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Lyapunov-type inequalities for fractional differential equations

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Abstract

The principal aim of this paper is to discuss Lyapunov-type inequalities for fractional differential equations with fractional boundary conditions. Some new Lyapunov's inequalities are established, which almost generalize and improve some earlier results in the literature.

Keywords: Lyapunov-type inequalities; Fractional differential equations; Riemann-Liouville derivative; Green's function; Boundary conditions.

1. Introduction

We recall the well-known Lyapunov inequality for Hill's equation

$$x''(t)+q(t)x(t)=0 \tag{1.1}$$

where $q(t)\in L^1[a,b]$ is a real-valued function. If $x(t)$ is a nontrivial solution of Eq. (1.1) such that $x(a)=x(b)=0,$ where $a<b$ are two consecutive zeros of $x(t),$ then the following inequality holds:

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}, \tag{1.2}$$

where the constant 4 is sharp [1], and Ineq. (1.2) is known as Lyapunov inequality.
The Lyapunov-type inequality and its generalizations have been used as a useful tool in oscillation theory, eigenvalue problems, disconjugacy, and many other areas of differential and difference equations, see for instance [2-5, 8-10] and the references cited therein.

The study of Lyapunov-type inequalities for fractional differential functions has begun recently. Ferreira [7] first study Lyapunov inequality in this direction, where he derived a Lyapunov-type inequality as follows.
Theorem 1.1 (see [7]) Consider the fractional boundary value problem

$$\begin{cases} {}_aD^\alpha x(t)+q(t)x(t)=0, a<t<b, \\ x(a)=x(b)=0, \end{cases} \tag{1.3}$$

where ${}_aD^\alpha$ is the (left) Riemann-Liouville derivative of order $\alpha\in(1,2]$ and $q(t):[a,b]\to\mathbb{R}$ is a continuous function. If Eq. (1.3) has a nontrivial solution, then

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)(\frac{4}{b-a})^{\alpha-1}. \tag{1.4}$$

In 2017, Agarwall et al. [6] studied Lyapunov type inequalities for mixed nonlinear fractional differential equations with a forcing term

$$\begin{cases} {}_aD^{\alpha}(x(t)) + p(t)|x(t)|^{\mu-1}x(t) + q(t)|x(t)|^{\gamma-1}x(t) = f(t), \\ x(a) = x(b) = 0, \end{cases}$$

where $0 < \alpha \le 2, 0 < \gamma < 1 < \mu < 2.$

In this paper, we obtain Lyapunov-type inequalities for the Riemann-Liouville fractional nonlinear differential equations with a forcing term

$$\begin{cases} D^{\beta}_{a^+}(r(t)D^{\alpha}_{a^+}x(t)) + \sum_{k=1}^{n} p_k(t)|x(t)|^{\mu_k-1}x(t) = f(t), a < t < b, \\ x(a) = x'(a) = x'(b) = 0, D^{\alpha}_{a^+}x(a) = D^{\alpha}_{a^+}x(b) = 0, \end{cases} \tag{1.5}$$

where $0 < \mu_k < 2, 1 \le k \le n, \alpha \in (2,3], \beta \in (1,2]. p_k(t), f(t):[a,b] \to \mathbb{R}$ are continuous functions for all $k = 1, 2, \cdots, n,$ and $r(t) \in C[a,b]$ such that $r(t) > 0.$

The organization of the rest of this paper is as follows: the next Section recalls some definitions and Lemmas which will play an important role in the proof of our main results. In Section 3, we establish some new Lyapunove-type inequalities for Eq. (1.5), and a example illustrating the result is also given in Section 4.

2. Preliminaries

In this section, we introduce some preliminaries.

Definition 2.1 The Riemann-Liouville fractional derivative of order $\alpha \ge 0$ is defined by

$$(D^0_{a^+}f)(t) = f(t)$$

and

$$(D^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds, t \in [a,b],$$

where $n = [\alpha] + 1.$

Lemma 2.2[6] If A is positive, and B, z are nonnegative, then

$$Az^2 - Bz^{\alpha} + (2-\alpha)\alpha^{\alpha/(2-\alpha)}2^{2/(\alpha-2)}A^{-\alpha/(2-\alpha)}B^{2/(2-\alpha)} \ge 0$$

