Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 12, Issue 5
Published online: November 07, 2017|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Long memory in the Hybrid Time Series

Masad Awdh Alrasheedi

Faculty of Business Administration, Taibah University, Saudi Arabia.

Abstract

In this paper, consideration is given to the assessment of the availability of long-term memory in a time series with variable coefficients that depend on the Markov chain or the continuous Markov process. In the work,the author has succeeded in establishing sufficient conditions for the presence of a long-term memory based on a multifractal detrended fluctuation analysis and the Geweke-Porter-Hudak method. This is demonstrated with a real example which involved an analysis of the Erste Group for the period07.01.2000 – 23.10.2017, as a result of which it was possible to prove that this company.

Keywords: Hybrid systems; hybrid time series; structural breaks; Markov process.

1. Introduction

In recent years, the issue of long-term memory in a time series has received a lot of attention, as has been highlighted by some of the work carried out in this area [4, 12, 14, 20]. The impetus and rationale for these studies is due to several key factors:

1. Time series is one of most popular mathematical models for the description of the real process with 'memory' or dependence of the real process from itself in previous moments of time. 'Memory' in this context can be understood as some dependence for time series X in the times t and s. 'Memory' can be described by some distribution between the values of time series or by the autocorrelation function, $\rho(h)$.
2. Time series is a simple and convenient toolbox for evaluating various properties of the real process - stationarity, heteroscedasticity, etc. It can also be quite easily used to build forecasting for the real process.

The basic model for this work will be the ARFIMA (p, d, q)[1] [6, 13] models for the discrete random process X, which is given as follows

$$\Phi(L)(1-L)^d X_t = \Theta(L)\varepsilon_t, \tag{1}$$

where L – lag operator[2], $d \in (-0.5; 0.5)$; $\Phi(\cdot)$ and $\Theta(\cdot)$ polynomials with degree p and q respectively, which define autoregressive and moving average 'parts' of the process X:

$$\Phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p,$$
$$\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q;$$

ε_t are independent identically distributed (iid) random variables with mean 0 and variance σ^2 [8]. In some studies [5, 6], authors have used a stronger assumption about random variables $\varepsilon_t$in order to calculate the forecast for the time series:

$$\varepsilon_t \sim iid\, N(0, \sigma^2).$$

[1]Autoregressive fractionally integrated moving average model.
[2] Lag operator for the random process X_t is defined as follows:

$$LX_t = X_{t-1}$$

for any t.

In order to select optimal models of the different ARFIMA(p, d, q) models, the author has used AIC[3] information criteria.
Unfortunately, despite the relative simplicity of using the time series, there was a sharp change in its characteristics(trend, autocorrelation function etc.) at different intervals for the real processX_t. This has been referred to in the literature as **structural breaks** [2, 3] and can be used to provide further reasons such as crises, seasonality of the market, etc. Some authors have attempted to explain the presence of the structural breaks attributing it to in the time series as result of the change model of the process. Other works have explained these phenomena by using some additional exogenous variables. The Hybrid system [1, 18, 19] is one of the most popular methods for describing random processes with some exogenous variables. For analysis of the hybrid time series, artificial neural networks are commonly used (ANN) [10, 19]. This approach is very useful in the analysis and forecasting of the time series, but it does require a large number of observations in the time series for constructing the ANN and the speed of this algorithm is very small. In this article,the easiest method of describing the time series is considered, which is based on the switching (regime-switching) [7, 11] or hybrid processes.
The main model of this work is the hybrid (or switching) ARFIMA model, which is described in the form below:

$$\Phi_{\mathrm{m_t}}(L)(1-L)^{d(m_t)}X_t = \Theta_{\mathrm{m_t}}(L)\varepsilon_t, \qquad (2)$$

where m_t – is the homogeneous Markov chain with finite space of the states $S = \{1,2,..,N\}$ and the transition matrix is

$$P = \begin{pmatrix} p_{11}p_{12} & \cdots & p_{1N} \\ p_{21}p_{22} & \cdots & p_{2N} \\ & \cdots & \\ p_{N1}p_{N2} & \cdots & p_{NN} \end{pmatrix}.$$

