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Fractals in extended b-metric space.

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Abstract.

Iterated function systems are method of constructing fractals, which are based on the mathematical foundations laid by Hutchinson[1] and Barnsley[2]. Formally an Iterated function systems is a finite set of 'contraction mappings', on a complete metric space X. In this paper we construct a fractal set of Iterated function systems, which in our case are a collection of mappings defined in an extended b-metric space, of compact subsets of the space. We will prove that the Hutchinson operator defined with the help of a finite family of 'generalized F-contraction mappings' on a complete extended b- metric space is itself a generalized F- contraction mapping on a family of compact subsets of X. Then by successive application of a generalized F-Hutchinson operator we obtain a final fractal in an extended b- metric space.

Keywords: Iterated function system; fractal; extended b-metric space.

1. Introduction

Iterated function systems are method of constructing fractals, which are based on the mathematical foundations laid by Hutchinson[1] and Barnsley[2]. Formally an Iterated function systems is a finite set of 'contraction mappings', on a complete metric space X. Hutchinson showed that, for the metric space $\mathbb{R}^n$ such a system of function has a unique nonempty closed and bounded fixed set S, which is called the attractor of the Iterated Function System. The fixed set S has the following property

$$S = \bigcup_{i=1}^{N} f_i(S) .$$

In this context fixed point theory plays an important role. Lately many researchers have obtained several results by extending the scope of metric fixed point theory, either by generalizing the domain of the mapping or by extending the contractive conditions on the mappings. For example the concept of metric has been generalized in many ways. Czerwik [3] introduced the concept of b- metric space. In [4] this concept was generalized further, by introducing the concept of extended b- metric space. In [4] and [5] several results are gained. In [6] a fractal is constructed with the help of a finite family of F-contraction mappings, which are more general than contraction mappings, defined on a complete metric space. In this paper we construct a fractal set of Iterated Function Systems, which in our case are a collection of mappings defined in an extended b-metric space, of compact subsets of the space. We will prove that the Hutchinson operator defined with the help of a finite family of 'generalized F-contraction mappings' on a complete extended b- metric space is itself a generalized F- contraction mapping on a family of compact subsets of X. Then by successive application of a generalized F-Hutchinson operator we obtain a final fractal in an extended b-metric space.

Definition 1.1. [4] *Let X be a nonempty set and $\theta : X \times X \to [1, +\infty[$. A function $d_\theta : X \times X \to [0, +\infty[$ is called an **extended b-metric** if for all $x, y, z \in X$ it satisfies*

1. $d_\theta(x,y)=0 \Leftrightarrow x=y$

2. $d_\theta(x,y)=d_\theta(y,x)$

3. $d_\theta(x,z)\le \theta(x,z)\left[d_\theta(x,y)+d_\theta(y,z)\right]$

Example 1

Let $X=\{2,3,4\}$. *Define* $\theta: X\times X\to [1,+\infty[$ *and* $d_\theta: X\times X\to[0,+\infty[$ *such that* $\theta(x,y)=2+x+y$, $d_\theta(2,2)=d_\theta(3,3)=d_\theta(4,4)=0$, $d_\theta(2,3)=d_\theta(3,2)=30$, $d_\theta(2,4)=d_\theta(4,2)=200$, $d_\theta(3,4)=d_\theta(4,3)=2000$

Example 2

Let $X=C([a,b],\mathbb{R})$ *be the space of all continuous real valued functions defined on* $[a,b]$. *X is an extended b-metric space by considering* $d_\theta(x,y)=\sup_{t\in[a,b]}\left|x(t)-y(t)\right|^2$ *with* $\theta(x,y)=\left|x(t)\right|+\left|y(t)\right|+1,$ *where* $\theta: X\times X\to[1,+\infty[$.

It is obvious that the class of extended b-metric spaces is larger than b-metric spaces, because if $\theta(x,y)=b$, for $b\ge 1$ then we obtain the definition of a b-metric space.

Definition 1.2. [4] *Let* (X,d_θ) *be an extended b-metric space.*

1.*A sequence* $\{x_n\}$ *in* X *is said to converge to* $x\in X$, *if for every* $\varepsilon>0$ *there exist* $N=N(\varepsilon)\in\mathbb{N}$ *such that* $d_\theta(x_n,x)<\varepsilon$ *for all* $n\ge N$.

