0=\frac{1}{\dot{\Delta}_0\left(\lambda_n^0\right)}\{\beta^{+}\sin\pi n\mu^{+}(\pi)-\beta^{-}\sin\pi n\mu^{-}(\pi)+\frac{b_0(\pi)+H_1}{\beta}\left(\beta^{+}\cos\pi n\mu^{+}(\pi)+\beta^{-}\cos\pi n\mu^{-}(\pi)\right)$$
$$+\frac{1}{\beta}\int\limits_{-\mu^{+}(\pi)}^{\mu^{+}(\pi)}\left[K_{21}^0(\pi,t)+H_1K_{11}^0(\pi,t)\right]\cos\pi nt dt\}$$

from equality $\lambda_n^0=\widetilde{\lambda_n^0}$ is the interest $b_0(\pi)=\widetilde{b_0}(\pi)$ by using the expression , $b_0(x)$'s would

$$b_0(\pi)=-\frac{1}{2}\int\limits_0^{\pi}u^2(t)\,e^{-\frac{1}{2}\int\limits_t^{\pi}u(s)ds}dt$$

$$\int\limits_0^{\pi}u^2(t)\,e^{-\frac{1}{2}\int\limits_t^{\pi}u(s)ds}dt=\int\limits_0^{\pi}\widetilde{u}^2(t)\,e^{-\frac{1}{2}\int\limits_t^{\pi}\widetilde{u}(s)ds}dt \qquad (5.3).$$

From (5.3) is equal to $u(x)=Au_0(x)$ $\left(u_0(x)=\frac{x^{1-\alpha}}{1-\alpha}\right)$ can be written as

$$A^2\int\limits_0^{\pi}u_0^2(t)\,e^{-\frac{A}{2}\int\limits_t^{\pi}u_0(s)ds}dt=\widetilde{A}^2\int\limits_0^{\pi}u_0^2(t)\,e^{-\frac{\widetilde{A}}{2}\int\limits_t^{\pi}u_0(s)ds}dt$$

or

$$\left(A^2-\widetilde{A}^2\right)\int\limits_0^{\pi}u_0^2(t)\,e^{-\frac{A}{2}\int\limits_t^{\pi}u_0(s)ds}\left(\frac{1-e^{\frac{1}{2}(A-\widetilde{A})\int\limits_t^{\pi}u_0(s)ds}}{A-\widetilde{A}}\right)dt=0.$$

That is $A^2 = \widetilde{A}^2 \left(A, \widetilde{A} \geq 0 \right)$ or $A = \widetilde{A}$.

Using Lemma3 , we can prove the following two theorems .

Theorem.1: If $\lambda_n = \widetilde{\lambda_n}$, $\alpha_n = \widetilde{\alpha_n}$, then $q_0\left(x\right) = \widetilde{q_0}\left(x\right)$ are almost everywhere.

Theorem.2: If $\lambda_n = \widetilde{\lambda_n}$, $\mu_n = \widetilde{\mu_n}$, then $q_0\left(x\right) = \widetilde{q_0}\left(x\right)$ are almost everywhere.

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