Polynomials $\Phi_{\mathrm{m_t}}(L)$ and $\Theta_{\mathrm{m_t}}(L)$ for the equation (2) have the next form

$$\Phi_{\mathrm{m_t}}(L) = 1 - \phi_1(m_t)L - \cdots - \phi_p(m_t)L^p,$$
$$\Theta_{\mathrm{m_t}}(L) = 1 + \theta_1(m_t)L + \cdots + \theta_q(m_t)L^q.$$

The result produces a model with new explanatory (exogenous) variables – the Markov chain m_t. This type of system is often called the ‘hybrid system’ or the ‘system with Markov switching’.
The main task of the work will be to study the availability of ***long-term memory*** [5, 6] in the time series, which is described with the help of the hybrid ARFIMA model (2) andto consider concepts of the long term memory effect for the time series. Fundamental research [9, 15, 17] about the ‘long term memory effect’ of the random process has concentratedon the self-similarity of the random processes. To analyze such systems, Harold Edwin Hurst [15] proposed the method of the normalized range ($R\backslash S$ analysis), the key parameter of which is the Hurst exponent (or Hurst index),H. The presence of such dependencies is very well tracked precisely in the time series, which is constructed with the help of stochastic difference equations. For the time series long-memory effect,one can describe this through the autocorrelation function, using the following definition [5]:
Definition 1. Time series $X_t$has long-memory behavior, if

1. X_t - stationary[4];
2. Series

$$\sum_{h=1}^{\infty} |\rho(h)|$$

are not convergence – sum of series $|\rho(1)|, |\rho(2)|, \ldots, |\rho(h)|, \ldots$ equals infinity. In other words,the author of [6] expounds this formula in the next context

$$h^{2d-1}\rho(h) \to \gamma > 0$$

for $h \to \infty$.

Using the definition of the [5], one can use next separation of the time series (1) by the presence of the long term memory for X_t:

1. Time series with long memory, if $d > 0$.

[3]Akaike information criteria.
4 We assume strong stationarity:

$$EX_t = const, cov(X_t, X_{t+s}) = f(s),$$

where $f(s)$ – some function.

2. Time series with short memory, if $d < 0$.
3. Uncorrelated time series (time series without memory), if $d = 0$.

Notice, that the AR, MA and ARMA processes belong to the 3rd type with $d = 0$ or $= 0.5$.

2. Hurst index and its estimation

One of the main methods for estimation of the parameter H or d is the multifractal detrended fluctuation analysis (MFDFA) and the GPH method. These methods are based on different assumptions; therefore, as will be demonstrated below, the values of the estimates for the two methods are different using the MFDFA and GPH methods respectively. However, the average value (by stationary distribution of the switching process m_t) on the interval will be almost the same for both methods. Both methods are briefly described in the next section.

2.1. Multifractal detrended fluctuation analysis (MFDFA)

The DFA method is based on the concept of the stochastic self-similarity of a random process, which was first considered by [17] – process $X_t$called self-similar with index a, if for any $t_i \in R_1$ and $k > 0$:

$$\left(X_{t_1}, \dots, X_{t_n}\right) \sim k^{-a}\left(X_{kt_1}, \dots, X_{kt_n}\right),$$

where $\sim$ means equality by distribution. Notice, that the standard Brownoan motion value of the parameter is 0.5. Therefore, this definition is natural for defining the Hurst index for the process.

By [16]'s estimation of the parameter H by MFDFA analysis for the time series X_t, $t = 1, \dots, T$one is able to achieve this in several steps:

1. Calculate summing process

$$Y_t = \sum_{i=1}^{t} X_i - \bar{X},$$

where $\bar{X} = \frac{1}{T}\sum_{i=1}^{T} X_i$.

2. Separate into intervals (i.e.)$1, \dots, T$ by n subinterval. In each of the subintervals,one can calculate the trend Y_t^j of the degree p, where $j = 1, \dots, n$.It has been observed that several studies assume a sufficient use of the first degree trend – $p = 1$.
3. Calculate value of the *fluctuation function*:

$$F(n,q) = \sqrt[q]{\sum_{i=1}^{n}\left(\frac{1}{n_i}\sum_{j=1}^{n_i}\left(Y_t - Y_t^i\right)^2\right)^{\frac{q}{2}}}, \tag{3}$$

where n_i – number of observation of the Y_t in the i-th interval. For $q = 2$ method of estimation is known as thedetrended fluctuation analysis. For $q = 0$, the following formula can be used:

$$F(n,0) = \exp\left\{\sum_{i=1}^{n} \log\left(\frac{1}{n_i}\sum_{j=1}^{n_i}\left(Y_t - Y_t^i\right)^2\right)\right\}.$$

4. Calculate the value of the fluctuation function for different values n and estimate coefficients of the linear regression.

$$\log\big(F(n,q)\big) = a + b * log(n).$$

Using estimation of the parameters a and $b$one can find an/the estimation of the Hurst exponent

$$\hat{H} = \hat{b},$$

where $\hat{b}$ – is the estimation of the parameter b.

