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Some results on extended b-metric spaces and Pompeiu-Hausdorff metric.

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Abstract

In this paper we will show some new results about extended b- metric space. Given an extended b-metric space (X, d_θ) ,we may define a new extended b- metric space with Pompeiu- Hausdorff metric H on the set $H(X)$ of the collection of all nonempty compact subsets of X. We will show that if (X, d_θ) is a complete extended b- metric space then the Hausdorff extended b-metric space $(H(X), H)$ is also complete.

Keywords: Extended b- metric space; Pompeiu-Hausdorff metric; Complete Spaces.

1. Introduction

The Pompeiu Hausdorff distance measures the distance between subsets of a metric space. It was initiated by D. Pompeiu in [6]. Further Felix Hausdorff [7] studies the notion of set distance, in the natural setting of metric spaces and made a small modification. Informally it gives the largest length out of the set of all distances between each point of a set to the closest point of the second set. It is well known that given any metric space, the Pompeiu Hausdorff distance defines respectively a metric on the space of all nonempty compact subsets of the metric space. The idea of generalizing metric spaces into b-metric spaces was initiated from the works of Bourbaki [4], Czerwik [5.] In [1] the idea of b-metric space was generalized further by introducing the concept of extended b-metric space. In this paper we will extend the Pompeiu Hausdorff metric in an extended b- metric space.

Definition 1.1. [1] *Let* X *be a nonempty set and* $\theta : X \times X \to [1, +\infty[$ *. A function* $d_\theta : X \times X \to [0, +\infty[$ *is called an* ***extended b-metric*** *if for all* $x, y, z \in X$ *it satisfies*

1. $d_\theta(x, y) = 0 \Leftrightarrow x = y$

2. $d_\theta(x, y) = d_\theta(y, x)$

3. $d_\theta(x, z) \le \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$

It is obvious that the class of extended b-metric spaces is larger than b-metric spaces, because if $\theta(x, y) = b$, for $b \ge 1$ then we obtain the definition of a b-metric space.

Definition 1.2. [1] *Let* (X, d_θ) *be an extended b-metric space.*

1.*A sequence* $\{x_n\}$ *in* X *is said to converge to* $x \in X$ *, if for every* $\varepsilon > 0$ *there exist* $N = N(\varepsilon) \in \mathbb{N}$ *such that* $d_\theta(x_n, x) < \varepsilon$ *for all* $n \ge N$ *.*

2. A sequence $\{x_n\}$ *in* X *is said to be Cauchy, if for every* $\varepsilon > 0$ *there exist* $N = N(\varepsilon) \in \mathbb{N}$ *such that* $d_\theta(x_n, x_m) < \varepsilon$ *for all* $n, m \geq N$.

3. An extended b-metric space (X, d_θ) *is complete if every Cauchy sequence in X is convergent.*

Denote $B(a,r) = \{x \in X; d_\theta(x,a) < r\}$ and $B[a,r] = \{x \in X; d_\theta(x,a) \leq r\}$. We call them respectively the open ball and the closed ball.

Definition 1.3. *Let* (X, d_θ) *be an extended b-metric space . A subset A of X is called open if for any* $a \in A$ *, it exists* $\varepsilon > 0$ *,such that* $B(a,r) \subset A$ *. A subset of* B *of* X *is called closed if for any sequence* $\{x_n\}$*, such that* $\lim_{n\to\infty} x_n = x$ *and* $x_n \in B$ *for all* $n \in \mathbb{N}$*, then* $x \in B$.

In a b-metric space (X, d) are well known the following results

1. d_θ is not necessarily continous in each variable
2. An open ball is not necessarily an open set.

In an extended b- metric space (X, d_θ) we can say the same thing, since every b- metric space is an extended b-metric space.

Lemma 1.1.[2] *Let* (X, d_θ) *be an extended b-metric space. If* d_θ *is continous in one variable then* d_θ *is continous in the other variable.*

Lemma 1.2.[2] *Let* (X, d_θ) *be an extended b-metric space. If* d_θ *is continous in one variable then for each* $a \in X$ *and* $r > 0$ *we have*

1. $B(a,r)$ *is open*
2. $B[a,r]$ *is closed*

Definition 1.4: *Let* (X, d_θ) *be an extended b-metric space . A subset A of X is called*

1. compact if and only if for every sequence of elements of A there exists a subsequence that converges to an element of A.