2. A sequence $\{x_n\}$ *in* X *is said to be Cauchy, if for every* $\varepsilon>0$ *there exist* $N=N(\varepsilon)\in\mathbb{N}$ *such that* $d_\theta(x_n,x_m)<\varepsilon$ *for all* $n,m\ge N$.

3. An extended b-metric space (X,d_θ) *is complete if every Cauchy sequence in X is convergent.*

Denote $B(a,r)=\{x\in X; d_\theta(x,a)<r\}$ and $B[a,r]=\{x\in X; d_\theta(x,a)\le r\}$. We call them respectively the open ball and the closed ball.

Definition 1.3 .[5] *Let* (X,d_θ) *be an extended b-metric space .*

1. A subset A of X is called open if for any a∊A, it exists ε>0, such that $B(a,r)\subset A$

2. A subset B of X is called close if for any sequence x_n *, such that* $\lim_{n\to\infty} x_n = x$ *and* x_n *∊B for all n, then x∊B.*

In a b-metric space (*X*,*d*) are well known the following results

1. *d* is not necessarily continous in each variable
2. An open ball is not necessarily an open set.

In an extended b- metric space (X,d_θ) we can say the same thing, since every b- metric space is an extended b-metric space.

Definition 1.4: *Let* (X,d_θ) *be an extended b-metric space . A subset A of X is called:*

1. compact if and only if for every sequence of elements of A there exists a subsequence that converges to an element of A.

2. bounded if and only if $\delta(A) = \sup\{d_\theta(x, y) : x, y \in A\} < \infty$

Definition 1.5. *Let* (X, d_θ) *be an extended b-metric space and let* $H(X)$ denote the set of all nonempty compacts subsets of X . For $A, B \in H(X)$, let

$$H(A,B) = \max\left\{\sup_{a\in A}(d_\theta(a,B)), \sup_{b\in B}(d_\theta(b,A))\right\}$$

Where $d_\theta(x,B) = \inf\{d_\theta(x,y) : y \in B\}$ is the distance of a point x from the set B . The mapping H is said to be the Pompeiu-Hausdorff metric induced by d_θ. This type of metric was first introduced by Pompeiu [7] and was further modified by Hausdorff [8] in the natural setting of metric spaces.

Lemma 1.1.[5] *Let* (X, d_θ) *be an extended b-metric space. For any A,B,C,D sets of* H(X) we have

a. $\sup_{a\in A}(d_\theta(a,B))$ =0 if and only if $A \subseteq B$

b. If $B \subseteq C$ then $\sup_{a\in A}(d_\theta(a,C)) \le \sup_{a\in A}(d_\theta(a,B))$

c. $H(A \cup B, C \cup D) \le \max\{H(A,C), H(B,D)\}$

Lemma 1.2. [5] Let (X, d_θ) *be an extended b-metric space and CB(X) denote the set of all closed and bounded subsets of X. Then (CB(X),* $H_{\bar\theta}$*) is an extended b-metric space where the mapping* $\bar\theta : CB(X) \times CB(X) \to [1, +\infty)$ is such that $\bar\theta(A,B) = \sup\{\theta(a,b) : a \in A, b \in B\}$.

From now on we denote the set $H(X)$ by $H_{\bar\theta}(X)$ and the metric H by $H_{\bar\theta}$.

Lemma 1.3.[5] *Let* (X, d_θ) *be an extended b-metric space . If* d_θ *is continous in one variable then*

1. d_θ *is continous in the other variable.*
2. For each $a \in X$ *and r>0 we have that B(a,r) is open and B[a,r] is closed.*

Lemma 1.4.[9] *Let* (X, d_θ) *be an extended b-metric space where* $\theta : X \times X \to [1, +\infty[$ *a bounded function and* d_θ *is continous in one variable. If* (X, d_θ) *is a complete extended b-metric space then* $\left(H_{\bar\theta}(X), H_{\bar\theta}\right)$ *is also a complete extended b- metric space.*

Now we give the definition of a generalized contraction called F- contraction introduced by Wardowski [10]

Let $\mathcal{F}$ be the collection of all continuous mappings $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy the below conditions:

(F_1) F F is strictly increasing, i.e, for all $a, b \in \mathbb{R}^+$ such that $a < b$ implies that $F(a) < F(b)$.