To simplify the calculations, it is possible to use $q = 2$. In this case, formula (3) can be used to simplify the next form:

$$F(n,2) = \sqrt{\sum_{i=1}^{n} \frac{1}{n_i} \sum_{j=1}^{n_i} \left(Y_t - Y_t^i\right)^2}.$$

Hence, by MFDFA analysis it is possible to estimate the value of the parameter H and the value of the parameter d by the following formula:

$$d = H - 0.5.$$

The GPH method is based on the estimation of the spectral density of the process, which is described in detail in [12].

3. Main results

In this section, the author considers the main results of the study: namely, the theorem on the calculation of the Hurst index for hybrid time series based on the ergodic theorem for the Markov switching process m_t and the algorithm for calculating the Hurst index for the hybrid time series (2). It is worth pointing out that this algorithm contains some assumptions that can affect the final result – the Hurst index,H. Additionally, it should be noted that this result is marginal- that is, it can only be used for the time series with a large number of observations. The last set of results of this section areforthe theoretical results for the data of the *Erste Group*,for the period 07.01.2000 to 23.10.2017 – which consists of 4412 observations. It is also worth mentioning that real data, and not adjusted data was used in the study.

3.1. Main statement

In this section, the main result concerning the estimation of the Hurst index,H for the time series (2) is considered. The primary task is to obtain the given index from the corresponding indexes H_i calculated for the fixed states i and some characteristics of the Markov chain $m_t, t \geq 0$. This means by assumption, that it is possible to calculate the Hurst indexes H_i for each state $i \in \{1,..,N\}$ for the system

$$\Phi_i(L)(1-L)^{d(i)} X_t = \Theta_i(L)\varepsilon_t. \quad (4)$$

With regards to the Markov chain, the existing stationary distribution $\pi$is assumed – this is the standard assumption in the limit theorem with a switching process.

Theorem 1. Let next conditions hold

1. $m_t, t \geq 0$ ergodic Markov chain with finite numbers of possible states $\{1, \dots, N\}$ and stationary distribution $$\pi = (\pi_1, \dots, \pi_N).$$
2. For each fixed state of the switching Markov process $m_t \equiv i \in \{1, \dots, N\}$ for the ARFIMA process (5) value of the Hurst index, defined as H_i.
3. For any $i \in \{1, \dots, N\}$ solution of the equation $$\frac{\Phi_i(L)(1-L)^{d(i)}}{\Theta_i(z)} = 0$$ satisfy next condition: $|z_{ij}| > 1 + \varepsilon$, where $\varepsilon > 0$ – some positive constant.
4. $\varepsilon_t \sim iid$ with mean o and variance σ^2.

Then Hurst index for hybrid time series (2) is calculated using the follow formula:

$$H = \sum_{i=1}^{N} H_i \pi_i. \quad (5)$$

Proof. Firstly,it is noticed that by using assumption 1 of the Theorem, we can derive the next link using the following matrix:

$$\lim_{n\to\infty} P(n) = \lim_{n\to\infty} P^n = \begin{pmatrix} \pi_1 & \dots & \pi_N \\ \pi_1 & \dots & \pi_N \\ & \dots & \\ \pi_1 & \dots & \pi_N \end{pmatrix}.$$

Secondly, the MFDFA method value of the Hurst index H can be defined for $n$which is not particularly large – this means that the relationship between the logarithm of the fluctuation function $\log\left(F(n,q)\right)$ and the logarithm of the width of the window $log(n)$remain almost unchanged after some fixed n [16]. Using this fact, it is only possible to consider fixed-length segments of the trajectory of the time series X_t. By assumption 3 of the theorem, the ARFIMA(p,d,q) (2) process can be represented using AR(∞) in the next form:

$$\tilde{\Phi}_i(L)X_t = \varepsilon_t^i, \tag{6}$$

where

$$\tilde{\Phi}_i(L) = 1 - \tilde{\phi}_1^i(m_t)L - \cdots - \tilde{\phi}_n^i(m_t)L^n - \cdots.$$