2.*bounded if and only if* $\delta(A) = \sup\{d_\theta(x,y) : x, y \in A\} < \infty$.

3. *totally bounded if and only if for each* $\varepsilon > 0$ there exists a finite collection of open balls $B(x_i, \varepsilon)$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.

Denote $d_\theta(x, A) = \inf\{d_\theta(x,a) : a \in A\}$ and $H(X)$ the collection of all nonempty compact subsets of X .

Lemma 1.3. *Let* (X, d_θ) *be an extended b-metric space where* d_θ is a continuous function in one variable.

Let $x \in X$ and $A \in H(X)$ *then there exist* $a_x \in A$ *such that* $d_\theta(x, A) = d_\theta(x, a_x)$.

Proof: By definition of an infinum we can let $\{a_n\}$ be a sequence in A such that

$$d_\theta(x, A) \leq d_\theta(x, a_n) < d_\theta(x, A) + \frac{1}{n} .$$

Since A is a compact set then there exist a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to an element $a_x \in A$. Then we get

$$d_\theta(x,A) \le d_\theta(x,a_{n_k}) < d_\theta(x,A) + \frac{1}{n_k}. \qquad (1)$$

By the continuity of d_θ it follows that $\lim_{n_k\to\infty} d_\theta(x,a_{n_k}) = d_\theta(x,a_x)$. On taking limit as $n_k \to \infty$ in (1) we obtain

$$d_\theta(x,A) \le d_\theta(x,a_x) \le d_\theta(x,A).$$

□

Lemma 1.4. *Let* (X,d_θ) *be a complete extended b-metric space and* A *a closed subset of* X *then the set* A *is complete*

Proof: Is straightforward.

Lemma 1.5 (X,d_θ) *is a compact space if and only if it is a complete extended b-metric space and totally bounded.*

Proof: The proof is analogus to the case where (X,d_θ) is a metric space. The reader may find further details in [4] or in [7]

Proposition 1.1. *Let* (X,d_θ) *be an extended b-metric space where* $\theta: X\times X \to [1,+\infty[$ *is a bounded function (i.e., there exist s>1 such that for all* $(x,y)\in X\times X$, $\theta(x,y)\le s$ *). If* $\{x_k\}$ is a sequence in (X,d_θ) *with the property that* $d_\theta(x_k,x_{k+1}) < \frac{1}{(s+1)^k}$ for all k , then $\{x_k\}$ is a Cauchy sequence.

Proof: Let $\varepsilon > 0$ and choose positive integer $N > 1$ such that $(\frac{s}{s+1})^{N-1} < \frac{\varepsilon}{2}$. Then for all $n > m \ge N$ we find that

$$\begin{aligned} d_\theta(x_m,x_n) &\le s d_\theta(x_m,x_{m+1}) + s^2 d_\theta(x_{m+1},x_{m+2}) + \cdots + s^{n-m-1} d_\theta(x_{n-2},x_{n-1}) + s^{n-m-1} d_\theta(x_{n-1},x_n) \\ &< s\frac{1}{(s+1)^m} + s^2\frac{1}{(s+1)^{m+1}} + \cdots + s^{n-m-1}\frac{1}{(s+1)^{n-2}} + s^{n-m-1}\frac{1}{(s+1)^{n-1}} = \frac{s}{(s+1)^{m-1}} + \left(\frac{s}{s+1}\right)^{n-1}\frac{1}{s^m} \\ &< \frac{s}{(s+1)^{m-1}} + \left(\frac{s}{s+1}\right)^{n-1} < \left(\frac{s}{s+1}\right)^{m-1} + \left(\frac{s}{s+1}\right)^{n-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

It follows that $\{x_k\}$ is a Cauchy sequence.

□

2. Main Resuls

Definition 2.1. Let (X, d_θ) be an extended b-metric space. For A, B ∊ H(X), let $H_\theta(A,B) = \max\left\{\sup_{a\in A}(d_\theta(a,B)), \sup_{b\in B}(d_\theta(b,A))\right\}$. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d_θ.