(F_2) For every sequence $\{x_n\}$ of positive real numbers, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} F(x_n) = -\infty$ are equivalent.

(F_3) There exist $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Definition 1.6 [10]. Let (X,d) be a metric space. A mapping $f:X\to X$ is called F-contraction if for any $x,y\in X$ there exsist F∈ $\mathcal{F}$ and t>0 such that $t+F(d(f(x),f(y)))\le F(d(x,y))$, whenever $d(f(x),f(y))>0$.

It is clear that by using the strictly increasing property of F and the definition 1.5 we have that $d(f(x),f(y))\le d(x,y)$, $\forall x,y\in X$, such that $f(x)\ne f(y)$. This means that F-contraction is a contractive mapping, so it is also continuous. Furthermore, Wardowski proved that in a complete metric space (X,d) every F- contractive map has a unique fixed point in X and for every point $a\in X$, the sequence of iterates $\{a,f(a),f^2(a),...\}$ converges to the fixed point of f.

Let G be the set of all mapping $g:R_{\ge 0}\to R_{\ge 0}$ such that $\liminf_{x\to 0} g(x)>0$ for all $x\ge 0$.

Definition 1.7. Let (X,d_θ) be an extended b-metric space. A mapping $f:X\to X$ is called a generalized F-contraction if for any $x,y\in X$, there exist F∈ $\mathcal{F}$ and $g\in G$ such that $g(d_\theta(x,y))+F(d_\theta(f(x),f(y)))\le F(d_\theta(x,y))$ whenever $d_\theta(f(x),f(y))>0$.

2. Main Results

Theorem 2.1: Let (X,d_θ) be an extended b-metric space and let $f:X\to X$ be generalized F-contraction then the following statements hold:

1. f maps elements of $H_{\bar\theta}(X)$ to elements of $H_{\bar\theta}(X)$
2. If for any $A\in H_{\bar\theta}(X)$, $\overline{f}(A)=\{f(x):x\in A\}$ then $\overline{f}:H_{\bar\theta}(X)\to H_{\bar\theta}(X)$ is a generalized F-contraction mapping on $(H_{\bar\theta}(X),H_{\bar\theta})$.

Proof: Since a generalized F-contraction mapping is continuous, we obtain that the image of a compact subset of X under f is also compact. Thus we have $A\in H_{\bar\theta}(X)$ implies that $\overline{f}(A)\in H_{\bar\theta}(X)$.

Now for the second statement, let $A,B\in H_{\bar\theta}(X)$ such that $H_{\bar\theta}(\overline{f}(A),\overline{f}(B))\ne 0$. As $f:X\to X$ is a generalized F-contraction, we obtain $d_\theta(f(x),f(y))<d_\theta(x,y)$, for any $x,y\in X$ such that $x\ne y$.

So we have that $d_\theta(f(x),\overline{f}(B))=\inf_{y\in B}\{d_\theta(f(x),f(y))\}\le\inf_{y\in B}\{d_\theta(x,y)\}=d_\theta(x,B)$. Analogously we obtain

$$d_\theta(f(y),\overline{f}(A))\le d_\theta(y,A).$$

Furthermore

$$H_{\bar\theta}(\overline{f}(A),\overline{f}(B))=\max\{\sup_{x\in A}d_\theta(f(x),\overline{f}(B)),\sup_{y\in B}d_\theta(f(y),\overline{f}(A))\}<\max\{\sup_{x\in A}d_\theta(x,B)),\sup_{y\in B}d_\theta(y,A))\}=H_{\bar\theta}(A,B)$$

By applying the strictly increasing property of F we get that $F(H_{\bar\theta}(\overline{f}(A),\overline{f}(B)))<F(H_{\bar\theta}(A,B))$. Thus it exists a function $g:R_{\ge 0}\to R_{\ge 0}$ with $\liminf_{x\to 0} g(x)>0$ for all $x\ge 0$ such that

$$g(H_{\bar{\theta}}(A,B))+F(H_{\bar{\theta}}(\overline{f}(A),\overline{f}(B)))\leq F(H_{\bar{\theta}}(A,B))$$

.Thus $f:H_{\bar{\theta}}(X)\to H_{\bar{\theta}}(X)$ is a generalized F- contraction.