Using opinions about fixed-length segments, AR(∞) can be replaced by AR(n) process, where $n < \infty$. Therefore, the infinite AR model (6) is replaced by the next finite model:

$$\bar{\bar{\Phi}}_i(L)X_t = \varepsilon_t^i,$$

where

$$\bar{\bar{\Phi}}_i(L) = 1 - \tilde{\phi}_1^i(m_t)L - \cdots - \tilde{\phi}_n^i(m_t)L^n,$$

n is some fixed number for all $i \in \{1, \dots, N\}$.

Using this fact, it is relatively easy to check that the random R^{n+1} dimensional random process

$$Y_t = (X_{t+1}, \dots, X_{t+n}, m_{t+n}) = (Y_t^1, \dots, Y_t^{n+1})$$

is the Markov process, where transition probabilities are dependent from distribution of the ε_t.

Moving on,one can consider the function

$$f(Y_t) \;\; = \frac{\sum_{i=1}^{n} \log(i) \log\left(F(i,q)\right)}{\sum_{i=1}^{n} \log(i)^2} = H_{m_{t+n}}, \tag{7}$$

where $F(i, 2)$ is values of the fluctuation function which defined in the (3). Consequently, if one applies Birkhoff`s ergodic theorem, the next relationship/link is achieved

$$\lim_{k\to\infty} \frac{f(Y_t) + f(Y_{t+1}) + \cdots + f(Y_{t+k})}{k} = \sum_{i=1}^{N} f(Y_t | Y_t^{n+1} = i)\pi_i \, .$$

By rewriting this relationship using values of the Hurst indexes H_i, this gives

$$\lim_{k\to\infty} \frac{H_{m_{t+n}} + H_{m_{t+n}+1} + \cdots + H_{m_{t+n}+k}}{k} = \sum_{i=1}^{N} H_i \pi_i \, .$$

On the other hand, it is possible to calculate the value of the Hurst index,H by using MFDFA and as a result of using large numbers, this would give

$$\lim_{k\to\infty} \frac{H_{m_{t+n}} + H_{m_{t+n}+1} + \cdots + H_{m_{t+n}+k}}{k} = H.$$

Hence, the last two last formulas used provide the final proof of theorem 1.

Remark 1.The proof of theorem 1 relies only on the MFDFA method in defining the Hurst exponent,H. There is no doubt that this proof of Theorem1 will work for the GPH method as estimation of the parameter H in this method is also a function of Markov process Y_t.

3.2. Algorithm of the estimation *H*

Consider the algorithm for calculating the Hurst index H for the time series, which is based on Theorem 1. Note that for using this theorem, it is assumed that the time series has a large number of observations. In the model example case discussed below, the number of observations is 4412, so this condition is fulfilled. On the other hand, this algorithm will propose an approach to the determination of partial ARFIMA models (4) for all $i \in \{1, \dots, N\}$.

Algorithm:

1. Define a number of the states N for the switching Markov process m_t. In general, the following formula is used in order to define this value:

$$N = log(T),$$

where T – is the length of the time series (number of observations in the time series). In the example above, in order to promote a better understanding of the dynamic of the time series, we consider only$N = 3$.

2. Estimate number K for creating $N * K$ subintervals the time series. In each interval, the Hurst index $H_i$is estimated.
3. Using values of the H_i, these intervals are separated by N clusters. Notice that for clustering, two dimension data is used $(H_i^{MFDFA}, H_i^{GPH}), i = 1, \dots, NK$, where H_i^{MFDFA} – is the estimation by MFDFA analysis, and H_i^{GPH} – is the estimation by the GPH method.
4. By clustering, elements of the matrix P can be estimated. In order to achieve this aim, the number of transitions from one cluster to another is calculated.
5. For the estimation matrix $\hat{P}$, stationary distribution $\pi$is calculated based on the equation

$$\begin{cases} \pi\hat{P} = \pi, \\ \sum_{i=1}^{N} \pi_i = 1. \end{cases}$$

6. Calculate values of the H by formula (5), where H_i – is the average value of the Hurst exponent in each cluster, obtained in the 2.
7. For estimation of the models (5), the results of the clustering are utilized. Then, interval with maximal for each state $i \in \{1, \dots, N\}$ are selected for which $m_t = i$. Using this interval, the optimal ARFIMA model can be evaluated using AIC criteria.