Definiton 2.2. For any A ∊ H(X), and any positive number ε, let

$A_\varepsilon = \{x\in X : d_\theta(x,y)\le\varepsilon, \text{for some } y\in A\} = \{x\in X : d_\theta(x,A)\le\varepsilon\}$.

Remark 1: Notice that $\sup_{a\in A}(d_\theta(a,B))\le\varepsilon$ if and only if $A\subset B_\varepsilon$. By this last one we can give an equivalent definition for the mapping H as following

$$H_\theta(A,B)=\inf\left\{\varepsilon: A\subset B_\varepsilon \text{and } B\subset A_\varepsilon\right\}.$$

Proposition 2.1.[2] *Let* (X,d_θ) *be an extended b-metric space. For any A,B,C,D sets of* H(X) we have

a. $\sup_{a\in A}(d_\theta(a,B))$ =0 if and only if $A\subseteq B$

b. If $B\subseteq C$ then $\sup_{a\in A}(d_\theta(a,C))\le\sup_{a\in A}(d_\theta(a,B))$

c. $H(A\cup B,C\cup D)\le\max\{H(A,C),H(B,D)\}$

Proposition 2.2.[2]. Let (X,d_θ) *be an extended b-metric space and CB(X) denote the set of all closed and bounded subsets of X. Then (CB(X),* H_θ*) is an extended b-metric space where the mapping* $\theta: CB(X)\times CB(X)\to[1,+\infty)$ is such that

$$\theta(A,B)=\sup\left\{\theta(a,b): a\in A, b\in B\right\}$$

Definition 2.3. An extended b-metric space (X,d_θ) is complete if every Cauchy sequence must converge to a point in *X*.

In order to show that the space $(H(X),H_\theta)$ is complete whenever (X,d_θ) is complete we will choose an arbitrary Cauchy sequence $\{A_n\}$ in $H(X)$ and show that it converges to some $A\in H(X)$

Let A be the set of all points $x\in X$ such that there is a sequence $\{x_n\}$ that converges to *x* and $x_n\in A_n$ for all $n\in N$. We will show that A is the desired point of convergence of the sequence $\{A_n\}$.

But first we give some important propositions.

Proposition 2.3. If d_θ is continuous then the set A_ε is closed for all $A\in H(X)$.

Proof. Let $A\in H(X)$, $\varepsilon>0$ and x be an arbitrary limit point of A_ε. Then there exists a sequence $\{x_n\}\in A_\varepsilon$ that converges to x. Since $\{x_n\}\in A_\varepsilon$ for all n, by the definition of A_ε it follows that $d_\theta(x_n,A)\le\varepsilon$ for all *n*. By Lemma 1.3 there exist $a_n\in A$ such that $d_\theta(x_n,A)=d_\theta(x_n,a_n)$. Therefore $d_\theta(x_n,a_n)\le\varepsilon$ for all n. By the compactness of A it follows that each sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point $a\in A$. Also since $\{x_n\}$ converges to x then also its subsequence $\{x_{n_k}\}$ converges to $d_\theta(x,a)$. Thus by the continuity of d_θ we have that $d_\theta(x_{n_k},a_{n_k})$ converges to $d_\theta(x,a)$. Since $\{a_{n_k}\}$ and $\{x_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{x_n\}$ respectively, it follows that $d_\theta(x_{n_k},a_{n_k})\le\varepsilon$ for all k. Therefore $d_\theta(x,a)\le\varepsilon$, so $x\in A_\varepsilon$. Note that since x was an arbitrary limit point , then A_ε is a closed set since it contains all of its limit points.

□

Proposition 2.4. *Let* (X, d_θ) *be an extended b-metric space where* $\theta : X \times X \to [1, +\infty[$ *is a bounded function by a number* $s > 1$. Let $\{A_n\}$ be a Cauchy sequence in $H(X)$ and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$, for all k, then there exists a Cauchy sequence $\{a_n\}$ in X such that $a_n \in A_n$, for all n and $a_{n_k} = x_{n_k}$ for all k.