for any $\alpha \in (0,2)$ with equality holding if and only if $B = z = 0.$

Lemma 2.3[4] Let $\alpha > 0,$ if $D^{\alpha}_{a^+}u \in C[a,b],$ then

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(t) = u(t) + \sum_{k=1}^{n} c_k(t-a)^{\alpha-k},$$

where $n = [\alpha] + 1.$

Lemma 2.4[4] Let $2 < \alpha \le 3$ and $y \in C[a,b].$ Then the problem

$$\begin{cases} D^{\alpha}_{a^+}u(t) + y(t) = 0, a < t < b, \\ u(a) = u'(a) = u'(b) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_a^b G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1} - (t-s)^{\alpha-1}, a \le s \le t \le b, \\ (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1}, a \le t \le s \le b. \end{cases}$$

Lemma 2.5 Let $y \in C[a,b], 1 < \beta \le 2 < \alpha \le 3, r(t) > 0$ and $r(t) \in C[a,b].$ Then the problem

$$\begin{cases} D_{a^+}^{\beta}(r(t)D_{a^+}^{\alpha}x(t)) + y(t) = 0, a < t < b, \\ x(a) = x'(a) = x'(b) = 0, D_{a^+}^{\alpha}x(a) = D_{a^+}^{\alpha}x(b) = 0 \end{cases}$$

has a unique solution

$$x(t) = -\int_a^b G(t,s)(\frac{1}{r(s)}\int_a^b H(s,\tau)y(\tau)d\tau)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1} - (t-s)^{\alpha-1}, a \le s \le t \le b, \\ (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1}, a \le t \le s \le b, \end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} (\frac{b-s}{b-a})^{\beta-1}(t-a)^{\beta-1} - (t-s)^{\beta-1}, a \le s \le t \le b, \\ (\frac{b-s}{b-a})^{\beta-1}(t-a)^{\beta-1}, a \le t \le s \le b. \end{cases}$$

Proof. From Lemma 2.3 we have

$$r(t)D_{a^+}^{\alpha}x(t) = -\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds + c_1(t-a)^{\beta-1} + c_2(t-a)^{\beta-2},$$

where $c_i, i = 1, 2,$ are real constants. The condition $D_{a^+}^{\alpha}x(a) = 0$ implies that $r(a)D_{a^+}^{\alpha}x(a) = 0,$ which concludes $c_2 = 0.$ Then the condition $D_{a^+}^{\alpha}x(b) = 0$ implies that $r(b)D_{a^+}^{\alpha}x(b) = 0,$ which yields

$$c_1 = \frac{1}{\Gamma(\beta)}\int_a^b (\frac{b-s}{b-a})^{\beta-1}y(s)ds.$$

Then

$$r(t)D_{a^+}^{\alpha}x(t) = -\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)}\int_a^b (\frac{b-s}{b-a})^{\beta-1}y(s)ds,$$

this is,

$$r(t)D_{a^+}^{\alpha}x(t) = \int_a^b H(t,\tau)y(\tau)d\tau.$$

Therefore,

$$D_{a^+}^{\alpha}x(t) - \frac{1}{r(t)}\int_a^b H(t,\tau)y(\tau)d\tau = 0.$$

Setting

$$y_1(t) = -\frac{1}{r(t)}\int_a^b H(t,\tau)y(\tau)d\tau,$$

we have

$$\begin{cases} D_{a^+}^{\alpha}x(t) + y_1(t) = 0, a < t < b, \\ x(a) = x'(a) = x'(b) = 0. \end{cases}$$

Finally, applying Lemma 2.4, we obtain the desired result.

3. Main results

Our main results are the following Lyapunov-type inequalities.

Theorem 3.1 If Eq. (1.5) has a nontrivial continuous solution, then

$$[\int_a^b \frac{(b-s)^{\alpha-2}(s-a)}{r(s)}ds]^2 \int_d^b (b-s)^{\beta-1}(s-a)^{\beta-1}\sum_{k=1}^n |p_k(s)|ds \int_a^b (b-s)^{\beta-1} (s-a)^{\beta-1}[\sum_{k=1}^n \gamma_k |p_k(s)| + |f(s)|]ds > \frac{\Gamma^2(\alpha)\Gamma^2(\beta)(b-a)^{2\beta-2}}{4}, \tag{3.1}$$

where $\gamma_k = (2-\mu_k)\mu_k^{\mu_k/(2-\mu_k)} 2^{2/(\mu_k-2)}, k = 1, 2, \cdots, n.$