3.3. Real example

In order to see this example in real life, the study will consider the share price for the Erste Group Bank AG for the period 07.01.2000 – 23.10.2017. The data used in the example is unadjusted and has been taken from the seasonal component and trend; in other words, it is *real* data. For this model, the algorithm proposed in the previous paragraph will be gradually executed. Therefore, the initial values for the algorithm are:

$$T = 4412, N = 3, K = 20.$$

The dynamic of the share can be seen in Figure 1. From the values of the parameters N and K, $NK = 60$ subinterval is produced with width $\frac{T}{KN} \approx 73$.

In the third step, the estimation of the Hurst index,$H_i$is considered using the MFDFA and GPH methods respectively. The results produce two samples (for each method) of the estimations H in each subinterval.
For step 4 of the process, the next estimation of the transition matrix P is produced:

$$\hat{P} = \begin{pmatrix} 0.967 & 0.033 & 0 \\ 0 & 0.926 & 0.074 \\ 0 & 1 & 0 \end{pmatrix}.$$

In step 5, using matrix $\hat{P}$, it is possible to calculate stationary distribution of the Markov chain m_t:

$$\pi = (0; 0.931; 0.069).$$

As canbe seen from this distribution, the discrete Markov chain will most likely turn into state 2 – the probability of this event occurring is 0.931.
Before step 6, the values of the $H_i$for the two methods are calculated:

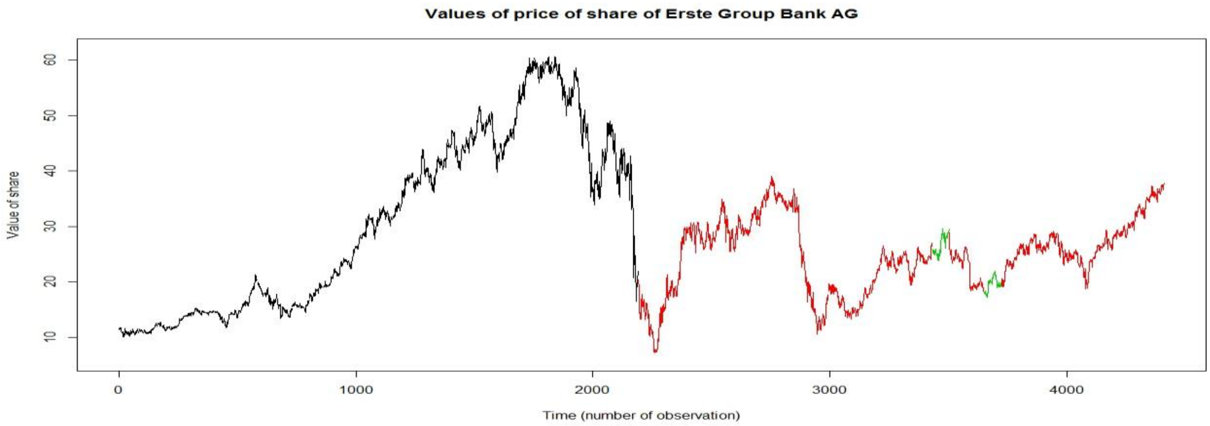

Figure 1.Dynamics of the share price for the Erste Group Bank AG for the period 07.01.2000 – 23.10.2017 (daily data – 4412 observations) for three states of the Markov chain (black – state 2, red – state 2 and green – state 3).

1. MFDFA method – $(0.3975165;\ 1.3978825;\ 1.9014966)$.
2. GPH method – $(0.4386114;\ 1.5397136;\ 0.7953074)$.

As can be observed, these methods produce results for different states of the Markov chain m_t. Now it is possible to complete step 6 of the example, by calculating the Hurst indexes for two methods using formula (5):

$$H^{MFDFA} = 0.3975165 * 0 + 1.3978825 * 0.931 + 1.9014966 * 0.069 = 1.432615,$$
$$H^{GPH} = 0.4386114 * 0 + 1.5397136 * 0.931 + 0.7953074 * 0.069 = 1.488375.$$

From the values above, one is able to determine that the results produced are very similar for the two methods. Furthermore, it is noted that before the crisis,the value of the Hurst exponent is less than 0.5, which means that the share price of the Erste Group Bank AG has a short memory in this period. On the other hand, after the crisis, the value of the Hurst index H increases. This fact is confirmed by model (2) with an additional exogenous variable in the model - the Markov chain m_t. From an economic point of view, such a change could lead to a sharp change in the company's policy and/or a sharp change in the market (in this case the financial market).