Proof. Let $\{x_{n_k}\}$ be a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$, for all k. Define $n_0 = 0$. For each n that satisfies $n_{k-1} < n \le n_k$, use Lemma 1.3 to choose $a_n \in A_n$ such that $d_\theta(x_{n_k}, A_n) = d_\theta(x_{n_k}, a_n)$. Then we find that

$$d_\theta(x_{n_k}, a_n) = d_\theta(x_{n_k}, A_n) \le \sup_{a \in A_{n_k}} \{d_\theta(a, A_n)\} \le H(A_{n_k}, A_n).$$

Note that since $x_{n_k} \in A_{n_k}$, then $d_\theta(x_{n_k}, a_{n_k}) = d_\theta(x_{n_k}, A_{n_k}) = 0$. It follows that $x_{n_k} = a_{n_k}$ for all k. Let $\varepsilon > 0$. Since $\{x_{n_k}\}$ is a Cauchy sequence in X, there exists a positive integer P such that

for all $k, j \ge P$. Since $\{A_n\}$ is a Cauchy sequence in $H(X)$, by definition there exists a positive integer $N \ge n_P$ such that $H_\theta(A_n, A_m) < \dfrac{\varepsilon}{(s + 2s^2)}$ for all $n, m \ge N$. Suppose that $j, k \ge P$. Then there exists integers $j, k \ge P$ such that $n_{k-1} < n \le n_k$ and $n_{j-1} < m \le n_j$. Note that

$$\begin{aligned}
d_\theta(a_n, a_m) &\le s d_\theta(a_n, x_{n_k}) + s^2 d_\theta(x_{n_k}, x_{n_j}) + s^2 d_\theta(x_{n_j}, a_m) \\
&= s d_\theta(x_{n_k}, A_n) + s^2 d_\theta(x_{n_k}, x_{n_j}) + s^2 d_\theta(x_{n_j}, A_m) \\
&\le s \sup\left\{ d_\theta(a, A_n) \middle| a \in A_{n_k} \right\} + s^2 d_\theta(x_{n_k}, x_{n_j}) + s^2 \sup\left\{ d_\theta(a, A_m) \middle| a \in A_{n_j} \right\} \\
&\le s H(A_{n_k}, A_n) + s^2 d_\theta(x_{n_k}, x_{n_j}) + s^2 H(A_{n_j}, A_m) \\
&< s \frac{\varepsilon}{(s + 2s^2)} + s^2 \frac{\varepsilon}{(s + 2s^2)} + s^2 \frac{\varepsilon}{(s + 2s^2)} = \varepsilon.
\end{aligned}$$

Thus $\{a_n\}$ is a Cauchy sequence in X such that $a_n \in A_n$ for all n and $a_{n_k} = x_{n_k}$ for all k.

□

From now on, the space (X, d_θ) is an extended b-metric space where $\theta : X \times X \to [1, +\infty[$ is a bounded function by a number $s > 1$ and d_θ is a continuous function.

In the next Proposition we will show that A is closed and nonempty in order to show that A is in $H(X)$.

Proposition 2.5. Let (X, d_θ) be a complete extended b-metric space and let $\{A_n\}$ be a sequence in $H(X)$ and let A be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence, then the set A is closed and nonempty.

Proof. At first we will show that A is nonempty. Let $\{A_n\}$ be a Cauchy sequence, thus it exists an integer n_1 such that $H_\theta(A_m, A_n) < \frac{1}{s+1}$ for all $m,n \geq n_1$. Similarly there exists an integer $n_2 > n_1$ such that $H_\theta(A_m, A_n) < \frac{1}{(s+1)^2}$ for all $m,n \geq n_2$. Continuing this process we have an increasing sequence $\{n_k\}$ such that $H_\theta(A_m, A_n) < \frac{1}{(s+1)^k}$ for all $m,n \geq n_k$. Let x_{n_1} be a fixed point in A_{n_1}. By Lemma 1.3 we can choose $x_{n_2} \in A_{n_2}$ such that $d_\theta(x_{n_1}, x_{n_2}) = d_\theta(x_{n_1}, A_{n_2})$. Note that

$$d_\theta(x_{n_1}, x_{n_2}) = d_\theta(x_{n_1}, A_{n_2}) \leq \sup\left\{d_\theta(a, A_{n_2}) \middle| a \in A_{n_1}\right\} \leq H_\theta(A_{n_1}, A_{n_2}) < \frac{1}{s+1}.$$