Theorem 2.2. Let (X,d_θ) be an extended b-metric space and let $\{f_i : i=1,2,...k\}$ be a finite family of generalized F-contraction of X to itself. Define $T:H_{\bar{\theta}}(X)\to H_{\bar{\theta}}(X)$ by $T(A)=\bigcup_{i=1}^{k} f_i(A)$, for each $A\in H_{\bar{\theta}}(X)$. Then T is a generalized F-contraction on $H_{\bar{\theta}}(X)$.

Proof: We will prove the statement for k=2. Let $f_1, f_2 : X\to X$ be two F-contractions. Take $A,B\in H_{\bar{\theta}}(X)$ with $H_{\bar{\theta}}(T(A),T(B)\neq 0$. From Lemma 1.1 (c), it follows that

$$g(H_{\bar{\theta}}(A,B)+F(H_{\bar{\theta}}(T(A),T(B)))=g(H_{\bar{\theta}}(A,B)+F\left(H_{\bar{\theta}}\left(f_1(A)\cup f_2(A), f_1(B)\cup f_2(B)\right)\right)\leq$$
$$g(H_{\bar{\theta}}(A,B)+F\left(\max\left\{H_{\bar{\theta}}\left(f_1(A), f_1(B)\right),H_{\bar{\theta}}\left(f_2(A), f_2(B)\right)\right\}\right)\leq F(H_{\bar{\theta}}(A,B)).$$

Definition 2.1. Let (X,d_θ) be an extended b-metric space. A mapping $T:H_{\bar{\theta}}(X)\to H_{\bar{\theta}}(X)$ is said to be a Ciric type generalized F-contraction if, for F∈ $\mathcal{F}$ and $g\in G$ such that, for any $A,B\in H(X)$ with $H_{\bar{\theta}}(T(A),T(B)\neq 0$, the following condition holds:

$$g(M_T(A,B))+F(H(T(A),T(B)))\leq F(M_T(A,B)), \tag{1}$$

where $$M_T(A,B)=\max\left\{\begin{array}{l}H_{\bar{\theta}}(A,B),H_{\bar{\theta}}(A,T(A)),H_{\bar{\theta}}(B,T(B)),\dfrac{H_{\bar{\theta}}(A,T(B))+H_{\bar{\theta}}(B,T(A))}{2\theta(A,T(B))},\\ H_{\bar{\theta}}\left(T^2(A),T(A)\right),H_{\bar{\theta}}\left(T^2(A),B\right),H_{\bar{\theta}}\left(T^2(A),T(B)\right)\end{array}\right\}.$$

Definition 2.2. Let (X,d_θ) be a complete extended b-metric space and let $\{f_i : i=1,2,...k\}$ be a finite family of generalized F-contraction of X to itself then $(X; f_1, f_2,..., f_k)$ is called generalized F-contractive iterated function system (IFS).

Therefore generalized F-contractive iterated function system (IFS) consist of a complete extended b-metric space and a finite family of generalized F-contraction.

Definition 2.3. Let (X,d_θ) be an extended b-metric space. Let $(X; f_1, f_2,..., f_k)$ be generalized F-contractive iterated function system. Let $T:H(X)\to H(X)$ be a mapping defined by $T(A)=\bigcup_{i=1}^{k} f_i(A)$ for each $A\in H(X)$. Then if T is a Ciric type generalized F-contraction we call it generalized F-Hutchinson operator.

Definition 2.4. A nonempty compact set $A\subseteq X$ is said to be an attractor of the generalized F-contractive IFS if

a. $T(A)=A$.

b. There is an open set $B\subseteq X$ such that $A\subseteq B$ and $\lim_{n\to\infty} T^n(C)=A$ for any compact set $C\subseteq B$, where the limit is taken with respect to the Hausdorff metric.

Proposition 2.1. Let (X,d_θ) be an extended b-metric space. Let $(X;f_1,f_2,...,f_k)$ be a generalized F-contractive iterated function system. Let $T:H_{\bar\theta}(X)\to H_{\bar\theta}(X)$ be a generalized F-Hutchinson operator . Let A_0 be an arbitrary element in $H_{\bar\theta}(X)$. And let $A_1=T(A_0),A_2=T(A_1),...,A_{m+1}=T(A_m)$ be a sequence of sets in $H_{\bar\theta}(X)$ then $\lim_{m\to\infty}H(A_{m+1},A_{m+2})=0$.