4. Conclusion

In this paper, a new approach was proposed to estimate the Hurst index in the hybrid time series (2) with Markov switching. The proof of the main statement is based on the ergodic theorem for the Markov processes. There is no doubt that this approach can be extended to more complex processes (semi-Markov processes, semimartingales, etc.). As a model example, the dynamics of the stock for the Erste Group Bank AG was considered, resulting in a sharp difference between the values of the Hurst index for this company,prior to and following the crisis. For estimations of the Hurst index, two methods were used – MFDFA and GPH. The data produced from these methods provided very similar results.

References

[1] C., Aladag, Egrioglub E., and Kadilar C. "Forecasting Nonlinear Time Series with A Hybrid Methodology." *Applied Mathematics Letters* 22, no. 9 (2009): 1467–70.

[2] D., Andrews. "Tests for Parameter Instability and Structural Change with Unknown Change Point: A Corrigendum." *Econometrica* 71, no. 1 (2003): 395–397.

[3] J., Bai, and Perron P. "Computation and Analysis of Multiple Structural Change Models." *Journal of Applied Econometrics* 18, no. 1 (2003): 1–22.

[4] R.T., Baillic. "Long Memory Processes and Fractional Integration in Econometrics." *Journal of Econometrics* 73, no. 59 (1996): 5–59.

[5] E.P.B., Box, Jenkince G.M., Reinsel G.S., and Ljung G.M. *Time Series Analysis: Forecasting and Control* . 5th ed. Wiley, 2015.

[6] P.J., Brockwell, and Davis R.A. *Introduction to Time Series and Forecasting* . 3rd ed. Springer International Publishing Switzerland, 2016.

[7] L, Calvet, and Adlai F. "How to Forecast Long-Run Volatility: RegimeSwitching and the Estimation of Multifractal Processes." *Journal of Financial Econometrics* 2 (2004): 49–83.

[8] R., Durrett. *Probability: Theory and Examples* . 4th ed. Cambridge University Press, 2013.

[9] A., Einstein. *Investigations on the Theory of the Brownian Movement*. Methuen & Co, 1926.

[10] C., Ferreira. *Gene Expression Programming: Mathematical Modeling By An Artificial Intelligence*. Springer-Verlag, 2006.

[11] C, Francq, and Zakoian C.M. "Stationarity of Multivariate Markov-Switching ARMA Models." *Journal of Econometrics* 102 (2001): 339–64.

[12] J., Geweke, and Porter-Hudak S. "The Estimation and Application of Long Memory Time Series Models." *Journal of Time Series Analysis* 4 (1983): 221–37.

[13] M.A., Hauser. "Maximum Likelihood Estimators for ARMA and ARFIMA Models: A Monte Carlo Study." *J. Statist. Planning Inference* 80 (1999): 229–255.

[14] T.O., Henry. "Long Memory in Stock International Evidence." *Applied Financial Economics* 12 (2002): 725–29.

[15] H.E., Hurst. "Long-Term Storage Capacity of Reservoirs." *Transactions of the American Society of Civil Engineers* 16, no. 1 (1951): 770–99.

[16] J.W., Kantelhardt, Zschiegner S.A., Koscielny-Bunde E., HavlinS., Bunde A., and Stanley H.E. . "Multifractal Detrended Fluctuation Analysis of Non Stationary Time Series." *Physica A* 316 (2002): 87–114.

[17] A.N., Kolmogorov. "The Wiener Spiral and Some Other Interesting Curves in Hilbert Space." *Dokl. Akad. Nauk SSSR* 26 (1940): 115–118.

[18] D., Plikynas. "Explaining International Investment Patterns: A Neural Network Approach." *Information Technology and Control* 34, no. 1 (2005): 42–50.

[19] G. , Zhang. "Time Series Forecasting Using A Hybrid ARMA and Neural Network Model." *Journal of Neurocomputing* 50 (2003): 159–75.

[20] P., Wilfredo. *Long-Memory Time Series: Theory and Methods*. Willey, 2007.