Proof. Let $x(t)$ be a nontrivial continuous solution, from Lemma 2.5, we have

$$x(t) = -\int_a^b G(t,s)(\frac{1}{r(s)}\int_a^b H(s,\tau)[\sum_{k=1}^n p_k(\tau)|x(\tau)|^{\mu_k-1}x(\tau) - f(\tau)]d\tau)ds.$$

Let $|x(c)| = \max_{t\in[a,b]}|x(t)|$. From [4] we have

$$0 \le G(t,s) \le G(b,s), (t,s) \in [a,b]\times[a,b],$$

$$0 \le H(t,s) \le H(s,s), (t,s) \in [a,b]\times[a,b].$$

As a consequence, we have

$$\begin{aligned}
|x(c)| &= \left|-\int_a^b G(c,s)(\frac{1}{r(s)}\int_a^b H(s,\tau)[\sum_{k=1}^n p_k(\tau)|x(\tau)|^{\mu_k-1}x(\tau) - f(\tau)]d\tau)ds\right| \\
&\le \int_a^b \frac{G(c,s)}{r(s)}\int_a^b H(s,\tau)[\sum_{k=1}^n |p_k(\tau)||x(\tau)|^{\mu_k} + |f(\tau)|]d\tau ds \\
&\le \int_a^b \frac{G(b,s)}{r(s)}\int_a^b H(\tau,\tau)[\sum_{k=1}^n |p_k(\tau)||x(\tau)|^{\mu_k} + |f(\tau)|]d\tau ds \\
&= \int_a^b \frac{G(b,s)}{r(s)}ds\int_a^b H(s,s)[\sum_{k=1}^n |p_k(s)||x(s)|^{\mu_k} + |f(s)|]ds \\
&\le \sum_{k=1}^n Q_k|x(c)|^{\mu_k} + F,
\end{aligned} \tag{3.2}$$

where

$$Q_k = \int_a^b \frac{G(b,s)}{r(s)}ds\int_a^b H(s,s)|p_k(s)|ds, k = 1, 2, \cdots, n,$$

$$F = \int_a^b \frac{G(b,s)}{r(s)}ds\int_a^b H(s,s)|f(s)|ds.$$

In Lemma 2.2 with $A = B = 1$, implies that

$$|x(c)|^{\mu_k} < |x(c)|^2 + \gamma_k, k = 1, 2, \cdots, n.$$

Using this inequality and Ineq. (3.2) we find the following quadratic inequality:

$$\sum_{k=1}^{n} Q_k |x(c)|^2 - |x(c)| + \sum_{k=1}^{n} \gamma_k Q_k + F > 0.$$

But this is only possible when

$$\sum_{k=1}^{n} Q_k (\sum_{k=1}^{n} \gamma_k Q_k + F) > \frac{1}{4},$$

which is the same as (3.1). This completes the proof of Theoren 3.1.

Theorem 3.2 If Eq. (1.5) has a nontrivial continuous solution, then

$$[\int_a^b \frac{(b-s)^{\alpha-2}(s-a)}{r(s)} ds]^2 \int_a^b \sum_{k=1}^{n} |p_k(s)| ds \int_a^b [\sum_{k=1}^{n} \gamma_k |p_k(s)| + |f(s)|] ds > \frac{4^{2\beta-3}\Gamma^2(\alpha)\Gamma^2(\beta)}{(b-a)^{2\beta-2}}, \quad (3.3)$$

where $\gamma_k = (2-\mu_k)\mu_k^{\mu_k/(2-\mu_k)} 2^{2/(\mu_k-2)}, k = 1, 2, \cdots, n.$

Proof. Let $\psi(s) = (b-s)(s-a), s \in [a,b].$ Observe that the function $\psi(s)$ has a maximum at the point $s_1 = \frac{a+b}{2}$, that is,

$$\psi_{\max}(s) = \psi(s_1) = \frac{(b-a)^2}{4}.$$

The desired result follows immediately from the last equality and inequality (3.1). This completes the proof of Theoren 3.2.