Simiarly we can choose $x_{n_3} \in A_{n_3}$ such that

$$d_\theta(x_{n_2}, x_{n_3}) = d_\theta(x_{n_2}, A_{n_3}) \leq \sup\left\{d_\theta(a, A_{n_3}) \middle| a \in A_{n_2}\right\} \leq H_\theta(A_{n_2}, A_{n_3}) < \frac{1}{(s+1)^2}.$$

By continuing this process we are able to obtain a sequence $\{x_{n_k}\}$ where each $x_{n_k} \in A_{n_k}$ for all k and

$$d_\theta(x_{n_k}, x_{n_{k+1}}) = d_\theta(x_{n_k}, A_{n_{k+1}}) \leq \sup\left\{d_\theta(a, A_{n_{k+1}}) \middle| a \in A_{n_k}\right\} \leq H_\theta(A_{n_k}, A_{n_{k+1}}) < \frac{1}{(s+1)^k}.$$

By Proposition 1.1 $\{x_{n_k}\}$ is a Cauchy sequence. Thus, since $\{x_{n_k}\}$ is a Cauchy sequence and $x_{n_k} \in A_{n_k}$ for all k, by Proposition 2.4 there exist a sequence $\{a_n\}$ in X such that $a_n \in A_n$, for all n and $a_{n_k} = x_{n_k}$ for all k. Since X is complete, the Cauchy sequence $\{a_n\}$ converges to a point $a \in X$. Since $a_n \in A_n$, for all n, then by the definition of the set A it follows that $a \in A$. It means that A is nonempty.

Now to prove that A is closed, let a be a limit point of A. Then by the definition of the limit point there exists sequence $y_k \in A \setminus \{a\}$ that converges to a. Since each $y_k \in A$ there exists a sequence $\{a_n{}^k\}$ such that $a_n{}^k$ converges to y_k and $a_n{}^k \in A_n$ for each n. It follows that there exists an integer n_1 such that $x_{n_1} \in A_{n_1}$ and $d_\theta(x_{n_1}, y_1) < 1$. Similarly there exist an integer $n_2 > n_1$ and a point $x_{n_2} \in A_{n_2}$ such that $d_\theta(x_{n_2}, y_2) < \frac{1}{2}$. By continuing this process we can construct an increasing sequence n_k of integers such that $d_\theta(x_{n_k}, y_k) < \frac{1}{k}$ for all k. Therefore ,we have that

$$d_\theta(x_{n_k}, a) \leq s(d_\theta(x_{n_k}, y_k) + d_\theta(y_k, a)).$$

Note that by taking limit as $k \to \infty$ to the above inequality it follows that the distance between $\{x_{n_k}\}$ and a converges to zero. Thus $\{x_{n_k}\}$ converges to a. This means that $\{x_{n_k}\}$ is a Cauchy sequence for which $x_{n_k} \in A_{n_k}$ for all k. By Proposition 2.4 there exists a Cauchy sequence $\{a_n\}$ in X such that $a_n \in A_n$, for all n and $a_{n_k} = x_{n_k}$. So it follows that $a \in A$, thus A is closed.

□

Proposition 2.6. Let $\{A_n\}$ be a sequence of totally bounded sets in X and let A be any subset of X. If for each $\varepsilon > 0$ there exists a positive integer N such that $A \subseteq A_N + \varepsilon$, then A is totally bounded.

Proof. Let $\varepsilon > 0$. Choose a positive integer N such that $A \subseteq A_N + \frac{\varepsilon}{4s^2}$. Since A_N is totally bounded, we can choose a finite set $\{x_i : 1 \le i \le k\}$ where $x_i \in A_N$ such that $A_N \subseteq \bigcup_{i=1}^{k} B_{d_\theta}(x_i, \frac{\varepsilon}{4s^2})$. Note that for each $a \in A$ from Lemma 1.3 there exists $x \in A_N$ such that $d_\theta(x,a) \le \frac{\varepsilon}{4s^2}$. Furthermore there exists $x_i \in A_N$ such that $d_\theta(x,x_i) \le \frac{\varepsilon}{4s^2}$. So we have that

$$d_\theta(a,x_i) \le s(d_\theta(a,x) + d_\theta(x,x_i)) \le s(\frac{\varepsilon}{4s^2} + \frac{\varepsilon}{4s^2}) = \frac{\varepsilon}{2s}. \quad (2)$$