Proof. We assume that $A_m\neq A_{m+1}$ for all $m\in N$, because if not we have that $A_k=A_{k+1}$ for some *k* implies $A_k=T(A_k)$ and then the proof is finished. From (1), we have $g\left(M_T(A_m,A_{m+1})\right)+F\left(H(A_{m+1},A_{m+2})\right)=g\left(M_T(A_m,A_{m+1})\right)+F\left(H(T(A_m),T(A_{m+1}))\right)\le F\left(M_T(A_m,A_{m+1})\right)$ where

$$M_T(A_m,A_{m+1})=\max\left\{H(A_m,A_{m+1}),H(A_m,T(A_m)),H(A_{m+1},T(A_{m+1})),\frac{H(A_m,T(A_{m+1}))+H(A_{m+1},T(A_m))}{2\theta(A_m,T(A_{m+1}))},\right.$$
$$\left.H\left(T^2(A_m),T(A_m)\right),H\left(T^2(A_m),A_{m+1}\right),H\left(T^2(A_m),T(A_{m+1})\right)\right\}=\max\left\{H(A_m,A_{m+1}),H(A_m,A_{m+1})),H(A_{m+1},A_{m+2})),\right.$$
$$\left.\frac{H(A_m,A_{m+2}))+H(A_{m+1},A_{m+1}))}{2\theta(A_m,A_{m+2}))},H\left(A_{m+2}),A_{m+1})\right),H\left(A_{m+2},A_{m+1}\right),H\left(A_{m+2},A_{m+2}\right)\right\}=\max\left\{H(A_m,A_{m+1}),H(A_{m+1},A_{m+2})\right\}$$

In case $M_T(A_m,A_{m+1})=H(A_{m+1},A_{m+2})$, we have the following inequality:

$$F(H(A_{m+1},A_{m+2}))\le F(H(A_{m+1},A_{m+2}))-g(H(A_{m+1},A_{m+2}))$$

,

Which is a contradiction because $g(H(A_{m+1},A_{m+2}))>0$. Thus $M_T(A_m,A_{m+1})=H(A_m,A_{m+1})$. We note that

$$F(H(A_{m+1},A_{m+2}))\le F(H(A_m,A_{m+1}))-g(H(A_m,A_{m+1}))<F(H(A_m,A_{m+1}))\ .$$

Therefore $\{H(A_{m+1},A_{m+2})\}$ is a decreasing sequence and consequently convergent. Now we will show that it converges to zero. By property of g, there exist $c>0$ and $n_0\in\mathbb{N}$ such that $g(H(A_m,A_{m+1}))>c$ for all $m\ge n_0$. Therefore

$$\begin{aligned}&F\left(H(A_{m+1},A_{m+2})\right)\le F\left(H(A_m,A_{m+1})\right)-g\left(H(A_{m+1},A_{m+2})\right)\\&\le F\left(H(A_{m-1},A_m)\right)-g\left(H(A_{m-1},A_m)\right)-g\left(H(A_m,A_{m+1})\right)\\&\le\cdots\le H(A_{n_0},A_{n_0+1})-\left[g\left(H(A_{n_0},A_{n_0+1})\right)+g\left(H(A_{n_0+1},A_{n_0+2})\right)+\cdots+g\left(H(A_m,A_{m+1})\right)\right]\end{aligned}$$

and we have $F\left(H(A_{m+1},A_{m+2})\right)\le F\left(H(A_{n_0},A_{n_0+1})\right)-(m-n_0+1)c$.

By taking limit as $m\to\infty$ in the above inequality, we have $\lim_{m\to\infty}F\left(H(A_{m+1},A_{m+2})\right)=-\infty$, which together with property (F_2) implies that $\lim_{m\to\infty}H(A_{m+1},A_{m+2})=0$.