Theoren 3.3 If Eq. (1.5) has a nontrivial continuous solution, then

$$(\int_a^b \frac{ds}{r(s)})^2 \int_a^b \sum_{k=1}^{n} |p_k(s)| ds \int_a^b [\sum_{k=1}^{n} \gamma_k |p_k(s)| + |f(s)|] ds > \frac{4^{2\beta-3}\Gamma^2(\alpha)\Gamma^2(\beta)(\alpha-1)^{2\alpha-2}}{(\alpha-2)^{2\alpha-4}(b-a)^{2(\alpha+\beta-2)}}, \quad (3.4)$$

where $\gamma_k = (2-\mu_k)\mu_k^{\mu_k/(2-\mu_k)} 2^{2/(\mu_k-2)}, k = 1, 2, \cdots, n.$

Proof. Let

$$\phi(s) = (b-s)^{\alpha-2}(s-a), s \in [a,b],$$

then

$$\phi'(s) = (b-s)^{\alpha-3}[b-s-(\alpha-2)(s-a)], s \in [a,b].$$

When $s_* = \frac{b+a(\alpha-2)}{\alpha-1}$, the function has $\phi'(s_*) = 0.$ So the function $\phi(s)$ has a maximum at the point $s_* = \frac{b+a(\alpha-2)}{\alpha-1}$, this is

$$\phi_{\max}(s) = \phi(s_*) = (\alpha-2)^{\alpha-2}(\frac{b-a}{\alpha-1})^{\alpha-1}.$$

The desired result follows immediately from the last equality and inequality (3.3). This completes the proof of Theoren 3.3.

For $\alpha=3,\beta=2,$ Eq. (1.5) becomes

$$\begin{cases}(r(t)x^{(3)}(t))''+\sum_{k=1}^{n}p_k(t)|x(t)|^{\mu_k-1}x(t)=f(t),a<t<b,\\ x(a)=x'(a)=x'(b)=0,x^{(3)}(a)=x^{(3)}(b)=0.\end{cases} \tag{3.5}$$

In this case, taking $\alpha=3,\beta=2$ in Theorem 3.1, we obtain the following result.

Corollary 3.4 If Eq. (3.5) has a nontrivial continuous solution, then

$$(\int_a^b\frac{(b-s)(s-a)}{r(s)}ds)^2\int_a^b(b-s)(s-a)\sum_{k=1}^{n}|p_k(s)|\ ds\int_a^b(b-s)(s-a)[\sum_{k=1}^{n}\gamma_k|p_k(s)|+|f(s)|]ds>(b-a)^2, \tag{3.6}$$

where $\gamma_k=(2-\mu_k)\mu_k^{\mu_k/(2-\mu_k)}2^{2/(\mu_k-2)},k=1,2,\cdots,n.$

Corollary 3.5 If Eq. (3.5) has a nontrivial continuous solution, then

$$(\int_a^b\frac{ds}{r(s)})^2\int_a^b\sum_{k=1}^{n}|p_k(s)|ds\int_a^b[\sum_{k=1}^{n}\gamma_k|p_k(s)|+|f(s)|]ds>\frac{256}{(b-a)^6}, \tag{3.7}$$

where $\gamma_k=(2-\mu_k)\mu_k^{\mu_k/(2-\mu_k)}2^{2/(\mu_k-2)},k=1,2,\cdots,n.$

4. Example

Example 4.1 Consider the fractional equation

$$\begin{cases}D_{0^+}^{\frac{5}{3}}(D_{0^+}^{\frac{7}{3}}x(t))+p_1(t)|x(t)|^{-\frac{1}{3}}x(t)+p_2(t)|x(t)|^{\frac{1}{3}}x(t)=f(t),t\geqslant 0,\\ x(0)=x'(0)=x'(b)=0,D_{0^+}^{\frac{7}{3}}x(0)=D_{0^+}^{\frac{7}{3}}x(b)=0,\end{cases}$$

where $p_1(t),p_2(t)$ and $f(t):[0,b]\to\mathbb{R}$ are continuous functions. If the solution $x(t)$ has consecutive zeros at 0 and $b>0,$ in view of Theorem 3.1 the following inequality must be satisfied

$$[\int_0^b(b-s)^{\frac{1}{3}}sds]^2\int_0^b(b-s)^{\frac{2}{3}}s^{\frac{2}{3}}(|p_1(s)|+|p_2(s)|)ds\int_0^b(b-s)^{\frac{2}{3}}s^{\frac{2}{3}}(2\times(\frac{1}{3})^{\frac{3}{2}}|p_1(s)|$$
$$+\frac{4}{27}|p_2(s)|+|f(s)|)ds>\frac{\Gamma^2(\frac{7}{3})\Gamma^2(\frac{5}{3})b^{\frac{4}{3}}}{4}.$$

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