This means that for some i, $B_{d_\theta}(x_i, \frac{\varepsilon}{2s}) \cap A \ne \varnothing$. By reordering the x_i' s, we may assume that $B_{d_\theta}(x_i, \frac{\varepsilon}{2s}) \cap A \ne \varnothing$ for $1 \le i \le p$ and $B_{d_\theta}(x_i, \frac{\varepsilon}{2s}) \cap A = \varnothing$ for $i > p$. Then for each $1 \le i \le p$, let $y_i \in B_{d_\theta}(x_i, \frac{\varepsilon}{2s}) \cap A$. We will show that $A \subseteq \bigcup_{i=1}^{p} B_{d_\theta}(y_i, \varepsilon)$. Let $a \in A$. As we mentioned before there exist x and x_i such that satisfy inequality (2) and $x \in B_{d_\theta}(x_i, \frac{\varepsilon}{2s})$. Let $y_i \in B_{d_\theta}(x_i, \frac{\varepsilon}{2s}) \cap A$. It follows that

$$d_\theta(a,y_i) \le s(d_\theta(a,x_i) + d_\theta(x_i,y_i)) \le s(\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s}) = \varepsilon .$$

Finally since for each $a \in A$ we found y_i for some $1 \le i \le p$ such that $a \in B_{d_\theta}(y_i, \varepsilon)$, it means that A is totally bounded.

□

Theorem 2.1. *Let (X, d_θ) be a complete extended b-metric space, then also $(H(X), H)$ is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $H(X)$. By Lemma 1.5 we know that $\{A_n\}$ are totally bounded and complete sets. Define A to be the set of all points $x \in X$ such there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We need to show that $A \in H(X)$ and $\{A_n\}$ converges to A. By Proposition 2.5, the set A is closed and nonempty. Let $\varepsilon > 0$. Since $\{A_n\}$ is Cauchy then there exists a positive integer N such that $H_\theta(A_n, A_m) < \varepsilon$ for all $m,n \ge N$. By Remark 1 then $A_m \subseteq (A_n)_\varepsilon$ for all $m,n \ge N$. Now we will show that $A \subseteq (A_n)_\varepsilon$ for all $n \ge N$. Fix $n \ge N$ and let $a \in A$. By the definition of the set A there exists a sequence $\{x_i\}$ such that $x_i \in A_i$ for all i and $\{x_i\}$ converges to a. By Proposition 2.3 the set $(A_n)_\varepsilon$ is closed. Since

$x_i \in (A_n)_\varepsilon$ for all $i \geq N$, then it follows that $a \in (A_n)_\varepsilon$. This means that $A \subseteq (A_n)_\varepsilon$. By Proposition 2.6, the set A is totally bounded. Furthermore by Lemma 1.4 the set A is complete. Finally since it is totally bounded and complete it is compact. Thus we proved that $A \in H(X)$. Now to prove that $\{A_n\}$ converges to A let $\varepsilon > 0$. We must prove that there exists a positive integer N such that $H_\theta(A_n, A) < \varepsilon$ for all $n \geq N$. By Remark 1 we need to show that $A \subseteq (A_n)_\varepsilon$ and $A_n \subseteq A_\varepsilon$. But from the first part of our proof we know that there exists N such that $A \subseteq (A_n)_\varepsilon$ for all $n \geq N$. Now to prove that $A_n \subseteq A_\varepsilon$, let $y \in A_n$ and let $\varepsilon > 0$. Let $\varepsilon_1 = \dfrac{\varepsilon}{s+1}$