Theorem 2.1. Let (X,d_θ) be a complete extended b-metric space , where d_θ is a continuous function and $\theta:X\times X\to[1,+\infty[$ *is a bounded function by a number* $s>1$ *also* continuous in the first variable . Let

$(X; f_1, f_2, ..., f_k)$ be a generalized F-contractive iterated function system. Let $T: H_{\bar{\theta}}(X) \to H_{\bar{\theta}}(X)$ be a generalized F-Hutchinson operator. Let A_0 be an element in $H_{\bar{\theta}}(X)$ and let $A_1 = T(A_0), A_2 = T(A_1), ..., A_{m+1} = T(A_m)$ be a sequence of sets in $H_{\bar{\theta}}(X)$ such that $H_{\bar{\theta}}(A_m, A_{m+1}) \le \dfrac{1}{(s+1)^m}$ for all $m \in \mathbb{N}$. Then we have the following results:

a) The sequence of compact sets $\{A_0 = T(A_0), T^2(A_0), ...\}$ converges to a fixed point of T.

b) Operator T has a unique fixed point $U \in H_{\bar{\theta}}(X)$; i.e $U = T(U) = \bigcup_{i=1}^{k} f_i(U)$.

Proof: First we show that the sequence of compact sets $A_1 = T(A_0), A_2 = T(A_1), ..., A_{m+1} = T(A_m)$, such that $H_{\bar{\theta}}(A_m, A_{m+1}) \le \dfrac{1}{(s+1)^m}$ is a Cauchy sequence. By Lemma 1.2 we have that $H_{\bar{\theta}}$ is an extended b-metric where $\bar{\theta}$ is a bounded function by the same boundary $s > 1$ as θ . Let $\varepsilon > 0$ and choose a positive integer $N > 1$ such that $(\dfrac{s}{s+1})^{N-1} < \dfrac{\varepsilon}{2}$. Then for all $n > m \ge N$ we find that

$$H_\theta(A_m, A_n) \le sH_\theta(A_m, A_{m+1}) + s^2 H_\theta(A_{m+1}, A_{m+2}) + \cdots + s^{n-m-1} H_\theta(A_{n-2}, A_{n-1}) + s^{n-m-1} H_\theta(A_{n-1}, A_n)$$
$$< s\frac{1}{(s+1)^m} + s^2\frac{1}{(s+1)^{m+1}} + \cdots + s^{n-m-1}\frac{1}{(s+1)^{n-2}} + s^{n-m-1}\frac{1}{(s+1)^{n-1}} = \frac{s}{(s+1)^{m-1}} + \left(\frac{s}{s+1}\right)^{n-1}\frac{1}{s^m}$$
$$< \frac{s}{(s+1)^{m-1}} + \left(\frac{s}{s+1}\right)^{n-1} < \left(\frac{s}{s+1}\right)^{m-1} + \left(\frac{s}{s+1}\right)^{n-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By Lemma 1.4 $\left(H_{\bar{\theta}}(X), H_{\bar{\theta}}\right)$ *is a complete extended b- metric space* , thus we have that the sequence A_n is a convergent sequence to a set U , for some $U \in H_{\bar{\theta}}(X)$. Now we will show that U is a fixed point of T. Suppose, for a contradiction that $H_{\bar{\theta}}(U, T(U)) \ne 0$. By definition 1.5 it follows that

$$g(M_T(A_n, U)) + F(H(T(A_n), T(U))) \le F(M_T(A_n, U)) \tag{2}$$

where,

$$M_T(A_n, U) = \max\left\{ \begin{array}{l} H(A_n, U), H(A_n, T(A_n)), H(U, T(U)), \dfrac{H(A_n, T(U)) + H(U, T(A_n))}{2\theta(A_n, T(U))}, \\ H\left(T^2(A_n), T(A_n)\right), H\left(T^2(A_n), U\right), H\left(T^2(A_n), T(U)\right) \end{array} \right\}$$
$$= \max\left\{ \begin{array}{l} H(A_n, U), H(A_n, A_{n+1}), H(U, T(U)), \dfrac{H(A_n, T(U)) + H(U, A_{n+1})}{2\theta(A_n, T(U))}, \\ H\left(A_{n+2}, A_{n+1}\right), H\left(A_{n+2}, U\right), H\left(A_{n+2}, T(U)\right) \end{array} \right\}$$

Let us consider the following cases about the value of $M_T(A_n, U)$:

Case 1. $M_T(A_n,U)=H(A_n,U)$. Note that the operator T is continuous and since $\lim_{n\to\infty} A_n = U$ we get that $\lim_{n\to\infty} T(A_n)=T(U)$ and thus $\lim_{n\to\infty} H(T(A_n),T(U))=0$. By taking lower limit as $n\to\infty$ in (2),we have

$$\liminf_{n\to\infty} g(H(A_n,U))+F(H(U,T(U))\leq F(H(U,U))$$

But since $\liminf_{t\to 0} g(t)>0$ for all $t\geq 0$ we have $F(H(U,T(U)))<F(H(U,U))=F(0)$. By the strictly increasing property of F it follows that $H(U,T(U))<0$, which is a contradiction.

Case 2. $M_T(A_n,U)=H(A_n,A_{n+1})$. On taking lower limit as $n\to\infty$ in (2), we get

$$\liminf_{n\to\infty} g(H(A_n,A_{n+1}))+F(H(U,T(U))\leq F(H(U,U))$$

which is a contradiction.

Case 3. $M_T(A_n,U)=H(U,T(U))$, then, by taking lower limit as $n\to\infty$ in (2), we obtain

$$\liminf_{n\to\infty} g(H(U,T(U)))+F(H(U,T(U))\leq F(H(U,T(U)) \ .$$

But since $g(H(U,T(U)))>0$ we have that $F(H(U,T(U))<F(H(U,T(U))$, which is a contradiction.

Case 4. $M_T(A_n,U)=\dfrac{H(A_n,T(U))+H(U,A_{n+1})}{2\theta(A_n,T(U))}$, then

$$\liminf_{n\to\infty} g\left(\frac{H(A_n,T(U))+H(U,A_{n+1})}{2\theta(A_n,T(U))}\right)+F(H(U,T(U))\leq F\left(\frac{H(U,T(U))+H(U,U)}{2\theta(U,T(U))}\right),$$

The above inequality implies that $F(H(U,T(U))<F\left(\dfrac{H(U,T(U))+H(U,U)}{2\theta(U,T(U))}\right)=F\left(\dfrac{H(U,T(U))}{2\theta(U,T(U))}\right)$, which is a contradiction because of the strictly increasing property of the function F .

Case 5. $M_T(A_n,U)=H(A_{n+2},A_{n+1})$. By taking lower limit as $n\to\infty$ in (2) we get

$$\liminf_{n\to\infty} g(H(A_{n+2},A_{n+1}))+F(H(U,T(U))\leq F(H(U,U)) \ ,$$

which is a contradiction.

Case 6. $M_T(A_n,U)=H(A_{n+2},U)$ then

$$\liminf_{n\to\infty} g(H(A_{n+2},U))+F(H(U,T(U))\leq F(H(U,U)),$$

which gives a contradiction.

Case 7. $M_T(A_n,U)=H(A_{n+2},T(U))$ then , by taking lower limit as $n\to\infty$ in (2), we obtain

$$\liminf_{n\to\infty} g(H(A_{n+2},T(U)))+F(H(U,T(U))\leq F(H(U,T(U))),$$

which is a contradicition.

Finally we get that U is a fixed point of T. Now we will show that U is the unique fixed point of T. Suppose that also V is a a fixed point of T and $H(U,V)\neq 0$. Since T is a Cyric type generalized F-contraction we have the following inequality

$$g(M_T(U,V)) + F(H(T(U),T(V))) = g(M_T(U,V)) + F(H(U,V)) \leq F(M_T(U,V)),$$

where

$$M_T(U,V) = \max\left\{\begin{matrix} H(U,V), H(U,T(U)), H(V,T(V)), \dfrac{H(U,T(V)) + H(V,T(U))}{\theta(U,T(V))}, \\ H(T^2(U),T(U)), H(T^2(U),V), H(T^2(U),T(V)) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} H(U,V), H(U,U), H(V,V), \dfrac{H(U,V) + H(V,U)}{2\theta(U,V)}, \\ H(U,U), H(U,V), H(U,V) \end{matrix}\right\}$$

$$= H(U,V).$$

Thus we get $g(H(U,V)) + F(H(U,V)) \leq F(H(U,V))$, which is a contradiction. This completes the proof.

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