We must prove that there exist $a \in A$ such that $d_\theta(y,a) < \varepsilon$. Since $\{A_n\}$ is a Cauchy sequence we can choose a positive integer N such that $H_\theta(A_m, A_n) < \varepsilon_1$ for all $m,n \geq N$. Additionally since $\{A_n\}$ is a Cauchy sequence in $H(X)$, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $H_\theta(A_m, A_n) < \dfrac{\varepsilon_1}{(s+1)^{i+1}}$ for all $m,n > n_i$ and $n_1 > N$. Note that by using Lemma 1.3 and the fact that $A_n \subseteq (A_{n_1})_{\frac{\varepsilon_1}{s+1}}$ then there exist $x_{n_1} \in A_{n_1}$ such that $d_\theta(y, x_{n_1}) \leq \dfrac{\varepsilon_1}{s+1}$. Also since $A_{n_1} \subseteq (A_{n_2})_{\frac{\varepsilon_1}{(s+1)^2}}$, then there exist $x_{n_2} \in A_{n_2}$ such that $d_\theta(x_{n_1}, x_{n_2}) \leq \dfrac{\varepsilon_1}{(s+1)^2}$. Continuing this process we can choose a sequence $\{x_{n_i}\}$ such that $x_{n_i} \in A_{n_i}$ for all positive integers i and $d_\theta(x_{n_i}, x_{n_{i+1}}) \leq \dfrac{\varepsilon_1}{(s+1)^{i+1}}$. By Proposition 1.1 we know that $\{x_{n_i}\}$ is a Cauchy sequence. Since (X, d_θ) is a complete extended b-metric space then the sequence $\{x_{n_i}\}$ is also a convergent sequence. So there exist $a \in X$ such that x_{n_i} converges to a. By Proposition 2.4 we find that there exist a sequence $\{y_n\}$ such that converges to a, $y_n \in A_n$ for all n and $y_{n_i} = x_{n_i}$. This means that $a \in A$. Furthermore we notice that

$$
\begin{aligned}
d_\theta(y, x_{n_i}) &\leq d_\theta(y, x_{n_1}) + d_\theta(x_{n_1}, x_{n_2}) + d_\theta(x_{n_2}, x_{n_3}) + \cdots + d_\theta(x_{n_{i-1}}, x_{n_i}) \\
&\leq \frac{s\varepsilon_1}{s+1} + \frac{s^2\varepsilon_1}{(s+1)^2} + \frac{s^3\varepsilon_1}{(s+1)^3} + \cdots + \frac{s^{i-1}\varepsilon_1}{(s+1)^{i-1}} + \frac{s^{i-1}\varepsilon_1}{(s+1)^i} \\
&= \varepsilon_1 \left[\frac{\frac{s}{s+1}\left(1 - \left(\frac{s}{s+1}\right)^{i-1}\right)}{1 - \frac{s}{s+1}} \right] + \frac{s^{i-1}\varepsilon_1}{(s+1)^i} < \frac{\varepsilon_1 \frac{s}{s+1}}{1 - \frac{s}{s+1}} + \frac{s^i \varepsilon_1}{(s+1)^i} \\
&< \varepsilon_1 s + \varepsilon_1 = \varepsilon_1(s+1) = \frac{\varepsilon}{s+1}(s+1) = \varepsilon.
\end{aligned}
$$

By using the fact that $d_\theta(y, x_{n_i}) < \varepsilon$ for all i and d_θ is a continous function, it follows that $d_\theta(y,a) < \varepsilon$,thus $y \in A_\varepsilon$. Therefore we know that there exists N such that $A_n \subseteq A_\varepsilon$ for all $n \geq N$. So we have that $H_\theta(A_n, A) < \varepsilon$ for all $n \geq N$, meaning that $\{A_n\}$ converges to $A \in H(X)$. This completes the proof.

□

References

[1] T. Kamran, M. Samreen, Q. UL Ain (2017). A generalization of b-metric space and some fixed point theorems. MDPI *Mathematics Journal*, 5 ,19, 2017 doi:10.3390/math5020019

[2] Subashi, L. (2017). Some topological properties of extended b-metric space, *Proceedings of The 5th International Virtual Conference on Advanced Scientific Results (SCIECONF-2017)*, Vol. 5, pg 164-167. Zilina, Slovakia. doi: 10.18638/scieconf.2017.5.1.451

[3] Bourbaki, N. (1974) Topologie Generale; Herman: Paris, France, 1974.

[4] Czerwik, S. (1993). Contraction mappings in b-metric spaces. (pp 5–11). *Acta Math. Inform. Univ. Ostra*. 1.

[5] D. Pompeiu (1905). Sur la continuite' des fonctions de variables complexes (These). Gauthier- Villars, Paris, ;Ann.Fac.Sci.deToulouse 7:264-315.

[6] F. Hausdorff (1914). Grunclzuege der Mengenlehre. Viet, Leipzig.

[7] https://www.math.wisc.edu/~seeger/522/c13.pd.